Conditional Fredholm determinant for the $S$-periodic orbits in Hamiltonian systems

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Abstract

For $S$ being a symplectic orthogonal matrix on $\mathbb{R}^{2n}$, the $S$-periodic orbits in Hamiltonian systems are a solution which satisfies $x(0) = Sx(T)$ for some period $T$. This paper is devoted to establishing the theory of conditional Fredholm determinant in studying the $S$-periodic orbits in Hamiltonian systems. First, we study the property of the conditional Fredholm determinant, such as the Fréchet differentiability, the splittingness for the cyclic type symmetric solutions. Also, we generalize the Hill formula originally gotten by Hill and Poincaré. More precisely, let $M$ be the monodromy matrix of the $S$-periodic orbits, then we get the formula relating the characteristic polynomial of the matrix $SM$ and the conditional Fredholm determinant. Moreover, we study the relation of the conditional Fredholm determinant and the relative Morse index. Applications to the problem of linear stability for the $S$-periodic orbits are given.

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1. Introduction

The purpose of the paper is to establish the theory of conditional Fredholm determinant and use it to study the $S$-periodic orbits in Hamiltonian systems.

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The \( S \)-periodic solution is a kind of generalized periodic solution of Hamiltonian systems, precisely, it is the solution of the following system:

\[
\begin{align*}
\dot{z}(t) &= JH'(t, z(t)), \\
z(0) &= Sz(T),
\end{align*}
\]  

(1.1)  
(1.2)

where \( J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \) and \( I_n \) is the identity matrix on \( \mathbb{R}^n \), \( H(t, x) \in C^2(\mathbb{R}^{2n+1}, \mathbb{R}) \), and \( S \in \text{Sp}(2n) \cap O(2n) \). Here we denote by \( \text{Sp}(2n) \) the symplectic group, that is,

\[
\text{Sp}(2n) = \left\{ M \in \text{GL}(2n, \mathbb{R}) \mid M^T J M = J \right\},
\]

where \( \text{GL}(2n, \mathbb{R}) \) is the group of all the \( 2n \times 2n \) complex matrices, and \( O(2n) \) the orthogonal group on \( \mathbb{R}^{2n} \). The reason why we consider the \( S \)-boundary problem is from the following two aspects:

The first one is motivated by the study of the closed geodesics on Riemannian manifold. More precisely, for a fixed closed geodesic \( c \) in an \( (n+1) \)-dimensional Riemannian manifold \( M \), let \( \{e_1(0), \ldots, e_n(0)\} \) be an orthonormal basis of \( \dot{c}(0)^\perp \subset T_c(0)M \), then its parallel transport \( \{e_1(t), \ldots, e_n(t)\} \) \( (t \in [0, T]) \) along \( c \) gives an orthonormal basis of \( \dot{c}(t)^\perp \subset T_c(t)M \). Setting \( e_0(t) = \dot{c}(t)/|\dot{c}(t)| \), note that \( e_0(T), e_1(T), \ldots, e_n(T) \) can be expressed by an orthogonal matrix \( \bar{S} \) under the basis \( \{e_0(0), e_1(0), \ldots, e_n(0)\} \), that is,

\[
e_i(T) = \bar{S}e_i(0), \quad i = 0, 1, \ldots, n.
\]  

(1.3)

Let \( S = \begin{pmatrix} \bar{S} & 0 \\ 0 & \bar{S} \end{pmatrix} \), in the neighborhood of \( c \), under the coordinate \( \{e_0(t), e_1(t), \ldots, e_n(t)\} \), the closed geodesic \( c \) could be written as a solution of Hamiltonian system with the \( S \)-boundary condition and its linear stability is from a linear Hamiltonian system with \( S \)-boundary condition, please refer to [7] for the details.

The other reason comes from the recent research of \( N \)-body problems. When we want to get a cyclic type symmetric periodic solution or the quasi-periodic solution, it is natural to consider (1.1)–(1.2). To illustrate this, we introduce the cyclic type symmetric Hamiltonian systems simply. Suppose \( S^m = I_{2n} \), \( H(t - T/m, Sz) = H(t, z) \) (in autonomous case, \( H(Sz) = H(z) \)), the \( Z_m \)-invariant periodic solution is the solution which satisfies \( z(t) = Sz(t + T/m) \), it is just the \( S \)-periodic solution on the interval \( [0, T/m] \). Please refer to [6] for the details.

To study the property of an \( S \)-periodic boundary solution \( z \), we consider the linear Hamiltonian system

\[
\begin{align*}
\dot{\gamma}(t) &= JB(t)\gamma(t), \\
\gamma(0) &= I_{2n}.
\end{align*}
\]  

(1.4)  
(1.5)

where \( B(t) = B(t)^T = H''(t, z(t)) \) is a continuous path on \( [0, T] \) of real symmetric matrices. Let \( \gamma \equiv \gamma_z(t) \) be the corresponding fundamental solution,

\[
\begin{align*}
\dot{\gamma}(t) &= JB(t)\gamma(t), \\
\gamma(0) &= I_{2n}.
\end{align*}
\]  

(1.6)  
(1.7)
Then \( z(t) \) is solution of (1.4)–(1.5) if and only if \( z(0) \in \ker(S\gamma(T) - I_{2n}) \). Let \( \varphi_t \) be the Hamiltonian flow of (1.1), then the \( S \)-periodic solution \( z \) is a fixed point of the symplectic map \( S\varphi_T \), so we give the next definition for the stability.

**Definition 1.1.** \( z \) is called spectrally stable if all the eigenvalues of \( S\gamma z(T) \) are on the unit circle \( U \), and it is called linear stable if moreover \( S\gamma z(T) \) is semi-simple.

In studying the stability of closed geodesics and the cyclic type symmetric periodic solution, it is reduced to judge whether the eigenvalues of \( S\gamma z(T) \) locate in the unit circle, and Definition 1.1 is reasonable in the sense [6,7].

Due to the noncommutativity, the matrices equation (1.6)–(1.7) could not be solved directly. To understand the equation, we study its property from the viewpoint of operators. We denote operators

\[
A = -J \frac{d}{dt}, \quad B \text{ (defined by } Bz = B(t)z(t) \text{)}
\]

on Hilbert space \( E = L^2([0, T], C^{2n}) \), where we restricted the domain of \( A \) by

\[
D_S = \{z(t) \in W^{1,2}([0, T], C^{2n}) \mid z(0) = Sz(T)\},
\]

so \( A \) is a self-adjoint operator and has compact resolvent.

To study the solution of Hamiltonian system (1.4)–(1.5), we consider the operator \((A - B)(A + P_0)^{-1}\), where \( P_0 \) is the orthogonal projection onto \( \ker A \). It will be shown in Lemma 2.2 that \((A + P_0)^{-1}\) is not trace class, but Hilbert–Schmidt, and hence, in general, \((B + P_0)(A + P_0)^{-1}\) is not trace class, but Hilbert–Schmidt. In the traditional sense [12, Chapter 3], the Fredholm determinant

\[
det\left((A - B)(A + P_0)^{-1}\right) = det\left(id - (B + P_0)(A + P_0)^{-1}\right)
\]

cannot be defined because the Fredholm determinant \( det(id + D) \) can only be defined when \( D \) is a trace class operator, where \( id \) is the identity operator on a Hilbert space. In this paper, motivated by Denk [4], we define the conditional Fredholm determinant as follows. For \( N \in \mathbb{N} \), let

\[
W_N = \bigoplus_{\nu \in \sigma(A), |\nu| \leq N} \ker(A - \nu),
\]

and denote by \( P_N \) the orthogonal projection onto \( W_N \). It is easy to see that \( P_N \) is convergent to \( id \) in strongly operator topology.

Let \( GL(2n, C) \) be the set of \( 2n \times 2n \) matrices on \( C^{2n} \), and denote \( \mathcal{B} = C([0, T], GL(2n, C)) \).

**Definition 1.2.** For \( B \in \mathcal{B} \), we define

\[
det\left((A - B)(A + P_0)^{-1}\right) = \lim_{N \to \infty} det(P_N(A - B)P_N(A + P_0)^{-1}P_N).
\]

Notice that \( P_N(A - B)P_N(A + P_0)^{-1}P_N \) is a matrix on a finite-dimensional space, which ensures that the right-hand side of (1.8) is the limit of a sequence of the usual determinants. It will be shown in the next section that the determinant is well defined, which will be called the conditional Fredholm determinant. Although the definitions of the conditional Fredholm determinant and the usual one are different, their properties are very similar. Moreover, the conditional
Fredholm determinant is the classical one provided that \( B(A + P_0)^{-1} \) is a trace class operator. It seems that the Fredholm determinant \( \det((A - B)(A + P_0)^{-1}) \) is a useful tool in studying the \( S \)-periodic solutions. In this paper, the Fredholm determinant is used to study the stability of Hamiltonian systems.

In Section 4, we will show the following theorem, which is the main theorem in this paper. We call it Hill’s formula because of Hill’s work on some special ODE [5], and the strict mathematical framework is established by Poincaré [10]. Readers are referred to [2] for the history.

**Theorem 1.3.** There is a constant \( C(S) > 0 \), which depends only on \( S \), such that for any \( \nu \in \mathbb{C} \)

\[
\det((A - B - \nu J)(A + P_0)^{-1}) = C(S)e^{-\frac{1}{2} \int_0^T \text{Tr}(JB(t)) dt} \lambda^{-n} \det(S\gamma(T) - \lambda I_{2n}),
\]

where \( \lambda = e^{\nu T} \), \( C(S) \) is defined in (4.6).

It is worth being remarked that the left-hand side of (1.9) is kind of infinite determinant, however the right-hand side is the usual determinant of matrix on \( \mathbb{C}^{2n} \). As a special case, if we consider the periodic solution of the Hamiltonian system, and assume \( B \) is a path of real symmetric matrices, then it is proved in Section 4 that \( C(I_{2n}) = T^{-2n}, e^{-\frac{1}{2} \int_0^T \text{Tr}(JB(t)) dt} = 1 \), and hence

\[
\det((A - B - \nu J)(A + P_0)^{-1}) = T^{-2n}\lambda^{-n} \det(\gamma(T) - \lambda I_{2n}).
\]

It is worth being pointed out that, in [4], Denk obtained the Hill formula for periodic solution of ODE. In [2], Bolotin and Treschev studied the Hill formula for periodic solution of Lagrangian system. Notice that Lagrangian system is a kind of second order ODE, and the related operators are trace class, and the determinant can be defined in the traditional sense, and in the later work of the authors, we will consider the Hill formula for the second order ODE from the other view of point.

An operator is called to be non-degenerate, if its kernel is nontrivial. From Theorem 1.3, we could immediately obtain a criteria to judge the instability for the \( S \)-periodic solutions.

**Corollary 1.4.** In the case \( B(t) = H''(t, z(t)) \), the \( z \) is spectrally unstable if one of the following conditions is satisfied

1) \( A - B \) is non-degenerate, and \( \det((A - B)(A + P_0)^{-1}) < 0 \),

2) \( A - B - \sqrt{-1\pi T} J \) is non-degenerate, and \((-1)^n \det((A - B - \sqrt{-1\pi T} J)(A + P_0)^{-1}) < 0 \).

We only explain this for condition 1), and condition 2) is similar. In fact, since \( S\gamma(T) \in \text{Sp}(2n) \), if \( \lambda \) is an eigenvalue of \( S\gamma(T) \), then so are \( \lambda^{-1}, \bar{\lambda} \), and \( \bar{\lambda}^{-1} \) (some of them may be equal to each other), moreover, they have the same geometric and algebraic multiplicities. If 1 is not an eigenvalue of \( S\gamma(T) \), then \( \det(S\gamma(T) - I_{2n}) < 0 \) implies that there exists a positive eigenvalue of \( S\gamma(T) \) which is bigger than 1, thus the solution is spectrally unstable. For the Lagrangian systems, it was proved in [2].

Suppose the Hamiltonian function \( H(t, z) \in C^2(\mathbb{R}^{2n+1}, \mathbb{R}) \) satisfies \( H(t - T/m, Sz) = H(t, z) \) (in autonomous case, \( H(Sz) = H(z) \)). Please note that if \( z(t) \) is a solution of (1.1) and satisfies the boundary condition \( z(0) = Sz(T/m) \), setting \( Q = Sm \), then it is obvious that \( z(t) \)
is a solution of (1.1) on \([0, T]\), and satisfies \(z(0) = Qz(T)\). Obviously \(z(t)\) is the cyclic type symmetric solution which satisfies \(z(t) = Sz(t + T/m)\). As a special case, \(S = I_{2n}\) means that \(z(t)\) is a \(T\) periodic solution of the Hamiltonian system with multiplicity \(m\). A natural question is arisen: what is the relation amongst the corresponding Fredholm determinants? We answer this question and prove the decomposition formula of Fredholm determinant in Section 5.

In studying the periodic solution of Hamiltonian system, a very successful index theory for such symplectic paths is introduced by Conley and Zehnder \([3]\) and developed by Long and others (see \([8]\) for the details). The index can be defined for the solution with \(S\)-boundary condition \([6]\). This index can be considered as an intersection number of paths in \(\text{Sp}(2n)\) with the singular set, on the other hand, it is just the relative Morse index \(I(A, A - B)\), see Section 6 for the details. The sign of conditional Fredholm determinant depends on that \(I(A, A - B)\) is even or odd. We let \(P_B\) be the orthogonal projection onto \(\ker(A - B)\), then \(\det((A - B + P_B)(A + P_0)^{-1}) \neq 0\). The following theorem will be proved in Section 6, which build the relationship between the sign of the conditional Fredholm determinant and the relative Morse index.

**Theorem 1.5.** \(\det((A - B + P_B)(A + P_0)^{-1}) > 0 (\leq 0)\) if and only if \(I(A, A - B)\) is even (odd).

This paper is organized as follows. In Section 2, we show that the conditional Fredholm determinant is well defined, and some other preliminaries are given. In Section 3, we study the properties of the conditional Fredholm determinant, and we will see that the function \(\det(\cdot)\) has the finite analyticity. In Section 4, we prove the Hill formula. In Section 5, we prove the decomposition formula for the cyclic type symmetric solution. Finally, in Section 6, we make clear the relation of Fredholm determinant and the relative Morse index.

2. Preliminaries

In this section, we consider the linear Hamiltonian system

\[
\begin{align*}
\dot{z}(t) &= JB(t)z(t), \\
z(0) &= Sz(T),
\end{align*}
\]

where \(B(t) = B(t)^T\) is temporarily assumed to be a continuous real symmetric path on \([0, T]\), \(S \in \text{Sp}(2n) \cap O(2n)\). As mentioned in Introduction, let \(\gamma(t)\) be the corresponding fundamental solution, then \(z(t)\) is solution of (2.1)–(2.2) if and only if \(z(0) \in \ker(S\gamma(T) - I_{2n})\). Recall that, we denote \(A = -J \frac{d}{dt}\) which is densely defined in the Hilbert space \(E = L^2([0, T], C^{2n})\) with the domain

\[
D_S = \{z(t) \in W^{1,2}([0, T], C^{2n}) \mid z(0) = Sz(T)\},
\]

and the operator \(B\) on \(E\) is defined by \(Bz = B(t)z(t)\), for any \(z(t) \in E\). Since \(B\) is a continuous real symmetric path, \(A, B\) and \(A - B\) are all self-adjoint operators. It is obvious that \(z(t)\) is a solution of (2.1)–(2.2) if and only if \(z \in \ker(A - B)\), therefore,

\[
\dim \ker(S\gamma(T) - I_{2n}) = \dim \ker(A - B). \tag{2.3}
\]
Recall that, when $S = I_{2n}$, that is, (2.1)–(2.2) is a periodic boundary problem, a complex $\mu$ is called the Floquet exponent of (2.1)–(2.2) if $e^{\mu T}$ is eigenvalue of $\gamma(T)$. In what follows, $\mu$ is called the $S$-Floquet exponent of (2.1)–(2.2) if and only if $e^{\mu T}$ is an eigenvalue of $S\gamma(T)$.

Now, for any complex number $\nu$, we consider the revised Hamiltonian system

$$\dot{z}(t) = JB(t)z(t), \quad z(0) = e^{-\nu T}Sz(T).$$

Set $y(t) = e^{-\nu t}z(t)$, then $z(t)$ is the solution of (2.4)–(2.5) if and only if $y(t)$ is the solution of

$$\dot{y}(t) = -\nu y(t) + JB(t)y(t), \quad y(0) = Sy(T),$$

and this is equivalent to that $S\gamma(T)y(0) = y(0)$. To illustrate the solution of (2.4)–(2.5), we consider the operator $A - B - \nu J$ on $H$ with domain $D_S$, and $\nu$ is called the $J$-eigenvalue of $A - B$ if $\ker(A - B - \nu J)$ is nontrivial. Note that $A - B - \nu J$ has compact resolvent, and it is self-adjoint if and only if $\nu$ is an imaginary number. Some direct calculations imply the following lemma.

**Lemma 2.1.**

$$\dim \ker(A - B - \nu J) = \dim \ker(S\gamma(T) - e^{\nu T}I_{2n}). \quad (2.8)$$

Denote by $P_0$ the orthogonal projection onto $\ker A$, which is a finite rank operator, so $(A + P_0)^{-1}$ is compact by the Sobolev Imbedding Theorem. Moreover, it is Hilbert–Schmidt, in what follows, Hilbert–Schmidt is simplified as H–S. An operator $T$ on a Hilbert space is called an H–S operator if for any orthonormal basis $\{e_j\}_{j=1}^{\infty}$, $\sum_{j=1}^{\infty} \|Te_j\|^2 < \infty$. It is well known that, if $T$ is an H–S operator, then $\left(\sum_{j=1}^{\infty} \|Te_j\|^2\right)^{\frac{1}{2}}$ is independent of the choice of the orthonormal basis, and let $\|T\|_2 = \left(\sum_{j=1}^{\infty} \|Te_j\|^2\right)^{\frac{1}{2}}$ be the H–S norm of $T$.

In this paper, we always consider the operator $(A - B - \nu J)(A + P_0)^{-1}$ which has a natural relationship with the system (2.6)–(2.7). To simplify the notation, denote

$$L(\nu) = (A - B - \nu J)(A + P_0)^{-1}, \quad (2.9)$$

then $L(\nu) = \text{id} - (P_0 + B + \nu J)(A + P_0)^{-1}$. Since the set of all the H–S operators is an ideal of the algebra of all the bounded operators, we have $L(\nu) - \text{id}$ is an H–S operator.

From [12], for an H–S operator $F$ we could define the regularized determinant by

$$\det_2(\text{id} + F) = \det((\text{id} + F)e^{-F}). \quad (2.10)$$

Note that, for a fixed H–S operator $F$, if there is a sequence of orthogonal projections $\{P_k\}_{k=1}^{\infty}$, such that
1) Range$(P_k) \subseteq \text{Range}(P_m)$, if $k \leq m$,
2) $P_k$ converge to $id$ in the strong operator topology,
3) as $k \to \infty$, the limit of Tr$(P_k F P_k)$ exists,

then we could define

$$\det(id + F) = \lim_{k \to \infty} \det(id + P_k F P_k). \quad (2.11)$$

Obviously this definition depends on the choice of $\{P_k\}_{k=1}^\infty$. If $\{P_k\}$ is fixed, then the determinant is well defined. In fact, for an H–S operator $F$, the det$_2(id + F)$ is always well defined, and if $\|F_k - F\|_2 \to 0$, then det$_2(id + F) = \lim_{k \to \infty} \det_2(id + F_k)$. Notice that $\|P_k F P_k - F\|_2 \to 0$, and hence

$$\det_2(id + F) = \lim_{k \to \infty} \det_2(id + P_k F P_k) = \lim_{k \to \infty} \det((id + P_k F P_k)e^{-P_k F P_k}) = \lim_{k \to \infty} \det(id + P_k F P_k)e^{-\text{Tr}(P_k F P_k)}.$$

This implies that

$$\lim_{k \to \infty} \det(id + P_k F P_k) = \det_2(id + F) \lim_{k \to \infty} e^{\text{Tr}(P_k F P_k)}. \quad (2.12)$$

To simplify the notation, we give the following definition.

**Definition 2.2.** Under the above assumption, an H–S operator $F$ acting on $H$ is called to have the trace finite condition, if the limit $\lim_{k \to \infty} P_k F P_k$ exists and the limit is finite.

We summarize the above reasoning as the following lemma.

**Lemma 2.3.** Under the above assumption, if $F$ is an operator with trace finite condition, then

$$\lim_{k \to \infty} \det(id + P_k F P_k) = \det_2(id + F) \lim_{k \to \infty} e^{\text{Tr}(P_k F P_k)}. \quad (2.13)$$

In this paper, $\{P_k\}_{k=1}^\infty$ is chosen as $\{P_N\}_{N=1}^\infty$, where $P_N$ is defined as in Definition 1.2. To continue, we give the characterization of the spectrum of $A$ with $S$-boundary conditions and obtain the following lemma.

**Lemma 2.4.** The spectrum of $A$ is periodic with the period $2k\pi/T$. Precisely, there exist $\nu_1 \leq \nu_2 \leq \cdots \leq \nu_n \in [0, 2\pi/T)$, such that $\sigma(A) = \bigcup_{i=1}^n \{\nu_i + 2k\pi/T\}_{k \in \mathbb{Z}}$, and each point is an eigenvalue of $A$ of multiplicity two.

**Proof.** Since $A$ is a self-adjoint operator and has a compact resolvent, its spectrum locates in $\mathbb{R}$ and all of them are eigenvalues, moreover, $A$ has no nontrivial Jordan block. Direct computation shows that $\nu \in \sigma(A)$ if and only if ker$(Se^{\nu JT} - I_{2n}) \neq 0$, and the corresponding eigenvector
is given by $e^{\nu J} \xi$, where $\xi \in \ker(S e^{\nu J^T} - I_{2n})$. Since $S \in \text{Sp}(2n) \cap O(2n)$, $SJ = JS$, direct computation shows that $S = \begin{pmatrix} R & -Q \\ Q & R \end{pmatrix}$, where both $R$ and $Q$ are real matrices. Note that

$$\frac{\sqrt{2}}{2} \begin{pmatrix} I_n & \sqrt{-1} I_n \\ I_n & -\sqrt{-1} I_n \end{pmatrix} \begin{pmatrix} R & Q \\ Q & R \end{pmatrix} \frac{\sqrt{2}}{2} \begin{pmatrix} I_n & I_n \\ -\sqrt{-1} I_n & \sqrt{-1} I_n \end{pmatrix} = \begin{pmatrix} R + \sqrt{-1} Q \\ R - \sqrt{-1} Q \end{pmatrix}.$$ 

It follows that $S$ is unitarily equivalent to $\begin{pmatrix} R + \sqrt{-1} Q \\ R - \sqrt{-1} Q \end{pmatrix}$, where $R + \sqrt{-1} Q$ and $R - \sqrt{-1} Q$ are both unitary matrices. And hence, there is an $n \times n$ unitary matrix $U$ such that

$$U^* (R + \sqrt{-1} Q) U = \begin{pmatrix} e^{-\sqrt{-1} \theta_1} \\ & e^{-\sqrt{-1} \theta_2} \\ & & \ddots \\ & & & e^{-\sqrt{-1} \theta_n} \end{pmatrix},$$

and

$$\tilde{U}^* (R - \sqrt{-1} Q) \tilde{U} = \begin{pmatrix} e^{\sqrt{-1} \theta_1} \\ & e^{\sqrt{-1} \theta_2} \\ & & \ddots \\ & & & e^{\sqrt{-1} \theta_n} \end{pmatrix},$$

where $0 \leq \theta_1 \leq \theta_2 \leq \cdots \leq \theta_n < 2\pi$. Furthermore, set $V = \frac{\sqrt{2}}{2} \begin{pmatrix} I_n \\ -\sqrt{-1} I_n \sqrt{-1} I_n \end{pmatrix} (U \quad \tilde{U})$, then

$$V^* J V = \begin{pmatrix} \sqrt{-1} I_n \\ -\sqrt{-1} I_n \end{pmatrix}.$$ 

In what follows, we write $\hat{J} = \begin{pmatrix} \sqrt{-1} I_n \\ -\sqrt{-1} I_n \end{pmatrix}$. Thus, $\nu \in \sigma(A)$ if and only if

$$V^* S V V^* e^{\nu J^T} V \xi = I_{2n} \xi,$$

that is,

$$\left( U^* (R + \sqrt{-1} Q) U \\ \tilde{U}^* (R - \sqrt{-1} Q) \tilde{U} \right) \begin{pmatrix} e^{\sqrt{-1} \nu I_n T} \\ e^{-\sqrt{-1} \nu I_n T} \end{pmatrix} \xi = I_{2n} \xi. \quad (2.14)$$

It is easy to verify that there is a nonzero $\xi$ satisfying (2.14) if and only if $\nu = \frac{\theta_j}{\pi}$, for some $j = 1, 2, \ldots, n$. Let $\nu_j = \frac{\theta_j}{\pi}$, then $0 \leq \nu_1 \leq \nu_2 \leq \cdots \leq \nu_n < \frac{2\pi}{\pi}$, and $\sigma(A) = \bigcup_{j=1}^{n} \{ \nu_j + \frac{2k\pi}{\pi} \mid k \in \mathbb{Z} \}$. Again, by Eq. (2.14), all the eigenvalues are of multiplicity 2. 

In what follows, we always denote $V = \frac{\sqrt{2}}{2} \begin{pmatrix} I_n \\ -\sqrt{-1} I_n \sqrt{-1} I_n \end{pmatrix} (U \quad \tilde{U})$. 


Lemma 2.5. Let \( \nu_j + 2k\pi/T \) be an eigenvalue of \(-\hat{J} \frac{d}{dt}\hat{J}\), then the corresponding eigenvectors are 
\[ e^{(\nu_j + 2k\pi/T)\hat{J}} \xi_{jl}, \quad l = 1, 2, \] where 
\[ \xi_{j1} = (0, \ldots, 0, 1, 0, \ldots, 0), \quad \text{and} \quad \xi_{j2} = (0, \ldots, 0, 1, 0, \ldots, 0). \]

Proof. It is easy to check that 
\[ -\hat{J} \frac{d}{dt} e^{(\nu_j + 2k\pi/T)\hat{J}} \xi_{jl} = (\nu_j + 2k\pi/T) e^{(\nu_j + 2k\pi/T)\hat{J}} \xi_{jl}. \]
Set \( \hat{S} = V^*SV \), since \( \hat{S} e^{(\nu_j + 2k\pi/T)\hat{J}} \xi_{jl} = \xi_{jl} \), we have that \( \xi_{jl} \in \ker(\hat{S} e^{(\nu_j + 2k\pi/T)\hat{J}} - I_{2n}) \), and hence we can take 
\[ \xi_{j1} = (0, \ldots, 0, 1, 0, \ldots, 0), \quad \text{and} \quad \xi_{j2} = (0, \ldots, 0, 1, 0, \ldots, 0). \]

Lemma 2.6. Let \( B(t) \) be a real symmetric path on \([0, T]\), and write \( \hat{B}(t) = V^*B(t)V = (\hat{B}_{kl}(t))_{2n \times 2n} \), then 
\[ \hat{B}_{jj}(t) = \hat{B}_{n+j,n+j}(t), \quad \text{for} \quad j = 1, \ldots, n. \tag{2.15} \]

Proof. Write \( B(t) \) as 
\[ B(t) = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}, \]
then, some direct computation shows that 
\[ \frac{\sqrt{2}}{2} \begin{pmatrix} I_n & -\sqrt{-1}I_n \\ I_n & \sqrt{-1}I_n \end{pmatrix} B \frac{\sqrt{2}}{2} \begin{pmatrix} I_n & I_n \\ -\sqrt{-1}I_n & \sqrt{-1}I_n \end{pmatrix} \]
\[ = \frac{1}{2} \begin{pmatrix} a + \sqrt{-1}(b^* - b) + c & a + \sqrt{-1}(b^* + b) - c \\ a - \sqrt{-1}(b^* + b) - c & a - \sqrt{-1}(b^* - b) + c \end{pmatrix}. \]
It follows that 
\[ V^*B(t)V = \frac{1}{2} \begin{pmatrix} U^*(a + \sqrt{-1}(b^* - b) + c)U & U^*(a + \sqrt{-1}(b^* + b) - c)U \\ U^*(a - \sqrt{-1}(b^* + b) - c)U & U^*(a - \sqrt{-1}(b^* - b) + c)U \end{pmatrix}. \tag{2.16} \]
Since \( B(t) \) is symmetric, so is \( \hat{B}(t) \), that is, all of \( \hat{B}_{jj}(t) \) are real functions. Noting that both \( b \) and \( b^* \) are real matrices, we have that \( b^* - b \) vanishes at the diagonal, and hence, by (2.16) 
\[ \hat{B}_{jj}(t) = \hat{B}_{n+j,n+j}(t). \]

Since the \( V \) is a unitary operator, and the unitary equivalence will not change any information of the operator, including the trace, the eigenvalues, etc. In what follows, to simplify the notation, we will not distinguish \( \hat{B}(t) \) and \( B(t) \), unless we point it out specially. The remaining part of the section is devoted to the calculation of \( Tr(B(A + P_0)^{-1}) \), to this end, we need the following lemma.
Lemma 2.7. If \( v \neq \frac{2k\pi}{T} \), \( k \in \mathbb{Z} \), then

\[
\lim_{N \to \infty} \sum_{|k| \leq N} \frac{1}{v + 2k\pi/T} = \frac{T}{2} \frac{1 + \cos Tv}{\sin Tv}.
\]  

(2.17)

**Proof.** Direct computation shows that

\[
\sum_{|k| \leq N} \frac{1}{v + 2k\pi/T} = \frac{T}{2\pi} \left( \frac{1}{\pi v} + \sum_{k=1}^{N} \left( \frac{1}{\pi (v + k)} + \frac{1}{\pi (v - k)} \right) \right)
\]

(2.18)

\[
= \frac{T}{2\pi} \left( \frac{1}{\pi v} + \frac{Tv}{\pi} \sum_{k=1}^{N} \left( \frac{1}{(\pi v)^2 - k^2} \right) \right).
\]

(2.19)

Since

\[
\frac{\pi \cos \pi z}{\sin \pi z} = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2},
\]

(2.20)

we have

\[
\lim_{N \to \infty} \sum_{|k| \leq N} \frac{1}{v + 2k\pi/T} = \frac{T}{2\pi} \left( \frac{1}{\pi v} + \frac{2T \cos \frac{Tv}{2}}{2 \sin \frac{Tv}{2}} \right) = \frac{T}{2} \cos \frac{Tv}{2} \frac{\cos \frac{Tv}{2}}{\sin \frac{Tv}{2}} = \frac{T}{2} \frac{1 + \cos Tv}{\sin Tv}. \quad \square
\]

With this preparation, the following trace formula is obtained. Recall that

\[
W_N = \bigoplus_{|v| \leq N} \ker(A - v),
\]

and we let \( P_N \) be the orthogonal projections onto \( W_N \). We also set \( W_0 = \ker(S - I_{2n}) \) which could be identified with \( \ker(A) \).

**Proposition 2.8.** Let \( 0 = v_1 = \cdots = v_{k_0} < v_{k_0+1} \leq \cdots \leq v_n < \frac{2\pi}{T} \) be the eigenvalues of \( A \) with \( S \)-boundary condition, then

\[
\lim_{N \to \infty} \text{Tr}(P_N B(A + P_0)^{-1} P_N) = \frac{2}{T} \sum_{j=1}^{k_0} \int_{0}^{T} B_{jj}(t) \, dt + \sum_{j=k_0+1}^{n} \frac{1 + \cos Tv_j}{\sin Tv_j} \int_{0}^{T} B_{jj}(t) \, dt.
\]

**Proof.** For \( 1 \leq j \leq n \), let

\[
M_j = \bigoplus_{k \in \mathbb{Z}} \ker \left( A - \left( v_j + \frac{2k\pi}{T} \right) \right).
\]


Then \( P_N P_{M_j} = P_{M_j} P_N \), where \( P_{M_j} \) is the orthogonal projection from \( E \) onto \( M_j \). Therefore,

\[
\lim_{N \to \infty} Tr(P_N B(A + P_0)^{-1} P_N) = \lim_{N \to \infty} \sum_{j=1}^{n} Tr(P_N P_{M_j} B(A + P_0)^{-1} P_{M_j} P_N)
\]

\[= \sum_{j=1}^{n} \lim_{N \to \infty} Tr(P_N P_{M_j} B(A + P_0)^{-1} P_{M_j} P_N).\]

Next, we compute \( \lim_{N \to \infty} Tr(P_N P_{M_j} B(A + P_0)^{-1} P_{M_j} P_N) \). Notice that, \( \{1/\sqrt{T} e^{\sqrt{-1}(v_j+2k\pi/T)\tilde{J}_t} \xi_{jl} \} \) is an orthonormal basis of \( M_j \). If \( v_j \neq 0 \), then

\[
\lim_{N \to \infty} Tr(P_N P_{M_j} B(A + P_0)^{-1} P_{M_j} P_N)
\]

\[= \lim_{N \to \infty} \sum_{k \leq N} \sum_{l=1}^{2} \left| B(t)(A - P_0)^{-1} \frac{1}{\sqrt{T}} e^{\sqrt{-1}(v_j+2k\pi/T)\tilde{J}_t} \xi_{jl}, \frac{1}{\sqrt{T}} e^{\sqrt{-1}(v_j+2k\pi/T)\tilde{J}_t} \xi_{jl} \right|
\]

\[= \frac{1}{T} \lim_{N \to \infty} \sum_{k \leq N} \sum_{l=1}^{2} \frac{1}{v_j + 2k\pi/T} \int_{0}^{T} \left| B(t)e^{\sqrt{-1}(v_j+2k\pi/T)\tilde{J}_t} \xi_{jl}, e^{\sqrt{-1}(v_j+2k\pi/T)\tilde{J}_t} \xi_{jl} \right| dt
\]

\[= \frac{2}{T} \lim_{N \to \infty} \sum_{k \leq N} \frac{1}{v_j + 2k\pi/T} \int_{0}^{T} B_{jj}(t) dt
\]

\[= \frac{1 + \cos T v_j}{\sin T v_j} \int_{0}^{T} B_{jj}(t) dt,
\]

where the third equality is from Lemma 2.6 and the last equality is from Lemma 2.7. If \( v_j = 0 \), then

\[
\lim_{N \to \infty} Tr(P_N P_{M_j} B(A + P_0)^{-1} P_{M_j} P_N) = \frac{2}{T} \int_{0}^{T} B_{jj}(t) dt.
\]

By the preceding reasoning,

\[
\lim_{N \to \infty} Tr(P_N B(A + P_0)^{-1} P_N)
\]

\[= \frac{2}{T} \sum_{j=1}^{k_0} \int_{0}^{T} B_{jj}(t) dt + \sum_{j=k_0+1}^{n} \frac{1 + \cos T v_j}{\sin T v_j} \int_{0}^{T} B_{jj}(t) dt. \quad \Box \ (2.21)
\]
Remark 2.9. In the proof of Proposition 2.8, we only use the real symmetricalness of $B(t)$ to prove $B_{j j}(t) = B_{n+j,n+j}(t)$. Therefore, we can also calculate the limit $\lim_{N \to \infty} \text{Tr}(P_N B(A + P_0)^{-1} P_N)$ in the case $B(t) \in \mathfrak{B} = C([0, T], GL(2n, \mathbb{C}))$, and the following formula is obtained:

$$\lim_{N \to \infty} \text{Tr}(P_N B(A + P_0)^{-1} P_N) = \frac{1}{T} \sum_{j=1}^{k_0} \int_0^T B_{j j}(t) \, dt + \frac{1}{2} \sum_{j=k_0+1}^{n} \frac{1 + \cos T v_j}{\sin T v_j} \int_0^T B_{j j}(t) \, dt + \frac{1}{2} \sum_{j=n+1+k_0}^{2n} \frac{1 + \cos T v_j}{\sin T v_j} \int_0^T B_{j j}(t) \, dt.$$

In what follows, we will call $\lim_{N \to \infty} \text{Tr}(P_N B(A + P_0)^{-1} P_N)$ the conditional trace of $B(A + P_0)^{-1}$, and we also use the notation $\text{Tr}(B(A + P_0)^{-1})$ although it is different from the usual trace.

By the discussion before Lemma 2.4, we can define

$$\det(L(v)) = \lim_{N \to \infty} \det(P_N L(v) P_N), \quad (2.22)$$

then we have

$$\det(L(v)) = \det_2(L(v)) e^{\text{Tr}(L(v) - id)}, \quad (2.23)$$

where

$$\text{Tr}(L(v) - id) = \lim_{N \to \infty} \text{Tr}(P_N (-B - P_0 - vJ)(A + P_0)^{-1} P_N). \quad (2.24)$$

Proposition 2.10. For any bounded operator $D$, such that $A + D$ is invertible, $\lim_{N \to \infty} \text{Tr}(P_N B(A + D)^{-1} P_N)$ exists, and $\det(id + B(A + D)^{-1})$ is well defined.

Proof. Let $\phi = (A + P_0)^{-1} - (A + D)^{-1}$, then direct computation shows that $\phi = (A + P_0)^{-1} (D - P_0)(A + D)^{-1}$, thus $\phi$ is a trace class operator, and hence $B\phi$ is a trace class operator; moreover, $\text{Tr}(B\phi) = \lim_{N \to \infty} \text{Tr}(P_N B\phi P_N)$. Since

$$P_N B(A + D)^{-1} P_N = P_N B(A + P_0)^{-1} P_N - P_N B\phi P_N,$$

then

$$\lim_{N \to \infty} \text{Tr}(P_N B(A + D)^{-1} P_N) = \lim_{N \to \infty} \text{Tr}(P_N B(A + P_0)^{-1} P_N) - \text{Tr}(B\phi).$$

Therefore,
\[
\det(id + B(A + D)^{-1}) = \det_2(id + B(A + D)^{-1}) \lim_{N \to \infty} e^{\text{Tr}(PNB(A+D)^{-1}P_N)}
\]

\[
= \det_2(id + B(A + D)^{-1}) \lim_{N \to \infty} e^{\text{Tr}(PNB(A+P_0)^{-1}P_N)}e^{-\text{Tr}(B\phi)}
\]
is well defined. \( \Box \)

3. The property for the Fredholm determinant

Recall that \( GL(2n, \mathbb{C}) \) is the set of \( 2n \times 2n \) matrices on \( \mathbb{C}^{2n} \), and denote \( \mathfrak{B} = C([0, T], GL(2n, \mathbb{C})) \) which is a Banach space under the operator norm. For \( B \in \mathfrak{B} \), \( f(B) = \det((A + B)(A + P_0)^{-1}) \) is well defined by the limit \( f_N(B) = \lim_{N \to \infty} \det((A + PN)(A + P_0))^{-1} \).

We will prove that \( \{f_N(B)\} \) can be controlled by the norm of \( B \), and hence \( f(B) \) can be controlled by the norm of \( B \). To do this, we need the following lemma. In the proof of the lemma, we will use the fact that \( PN(A + P_0)^{-1}P_N = (A + P_0)^{-1}PN \).

**Lemma 3.1.** There is a constant \( C \) such that, for any \( B \in \mathfrak{B} \) and any \( N \in \mathbb{N} \),

\[
|\text{Tr}PNB(A + P_0)^{-1}P_N| \leq C\|B\|. \tag{3.1}
\]

**Proof.** Notice that

\[
\text{Tr}PNB(A + P_0)^{-1}P_N
\]

\[
= \frac{1}{T} \sum_{j=1}^{k_0} \int_0^T [B(t)]_{jj} \, dt + \frac{1}{T} \sum_{j=k_0+1}^{n} \sum_{|v_j + \frac{2k\pi}{T}| \leq N} \frac{1}{v_j + 2k\pi/T} \int_0^T [B(t)]_{jj}(t) \, dt
\]

\[
+ \frac{1}{T} \sum_{j=n+1}^{n+k_0} \int_0^T [B(t)]_{jj} \, dt + \frac{1}{T} \sum_{j=n+k_0+1}^{2n} \sum_{|v_j + \frac{2k\pi}{T}| \leq N} \frac{1}{v_j + 2k\pi/T} \int_0^T [B(t)]_{jj}(t) \, dt.
\]

Firstly,

\[
\left| \sum_{j=1}^{k_0} \int_0^T [B(t)]_{jj} \, dt + \sum_{j=n+1}^{n+k_0} \int_0^T [B(t)]_{jj} \, dt \right| \leq 2k_0T\|B\|.
\]

Secondly, since \( \sum_{|v_j + \frac{2k\pi}{T}| \leq N} \frac{1}{v_j + 2k\pi/T} \) is convergent to \( \frac{T}{2} \frac{1+\cos T\nu_j}{\sin T\nu_j} \), there is a constant \( \tilde{C} > 0 \) such that for any \( N \),

\[
\left| \sum_{|v_j + \frac{2k\pi}{T}| \leq N} \frac{1}{v_j + 2k\pi/T} \right| < \tilde{C}.
\]

Therefore,
\[
\sum_{j=k_0+1}^{n} \sum_{|v_j + \frac{2k\pi}{T}| \leq N} \left| \frac{1}{v_j + \frac{2k\pi}{T}} \int_0^T [B(t)]_{jj}(t) dt \right| \leq (n - k_0)\tilde{C} T \| B \|, \\
\]
similarly
\[
\sum_{j=n+k_0+1}^{2n} \sum_{|v_j + \frac{2k\pi}{T}| \leq N} \left| \frac{1}{v_j + \frac{2k\pi}{T}} \int_0^T [B(t)]_{jj}(t) dt \right| \leq (n - k_0)\tilde{C} T \| B \|.
\]
Therefore,
\[
|TrP_NB(A + P_0)^{-1}P_N| \leq C \| B \|. \quad \Box
\]

Since \( Tr(B(A + P_0)^{-1}) \) is the limit of \( Tr(P_N B P_N(A + P_0)^{-1}) \), by Lemma 3.1, there is a constant \( C \), such that
\[
|TrB(A + P_0)^{-1}| \leq C \| B \|. \quad (3.2)
\]

**Proposition 3.2.** There are some positive constants \( C_1, C_2, C_3 \) such that, for any \( B \in \mathfrak{B} \) and any \( N \in \mathbb{N} \),
\[
|\det((A + P_N B P_N)(A + P_0)^{-1})| \leq \exp\left(C_1 \| B \|^2 + C_2 \| B \| + C_3\right). \quad (3.3)
\]

**Proof.** By (2.23) and Remark 2.9,
\[
\det((A + P_N B P_N)(A + P_0)^{-1})
\]
\[
= \det_2(id + (-P_0 + P_N B P_N)(A + P_0)^{-1})e^{Tr(-P_0 + P_N B P_N)(A + P_0)^{-1}}
\]
\[
= \det_2(id + (-P_0 + P_N B P_N)(A + P_0)^{-1})
\]
\[
\cdot e^{-Tr(P_N P_0(A + P_0)^{-1}P_N)}e^{Tr(P_N B P_N(A + P_0)^{-1})}.
\]
Note that \( P_0(A + P_0)^{-1} = P_0 \), thus,
\[
e^{Tr(P_N P_0(A + P_0)^{-1}P_N)} = e^{2k_0}
\]
and \( \| P_0(A + P_0)^{-1} \|_2 = 2k_0 \). Setting \( C_0 = \| (A + P_0)^{-1} \|_2 \), by [12, Theorem 9.2, p. 75],
\[
|\det_2(id + (-P_0 + P_N B P_N)(A + P_0)^{-1})| \leq e^{\Gamma_2 (\| -P_0 + P_N B P_N \| (A + P_0)^{-1})^2}
\]
\[
\leq e^{\Gamma_2 (\| -P_0(A + P_0)^{-1} \|_2 + \| P_N B P_N \| (A + P_0)^{-1})^2}
\]
\[
\leq e^{\Gamma_2 (\| P_0 \|_2 + \| B \| \| (A + P_0)^{-1} \|_2)^2}
\]
\[
= e^{\Gamma_2 (2k_0 + C_0 \| B \|^2)}
\]
\[
= e^{\Gamma_2 (C_0^2 \| B \|^2 + 2C_0 \| B \| + 4k_0^2)},
\]
where $\Gamma_2$ is a constant decided by $\text{det}_2$. Now, by Lemma 3.1, there is a constant $C$, such that
\[ e^{\text{Tr} P_N B P_N (A + P_0)^{-1}} \leq e^{C \|B\|}. \]
It follows that
\[ |\text{det}((A + P_N B P_N)(A + P_0)^{-1})| \leq e^{C_1 \|B\|^2 + C_2 \|B\| + C_3}, \]
which is the desired conclusion. $\square$

Obviously, by the above proposition,
\[ |\text{det}((A + B)(A + P_0)^{-1})| \leq e^{C_1 \|B\|^2 + C_2 \|B\| + C_3}. \]

Following [12], we call a function $f : X \to Y$ between Banach spaces, finitely analytic if and only if, for all $A_1, \ldots, A_n \in X$, $f(z_1 A_1 + \cdots + z_n A_n)$ is an entire function of $z_1, \ldots, z_n$ from $\mathbb{C}^n$ to $Y$.

**Lemma 3.3.** The function $f : (z_1, z_2, \ldots, z_m) \to \text{det}((A + z_1 B_1 + \cdots + z_m B_m)(A + P_0)^{-1})$ is an entire function on $\mathbb{C}^m$, that is, the function $\text{det}((A + \cdot)(A + P_0)^{-1})$ is a finitely analytic function from $\mathfrak{B}$ to $\mathbb{C}$.

**Proof.** Setting $B_N = P_N B P_N$, by (2.22), we have that
\[
\text{det}((A + z_1 B_1 + \cdots + z_m B_m)(A + P_0)^{-1}) = \lim_{N \to \infty} \text{det}((A + z_1 (B_1)_N + \cdots + z_m (B_m)_N)(A + P_0)^{-1}).
\]

By Proposition 3.2, we have that
\[
|\text{det}((A + z_1 (B_1)_N + \cdots + z_m (B_m)_N)(A + P_0)^{-1})| \leq e^{C_1 \|z_1 B_1 + \cdots + z_m B_m\|^2 + C_2 \|z_1 B_1 + \cdots + z_m B_m\| + C_3}
\leq e^{C_1 (\sum_{i=1}^m |z_i| \|B_i\|)^2 + C_2 \sum_{i=1}^m |z_i| \|B_i\| + C_3},
\]
which is uniformly bounded on any fixed compact subset of $\mathbb{C}^m$. Since for any $N > 0$, all of $(B_i)_N$, $i = 1, \ldots, m$, are of finite rank, by [11, Lemma 4, p. 323], $\det((A + z_1 (B_1)_N + \cdots + z_m (B_m)_N)(A + P_0)^{-1})$ is an entire function. By the standard theory of normal family of holomorphic functions (see e.g. [9, Proposition 3, p. 8]), there is a subsequence of $\{\text{det}((A + z_1 (B_1)_N + \cdots + z_m (B_m)_N)(A + P_0)^{-1})\}$, which is uniformly convergent to $\text{det}((A + z_1 B_1 + \cdots + z_m B_m)(A + P_0)^{-1})$ on any compact subset of $\mathbb{C}^m$, and hence $\text{det}((A + z_1 B_1 + \cdots + z_m B_m)(A + P_0)^{-1})$ is an entire function. $\square$

Let $G(x) = e^{C_1 x^2 + C_2 x + C_3}$, where $C_i > 0$, it is easy to check that $G(x)$ is a monotone function on $[0, +\infty)$. By Proposition 3.2 and Lemma 3.3, $|\text{det}((A + B)(A + P_0)^{-1})| \leq G(\|B\|)$. It follows
from [12, Theorem 5.1, p. 45] that det((A + B)(A + P_0)^{-1}) is Fréchet differentiable, and the Fréchet derivative satisfies

\[ \left| D(\det(A + B)(A + P_0)^{-1}) \right| \leq e^{C_1(\|B\|+1)^2+C_2(\|B\|+1)+C_3}. \] (3.4)

Moreover, Lemma 3.3 has a general version. Suppose \( \Omega \subseteq \mathbb{C}^m \) is an open set, and \( B(Z) \) is analytic map from \( \Omega \rightarrow \mathcal{B} \).

**Corollary 3.4.** Under the above condition, \( \det((A + B(Z))(A + P_0)^{-1}) \) is an analytic function from \( \Omega \) to \( \mathbb{C} \).

Inspired by [12], the determinant has more properties.

**Theorem 3.5.** For any H–S operators \( D, F \) which satisfy the trace finite condition, we have

1) \[ \det(id + D) = \det(id + D^*) \] (3.5)

2) \[ \det(id + D) \det(id + F) = \det(id + D + F + DF) \] (3.6)

3) \( \det(id + D) \neq 0 \) if and only if \( id + D \) is invertible,

4) for a complex number \( \lambda_0 \), set \( z_0 = -\lambda_0^{-1} \), then \( \lambda_0 \) is an eigenvalue of \( D \) of algebraic multiplicity \( k \) if and only if \( z_0 \) is a zero point of \( \det(id + zD) \) of order \( k \).

**Proof.**

1) This is from (2.11) directly.

2) By [12, Remarks, p. 76],

\[ \det_2(id + D)\det_2(id + F)e^{-\text{Tr}(DF)} = \det_2(id + D + F + DF). \] (3.7)

Eq. (2.12) implies that \( \det(id + \Phi) = \det_2(id + \Phi)e^{\lim_{N \rightarrow \infty} \text{Tr}(P_N\Phi P_N)} \), for \( \Phi = D, F, D + F + DF \). And hence some simple computations show the desired result.

3) Note that \( \det(id + D) \neq 0 \) if and only if \( \det_2(id + D) \neq 0 \), and this is equivalent to that \( (id + D)e^{-D} \) is invertible, which is equivalent to that \( id + D \) is invertible.

4) For \( k_0 > k \), let \( M = \ker(D - \lambda_0 id)^{k_0} \) and \( P_M \) be the projection on \( M \). Then \( D = DP_M + D(id - P_M) \), and \( \lambda_0 \) is not the eigenvalue of \( D(id - P_M) \). Since \( DP_M \) is a finite rank operator, we have \( \det(id + D P_M) \) and \( \det(id + D(id - P_M)) \) are well defined. From 3), \( \det(id + zD(id - P_M)) \neq 0 \) for \( z \) near \( z_0 \). By 2)

\[ \det(id + zD) = \det(id + zDP_M)\det(id + zD(id - P_M)) \] (3.8)

by the Jordan theorem in matrix theory, \( \det(id + zDP_M) = (1 - z/z_0)^k \), which is obvious of order \( k \). \qed
Theorem 3.5 has a direct corollary, which is interesting itself. Recall that \( \nu_0 \) is a \( J \)-eigenvalue of \( A - B \) if \( \ker(A - B - \nu_0 J) \) is nontrivial.

**Corollary 3.6.** Suppose \( \nu_0 \) is a \( J \)-eigenvalue of \( A - B \). Then \( \nu_0 \) is a zero point of the analytic function \( \det(L(\nu)) \) of order \( k \) if and only if \(-\nu_0 \) is an eigenvalue of \( \frac{d}{dt} - JB \) of algebraic multiplicity \( k \).

**Proof.** Suppose \( A - B \) is invertible, otherwise use \( A - B + \varepsilon J \) instead of \( A - B \). By the product formula we have

\[
\det((A - B - \nu J)(A + P_0)^{-1}) = \det((A - B - \nu J)(A - B)^{-1}) \det((A - B)(A + P_0)^{-1}).
\]  

(3.9)

By Theorem 3.5, \( \det((A - B)(A + P_0)^{-1}) \neq 0 \), and hence, the zero order of \( \det(L(\nu)) \) is equal to the zero order of \( \det((A - B - \nu J)(A - B)^{-1}) \). Direct computation shows that

\[
\det((A - B - \nu J)(A - B)^{-1}) = \det(ID + v(JA - JB)^{-1}) = \det\left(ID + v\left(\frac{d}{dt} - JB(t)\right)^{-1}\right).
\]

By Theorem 3.5, \( \nu_0 \) is a zero point of \( \det(L(\nu)) \) of order \( k \) if and only if \( -\nu_0^{-1} \) is an eigenvalue of \( \left(\frac{d}{dt} - JB(t)\right)^{-1} \) of algebraic multiplicity \( k \). Obviously, this is equivalent to that \(-\nu_0 \) is an eigenvalue of \( \left(\frac{d}{dt} - JB(t)\right) \) of algebraic multiplicity \( k \). \( \square \)

Suppose \( E = E_1 \oplus \cdots \oplus E_k \), and \( E_i \) is an invariant subspace of \( A \) and \( B \). Let \( A_i = A|_{E_i} \), \( B_i = B|_{E_i} \) and \( P_0^{(i)} = P_0|_{E_i} \) then \( A = A_1 \oplus \cdots \oplus A_k \), \( B = B_1 \oplus \cdots \oplus B_k \), \( P_0 = P_0^{(1)} \oplus \cdots \oplus P_0^{(k)} \). If the Fredholm determinant \( \det((A_i - B_i)(A_i + P_0^{(i)})^{-1}) \) is well defined for any \( i \). The next lemma is obvious.

**Lemma 3.7.**

\[
\det((A - B)(A + P_0)^{-1}) = \prod_{i=1}^{k} \det((A_i - B_i)(A_i + P_0^{(i)})^{-1}).
\]  

(3.10)

In the preceding part, we consider the \( S \)-boundary condition satisfies that \( S \in Sp(2n) \cap O(2n) \). Next, suppose that \( S \) is a unitary matrix on \( \mathbb{C}^{2n} \) and satisfies \( S^*JS = J \) (complex symplectic matrix), and consider the linear Hamiltonian system (2.1)–(2.2) under the complex \( S \)-boundary condition, \( A, B \) and \( P_0 \) are defined as above, then same reasoning as above shows that the trace and the determinant can be well defined. Now, we consider a kind of simple boundary condition. Let \( S \in Sp(2n) \cap O(2n) \), and \( \omega \) be a complex number modulo 1, then \( \omega S \) is a complex symplectic matrix. We consider the following Hamiltonian system,

\[
\dot{z}(t) = JB(t)z(t),
\]  

(3.11)

\[
z(0) = \omega Sz(T).
\]  

(3.12)
Let $A_\omega$, $B_\omega$ and $P_0^\omega$ be the operators corresponding to $A$, $B$ and $P_0$ respectively under the $\omega S$-boundary condition, then $A_\omega$ is a self-adjoint operator with the domain $D_\omega S$. In this case, both the conditional trace $\text{Tr}(B_\omega + P_0^\omega)(A_\omega + P_0^\omega)^{-1}$ and the determinant $\det((A_\omega - B_\omega)(A_\omega + P_0^\omega)^{-1})$ can be written down clearly.

**Proposition 3.8.** For any imaginary number $\nu$, let $P_{\nu,0}$ be the orthogonal projection onto $\ker(A - \nu J)$. Set $\omega = e^{\nu t}$, then

$$\text{Tr}
((B_\omega + P_0^\omega)(A_\omega + P_0^\omega)^{-1}) = \text{Tr}
((B + P_{\nu,0})(A - \nu J + P_{\nu,0})^{-1}),$$

and

$$\det((A_\omega - B_\omega)(A_\omega + P_0^\omega)^{-1}) = \det((A - \nu J - B)(A - \nu J + P_{\nu,0})^{-1}).$$

**Proof.** Since $\nu$ is an imaginary number, $e^{\nu t}$ is a unitary operator on $E$ and $e^{\nu t} D_S = D_\omega S$. Some simple calculations show that

$$e^{-\nu t} A_\omega e^{\nu t} = A - \nu J.$$ 

Given $N \in \mathbb{N}$, let $P_N^\omega$ be the orthogonal projection onto the subspace spanned by $\{\ker(A_\omega - \mu) \mid |\mu| \leq N\}$, and $P_{\nu,N}$ be the orthogonal projection onto the subspace spanned by $\{\ker(A_\omega - \nu J - \mu) \mid |\mu| \leq N\}$, then it is not difficult to show that $e^{-\nu t} P_N^\omega e^{\nu t} = P_{\nu,N}$. By the definition of the conditional trace,

$$\text{Tr}
((B_\omega + P_0^\omega)(A_\omega + P_0^\omega)^{-1})
= \lim_{N \to \infty} \text{Tr}
(P_N^\omega(B_\omega + P_0^\omega)(A_\omega + P_0^\omega)^{-1} P_N^\omega)
= \lim_{N \to \infty} \text{Tr}
((e^{-\nu t} P_N^\omega e^{\nu t}) \cdot (e^{-\nu t} (B_\omega + P_0^\omega)e^{\nu t}) \cdot (e^{-\nu t} (A_\omega + P_0^\omega)^{-1} e^{\nu t}) \cdot (e^{-\nu t} P_N^\omega e^{\nu t}))
= \lim_{N \to \infty} \text{Tr}
(P_{\nu,N}(B + P_{\nu,0})(A - \nu J + P_{\nu,0})^{-1} P_{\nu,N}).$$

Note that $A$ and $A - \nu J$ commute, which implies that $P_N$ and $P_{\nu,N}$ commute. It follows that $P_{\nu,N} \cdot P_N = P_N \cdot P_{\nu,N}$ are orthogonal projections for all $N$. Moreover, there exists a constant $C$, such that for any $N$,

$$\dim \text{Ran}(P_{\nu,N} - P_N) < C,$$

and this implies

$$\lim_{N \to \infty} \text{Tr}
(P_{\nu,N}(B + P_{\nu,0})(A - \nu J + P_{\nu,0})^{-1} P_{\nu,N})
= \lim_{N \to \infty} \text{Tr}
(P_N(B + P_{\nu,0})(A - \nu J + P_{\nu,0})^{-1} P_N),$$

thus, (3.13) is proved.
As above, it is easy to see that \((A_\omega - B_\omega)(A_\omega + P_0^\omega)^{-1}\) is unitarily equivalent to \((A - vJ - B)(A - vJ + P_0)^{-1}\), which implies that

\[
det_2((A_\omega - B_\omega)(A_\omega + P_0^\omega)^{-1}) = det_2((A - vJ - B)(A - vJ + P_0)^{-1}),
\]

therefore, (3.14) follows from (3.13) and Lemma 2.3. \(\square\)

Moreover, if \(B(t)\) is path of Hermitian matrices, then \((A_\omega - B_\omega)\) is self-adjoint, and in this case, the determinant \(det((A_\omega - B_\omega)(A_\omega + P_0^\omega)^{-1})\) is real. By taking conjugate, we have

**Proposition 3.9.** Let \(B(t)\) be a continuous path of symmetric real matrices, then

\[
det((A_\omega - B_\omega)(A_\omega + P_0^\omega)^{-1}) = det((A_\omega - B_\omega)(A_\omega + P_0^\omega)^{-1}).
\]

**Proof.** Let \(\tau\) be the conjugate operator, that is \(\tau(x(t)) = \bar{x}(t)\), then \(\tau A_\omega \tau = A_\bar{\omega}\) and \(\tau B_\omega \tau = B_\bar{\omega}\), especially \(\tau P_0^\omega \tau = P_N^\omega\). So for any \(N\),

\[
det(P_0^\omega (A_\omega - B_\omega)(A_\omega + P_0^\omega)^{-1} P_0^\omega) = det(P_N^\omega (A_{\bar{\omega}} - B_{\bar{\omega}})(A_{\bar{\omega}} + P_0^{\bar{\omega}})^{-1} P_N^{\bar{\omega}}),
\]

and we get the result by taking the limit. \(\square\)

**4. The Hill formula**

Following [2], for each \(\nu\) such that \((A - \nu J)\) is invertible, we set

\[
R(\nu) = det((A - B - \nu J)(A - \nu J)^{-1}), Q(\nu) = det((A - \nu J)(A + P_0)^{-1}),
\]

which is well defined by Proposition 2.10. Since \(P_N\) commutes with \((A - \nu J)(A + P_0)^{-1}\), we have

\[
det(L(\nu)) = R(\nu)Q(\nu).
\]

To compute \(det(L(\nu))\), we firstly compute \(Q(\nu)\). Recall that \(\sigma(A) = \bigcup_{k=1}^{\nu_n} \{v_k + 2k\pi / T\}_{k \in \mathbb{Z}}\), where \(0 \leq \nu_1 \leq \nu_2 \leq \cdots \leq \nu_n < 2\pi / T\), and the multiplicity is two. Suppose \(\nu_1 = \cdots = \nu_{k_0} = 0, 0 < \nu_{k_0+1} \leq \cdots \leq \nu_n < 2\pi / T\).

Set \(M_j = \bigoplus_{k \in \mathbb{Z}} \ker(A - \nu_j - 2k\pi / T)\), and let \(Q_j(\nu) = det((A - \nu J)|_{M_j}(A + P_0)^{-1}|_{M_j})\), then by Lemma 3.7, \(Q(\nu) = \prod_{1 \leq j \leq n} Q_j(\nu)\). Obviously \(Q_1(\nu) = \cdots = Q_{k_0}(\nu)\), some direct calculations show that

\[
Q_1(\nu) = v^2 \prod_{k \neq 0} (1 + v^2 / (2k\pi / T)^2)
\]

\[
= \frac{4}{T^2} (vT/2)^2 \prod_{k \neq 0} (1 + (vT / 2)^2 / (k\pi)^2)
\]

\[
= \frac{4}{T^2} \sinh^2(vT / 2)
\]

\[
= \frac{2}{T^2} (\cosh(vT) - 1),
\]

(4.2)
where the product formula \( \sinh(z) = z \prod_{k=1}^{\infty} (1 + z^2/(\pi^2 k^2)) \) is used to get the third equality. For \( k_0 + 1 \leq j \leq n \), by some computation, we have

\[
Q_j(\nu) = \prod_{k \in \mathbb{Z}} \left( 1 - \frac{\nu^2}{(x + 2k\sqrt{-1\pi/T})^2} \right) = \cosh(\nu T) - \cosh(\sqrt{-1\nu_j T})^{1-\cosh(\sqrt{-1\nu_j T})},
\]

where the last equality is from the Euler formula

\[
\prod_{k \in \mathbb{Z}} \left( 1 - \frac{\nu^2}{(x + 2k\sqrt{-1\pi/T})^2} \right) = \cosh(\nu T) - \cosh(xT)^1 - \cosh(xT),
\]

By (4.3)–(4.4), we have

\[
Q(\nu) = \left( \frac{2}{T^2} \right)^{k_0} (\cosh(\nu T) - 1)^{k_0} \prod_{j=k_0+1}^{n} \left( \cosh(\nu T) - \cosh(\sqrt{-1\nu_j T}) \right)^{1 - \cosh(\sqrt{-1\nu_j T})},
\]

which could be considered as a function of \( \lambda = e^{\nu T} \). To simplify the notation, we set

\[
C(S) = 2^{-n} \left( \frac{2}{T^2} \right)^{k_0} \prod_{j=k_0+1}^{n} \frac{1}{1 - \cosh(\sqrt{-1\nu_j T})},
\]

which depends only on \( S \). Obviously, \( C(S) > 0 \). If \( S = I_{2n} \), i.e. for the periodic boundary conditions, \( C(S) = T^{-2n} \), and if \( \ker(S - I_{2n}) = 0 \), then

\[
C(S) = \det(S - I_{2n})^{-1}.
\]

Next, setting \( \lambda_j = e^{\nu T} \), some direct calculations imply that

\[
Q(\lambda) = C(S)2^n \prod_{j=1}^{n} (\cosh(\nu T) - \cosh(\sqrt{-1\nu_j T}))
\]

\[
= C(S) \prod_{j=1}^{n} (\lambda + \lambda^{-1} - \lambda_j - \lambda_j^{-1})
\]

\[
= C(S)\lambda^{-n} \prod_{j=1}^{n} (\lambda - \lambda_j)(\lambda - \lambda_j^{-1})
\]

\[
= C(S)\lambda^{-n} \det(\lambda I_{2n} - S),
\]

where the last equality follows from the fact that \( \sigma(S) = \bigcup_{j=1}^{n} \{ \lambda_j, \lambda_j^{-1} \} \).

Next, we will deal with \( R(\nu) = \det(id - B(A - \nu J)^{-1}) \). For any fixed \( k \in \mathbb{Z} \), the periodic function \( e^{2\pi k \sqrt{-1\nu}/T} \) could be considered as a unitary operator on \( E \), which preserves the domain of \( A \). After some calculation, we obtain that
\[ e^{-2\pi k \sqrt{-1} t/T} (id - B(A - \nu J)^{-1}) e^{2\pi k \sqrt{-1} t/T} = id - B(A - (\nu + 2\pi k \sqrt{-1}/T)J)^{-1}. \] (4.9)

Therefore,

\[ \det2(id - B(A - \nu J)^{-1}) = \det2(id - B(A - (\nu + 2\pi k \sqrt{-1}/T)J)^{-1}). \] (4.10)

To continue, we need the following lemma. In the following lemma, we will use the exact forms of \( \hat{B} \) and \( B \), so temporarily, we will distinguish them.

**Lemma 4.1.** For \( \nu \neq \pm \sqrt{-1}(\nu_j + 2l\pi/T), \) set \( \lambda = e^{\nu T}, \) then

\[ \text{Tr}(B(A - \nu J)^{-1}) = \frac{1}{2} \sqrt{-1} \sum_{j=1}^{n} \left( \int_{0}^{T} \hat{B}_{jj}(t) dt \frac{\lambda_j + 1}{\lambda_j \lambda - 1} + \int_{0}^{T} \hat{B}_{n+j,n+j}(t) dt \frac{\lambda_j + \lambda}{\lambda_j - \lambda} \right), \] (4.11)

where \( \nu_j \) are eigenvalues of \( A \) with the \( S \)-boundary condition and \( \lambda_j = e^{\sqrt{-1} \nu_j T}. \) Moreover, for any fixed \( k \in \mathbb{Z}, \)

\[ \text{Tr}(B(A - (\nu + 2\pi k \sqrt{-1}/T)J)^{-1}) = \text{Tr}(B(A - \nu J)^{-1}). \] (4.12)

and

\[ \lim_{\text{Re} \nu \to \infty} \text{Tr}(B(A - \nu J)^{-1}) = \frac{1}{2} \int_{0}^{T} \text{Tr}(JB(t)) dt. \] (4.13)

**Proof.** Using the notations in Lemma 2.4, we have that

\[ V^*(A - \nu J)V = \left( \begin{array}{c} -\sqrt{-1}I_n \frac{d}{dt} - \sqrt{-1}I_n \nu \\ \sqrt{-1}I_n \frac{d}{dt} + \sqrt{-1}I_n \nu \end{array} \right). \]

Next, write

\[ \hat{B}(t) = V^*B(t)V. \]

It follows that

\[ \text{Tr}(B(A - \nu J)^{-1}) = \text{Tr}\hat{B}(t) \left( \begin{array}{c} -\sqrt{-1}I_n \frac{d}{dt} - \sqrt{-1}I_n \nu \\ \sqrt{-1}I_n \frac{d}{dt} + \sqrt{-1}I_n \nu \end{array} \right)^{-1} \]

\[ = \lim_{N \to \infty} \text{Tr}_P \hat{B}(t) \left( \begin{array}{c} -\sqrt{-1}I_n \frac{d}{dt} - \sqrt{-1}I_n \nu \\ \sqrt{-1}I_n \frac{d}{dt} + \sqrt{-1}I_n \nu \end{array} \right)^{-1} P_N. \]

Let \( 0 < \nu_j < 2\pi \) be an eigenvalue of \( -\sqrt{-1}I_n \frac{d}{dt} \) with \( (R + \sqrt{-1}Q) \)-boundary condition, then \( \nu_j \) is also an eigenvalue of \( \sqrt{-1}I_n \frac{d}{dt} \) with \( (R - \sqrt{-1}Q) \)-boundary condition. By Lemma 2.7,
\[
\sum_l \int_0^T \widehat{B}_{jj}(t) \, dt \left( \nu_j - \nu \sqrt{-1} + \frac{2l\pi}{T} \right)^{-1} \\
= \int_0^T \widehat{B}_{jj}(t) \, dt \cdot \frac{T \left( 1 + \cos(T(\nu_j - \nu \sqrt{-1})) \right)}{2 \sin(T(\nu_j - \nu \sqrt{-1}))} \\
= \frac{T}{2} \sqrt{-1} \int_0^T \widehat{B}_{jj}(t) \, dt \frac{e^{\sqrt{-1}T(\nu_j - \nu \sqrt{-1})} + 1}{e^{\sqrt{-1}T(\nu_j - \nu \sqrt{-1})} - 1}. \quad (4.14)
\]

Similarly,
\[
\sum_l \int_0^T \widehat{B}_{n+j,n+j}(t) \, dt \left( \nu_j + \nu \sqrt{-1} + \frac{2l\pi}{T} \right)^{-1} \\
= \frac{T}{2} \sqrt{-1} \int_0^T \widehat{B}_{n+j,n+j}(t) \, dt \frac{e^{\sqrt{-1}T(\nu_j + \nu \sqrt{-1})} + 1}{e^{\sqrt{-1}T(\nu_j + \nu \sqrt{-1})} - 1}. \quad (4.15)
\]

It follows that
\[
\text{Tr}(B(A - \nu J)^{-1}) = n \sum_{j=1}^n \frac{1}{2} \sqrt{-1} \int_0^T \widehat{B}_{jj}(t) \, dt \frac{e^{\sqrt{-1}T(\nu_j - \nu \sqrt{-1})} + 1}{e^{\sqrt{-1}T(\nu_j - \nu \sqrt{-1})} - 1} \\
+ \frac{1}{2} \sqrt{-1} \sum_{j=1}^n \int_0^T \widehat{B}_{n+j,n+j}(t) \, dt \frac{e^{\sqrt{-1}T(\nu_j + \nu \sqrt{-1})} + 1}{e^{\sqrt{-1}T(\nu_j + \nu \sqrt{-1})} - 1}. \quad (4.16)
\]

Since \(e^{(\nu + \frac{2\pi k\sqrt{-1}}{T})T} = e^{\nu T}\), we have
\[
\text{Tr}(B(A - (\nu + 2\pi k\sqrt{-1}/T)J)^{-1}) = \text{Tr}(B(A - \nu J)^{-1}). \quad (4.17)
\]

Now, since \(\lambda_j = e^{\sqrt{-1}\nu_j T}\), then \(\lambda_j, \bar{\lambda}_j\) are the eigenvalues of \(S\), and setting \(\lambda = e^{\nu T}\), the above equation can be simplified as
\[
\text{Tr}(B(A - \nu J)^{-1}) = \frac{1}{2} \sqrt{-1} \sum_{j=1}^n \left( \int_0^T \widehat{B}_{jj}(t) \, dt \frac{\lambda_j + \bar{\lambda}_j + 1}{\lambda_j \bar{\lambda}_j - 1} + \int_0^T \widehat{B}_{n+j,n+j}(t) \, dt \frac{\lambda_j + \bar{\lambda}_j}{\lambda_j - \bar{\lambda}_j} \right). \quad (4.18)
\]

Moreover,
\[
\lim_{\text{Re }\nu \to \infty} \text{Tr}(B(A - \nu J)^{-1})
\]
\[
= \lim_{|\lambda| \to \infty} \frac{1}{2} \sqrt{-1} \sum_{j=1}^{n} \left( \int_{0}^{T} \overline{B_{jj}(t)} \, dt \frac{\lambda_j \lambda + 1}{\lambda_j \lambda - 1} + \int_{0}^{T} \overline{B_{n+j,n+j}(t)} \, dt \frac{\lambda_j + \lambda}{\lambda_j - \lambda} \right)
\]
\[
= \frac{1}{2} \sqrt{-1} \int_{0}^{T} \sum_{j=1}^{n} \overline{B_{jj}(t)} \, dt - \frac{1}{2} \sqrt{-1} \int_{0}^{T} \sum_{j=1}^{n} \overline{B_{n+j,n+j}(t)} \, dt.
\] (4.19)

Now, recall that if we assume that \( B = \begin{pmatrix} a & b \\ b' & c \end{pmatrix} \), then
\[
\hat{B} = V^* B V
\]
\[
= \frac{1}{2} \begin{pmatrix} U^*(a + \sqrt{-1}b' - \sqrt{-1}b + c)U & U^*(a + \sqrt{-1}b' + \sqrt{-1}b - c)\bar{U} \\ \bar{U}^*(a - \sqrt{-1}b' - \sqrt{-1}b - c)U & \bar{U}^*(a - \sqrt{-1}b' + \sqrt{-1}b + c)\bar{U} \end{pmatrix}
\] (4.20)

Combining (4.19) and (4.20), we have
\[
\lim_{\text{Re }\nu \to \infty} \text{Tr}(B(A - \nu J)^{-1}) = \frac{1}{2} \sqrt{-1} \int_{0}^{T} \text{Tr}(U^*(a + \sqrt{-1}b' - \sqrt{-1}b + c)U) \, dt
\]
\[ - \frac{1}{2} \sqrt{-1} \int_{0}^{T} \text{Tr}(\bar{U}^*(a - \sqrt{-1}b' + \sqrt{-1}b + c)\bar{U}) \, dt
\]
\[ = \frac{1}{2} \sqrt{-1} \int_{0}^{T} \text{Tr}(a + \sqrt{-1}b' - \sqrt{-1}b + c) \, dt
\]
\[ - \frac{1}{2} \sqrt{-1} \int_{0}^{T} \text{Tr}(a - \sqrt{-1}b' + \sqrt{-1}b + c) \, dt
\]
\[ = \frac{1}{2} \int_{0}^{T} \text{Tr}(b - b') \, dt.
\]

Now, it is easy to see that \( JB = \begin{pmatrix} -b' & c \\ a & b \end{pmatrix} \), and hence,
\[
\lim_{\text{Re }\nu \to \infty} \text{Tr}(B(A - \nu J)^{-1}) = \frac{1}{2} \int_{0}^{T} \text{Tr}(J B(t)) \, dt.
\] (4.21)

The desired result is proved. \( \square \)
**Remark 4.2.** 1) If $B$ is a path of real symmetric matrices, then $\text{Tr}(JB(t)) = 0$, and hence, in this case

$$\lim_{\text{Re } \nu \to \infty} \text{Tr}(B(A - \nu J)^{-1}) = 0.$$ 

2) If moreover $S = I_{2n}$, that is, for periodic boundary problem, then $\nu_j = 0$, and hence, in this case,

$$\text{Tr}(B(A - \nu J)^{-1}) = 0.$$ 

**Lemma 4.3.** $R(\nu) \to e^{-\frac{1}{2} \int_0^T \text{Tr}(JB(t)) \, dt}$ as $\text{Re } \nu \to \infty$.

**Proof.** At first, we prove that $\|B(A - \nu J)^{-1}\|_2 \to 0$ as $\text{Re } \nu \to \infty$. In fact,

$$\|B(A - \nu J)^{-1}\|_2^2 = \text{Tr}((A - \nu J)^{s^{-1}}(A - \nu J)^{-1}) = \text{Tr}((A^2 - 2\text{Im } \nu \cdot AJ + |\nu|^2)^{-1}),$$

and hence $\|(A - \nu J)^{-1}\|_2^2 \to 0$ as $\text{Re } \nu \to \infty$. Noting that $B$ is a bounded operator, it follows that

$$\|B(A - \nu J)^{-1}\|_2 \leq \|B\| \|(A - \nu J)^{-1}\|_2 \to 0, \quad \text{as } \text{Re } \nu \to \infty.$$ 

This implies that $\det_2(id - B(A - \nu J)^{-1}) \to 1$ as $\text{Re } \nu \to \infty$. Moreover, by Lemma 4.1, $\text{Tr}(B(A - \nu J)^{-1})$ tends to $\frac{1}{2} \int_0^T \text{Tr}(JB(t)) \, dt$ as $\text{Re } \nu \to \infty$. Note that

$$R(\nu) = \det((A - B - \nu J)(A - \nu J)^{-1}) = \det_2(id - B(A - \nu J)^{-1}) e^{-\text{Tr}B(A - \nu J)},$$

and hence $R(\nu) \to e^{-\frac{1}{2} \int_0^T \text{Tr}(JB(t)) \, dt}$ as $\text{Re } \nu \to \infty$. $\Box$

Thanks to Lemma 4.1, we have

$$\det(id - B(A - \nu J)^{-1}) = \det(id - B(A - (\nu + 2\pi k i/T)J)^{-1}). \quad (4.22)$$

It follows that $R$ is a meromorphic function of $\lambda = e^{\nu T}$ and the only possible poles locate on $\lambda_j, \lambda_j^{-1}, j = 1, \ldots, n$, which are eigenvalues of $S$. Moreover, the order of the pole of $R(\lambda)$ at $\lambda_j$ is less or equal to the order of the zero point of $\det(\lambda I_{2n} - S)$ at $\lambda_j$.

With the above preparation, we are ready to prove Theorem 1.3. At first, we will prove it in some simple case, and then, we will prove the theorem by some perturbation theory.

**Proof of Theorem 1.3.** By the above discussion, $R(\lambda)$ is a meromorphic function, whose possible poles only locate on $\lambda_j, \lambda_j^{-1}, j = 1, \ldots, n$, it follows that $R(\lambda) \det(\lambda I_{2n} - S)$ is an analytic function in $\lambda$. Noting that $e^{-\frac{1}{2} \int_0^T \text{Tr}(JB(t)) \, dt} \neq 0$, by Lemma 4.3, $R(\lambda) \det(\lambda I_{2n} - S)$ is a polynomial of degree $2n$. Thus $\lambda^n \det(L(\lambda)) = \lambda^n R(\lambda) Q(\lambda) = C(S) R(\lambda) \det(\lambda I_{2n} - S)$ is a polynomial of degree $2n$, whose zeros only locate on the $J$-eigenvalues of $A - B$, and which are the eigenvalues of $S^\gamma(T) - \lambda I_{2n}$.
At first, we consider some simple case. \( B \) is called to be simple, if all the eigenvalues of \( S_\gamma(T) \) are distinct to each other. In this case, we immediately have

\[
\lambda^n \det(L(\lambda)) = C(S) e^{-\frac{1}{2} \int_0^T \text{Tr}(J B(t)) \, dt} \det(S_\gamma(T) - \lambda I_{2n}).
\] (4.23)

Next, we will get through the general case by some perturbation theory. Set \( K = -J \log(\gamma(T))/T \), then \( S_\gamma(T) = S e^{JKT} \). Notice that there is a matrix \( F \) whose norm is arbitrarily small, such that the eigenvalues of \( S e^{JKT} + F \) are distinct to each other. Now, let \( K_F = -JT^{-1} \log(S^{-1}(Se^{JKT} + F)) \). This implies that the norm of \( K_F - K \) is arbitrarily small and \( S e^{JK_FT} = S e^{JKT} + F \). By Floquet theory, setting \( P(t) = \gamma(t) \exp(-JKt) \), then \( P(t) \) is a bounded invertible operator on \( E \). Since \( P(0) = P(T) = I_{2n} \), \( P(t) \) preserves \( D_S \). Direct computation shows that

\[
P^{-1}(t) \left( \frac{d}{dt} - JB(t) \right) P(t) = \frac{d}{dt} - JK.
\] (4.24)

This implies that

\[
\frac{d}{dt} - JB(t) = P(t) \left( \frac{d}{dt} - JK_F \right) P^{-1}(t) - P(t) J(K - K_F) P^{-1}(t).
\] (4.25)

It is easy to see that \( -P(t) J(K - K_F) P^{-1}(t) \) is arbitrarily small. Therefore, any Hamiltonian system with \( S \)-boundary condition can be approximated by a simple Hamiltonian system. Now, on the one hand, \( C(S) e^{-\frac{1}{2} \int_0^T \text{Tr}(J B(t)) \, dt} \det(\lambda S_\gamma(T) - I_{2n}) \) is dependent on \( B(t) \) continuously; on the other hand, by Proposition 3.2, we have that \( \lambda^n \det(L(\lambda)) \) is also dependent on \( B(t) \) continuously. Combining with (4.23), the above reasoning implies that (4.23) is always true for any continuous path \( B(t) \). The proof of Theorem 1.3 is completed. \( \square \)

**Remark 4.4.** In Theorem 1.3, if we assume that \( B \) is a path of real symmetric matrices, then \( \text{Tr}(JB) = 0 \), and hence, Hill’s formula can be written as

\[
\det(L(\lambda)) = C(S) \lambda^{-n} \det(S_\gamma(T) - \lambda I_{2n}).
\] (4.26)

By Theorem 1.3 and Proposition 3.6, we have

**Corollary 4.5.** For a \( J \)-eigenvalue \( \nu_0 \) of \( A - B \), \( \nu_0 \) is a zero point of the analytic function \( \det(L(\nu)) \) of order \( k \) if and only if \( e^{\nu T} \) is an eigenvalue of \( S_\gamma(T) \) of algebraic multiplicity \( k \), and this is equivalent to that \(-\nu_0 \) is an eigenvalue of \( \frac{d}{dt} - JB \) of algebraic multiplicity \( k \).

**Remark 4.6.** Suppose that \( \lambda = e^{\nu T} \in \mathbf{U} \), and \( B(t) \) is a continuous path of real symmetric matrices, then the left of (4.26) is real since \( A - \nu J - B \) is a self-adjoint operator. In this case, \( \gamma(T) \in \text{Sp}(2n) \) and hence \( S_\gamma(T) \in \text{Sp}(2n) \), thus \( \lambda^{-n} \det(S_\gamma(T) - \lambda I_{2n}) \) is real for \( \lambda \in \mathbf{U} \) by the property of symplectic matrix [8, p. 37].
5. Decomposition for the cyclic type symmetric solution

We will consider the splittingness of Fredholm determinant under the cyclic type symmetry. For $Q, S \in \text{Sp}(2n) \cap O(2n)$, $S^m = Q$, the $Z_m$-group action for the generator $g \in Z_m$ is as follows:

$$g : D_Q \to D_Q, \quad z(t) \mapsto Sz\left(t + \frac{T}{m}\right).$$

Obviously $g$ is a unitary operator on $D_Q$ and $H$, and satisfies $g^m = I$, $g A = A g$. Suppose the Hamiltonian function $H(t, z) \in C^2(\mathbb{R}^{2n+1}, \mathbb{R})$ satisfies $H(t - T/m, S z) = H(t, z)$ (in autonomous case, $H(S z) = H(z)$), then the functional $f(z) = \int_0^T \left(-J \frac{dz(t)}{dt}, z(t)\right) dt - H(t, z(t)) dt$ is $Z_m$-invariant. The $Z_m$ invariant solution which satisfies $z(t) = Sz(t + T/m)$ is just the solution with $S$ boundary on the interval $[0, T/m]$. It is obvious for the $Z_m$ invariant solution $z$, $f''(z) g f'(z)$, this means $gB = Bg$.

Since $A$ and $B$ commute with $g$, both $A$ and $A - B$ could be decomposed relying on the invariant subspaces of $g$. Notice that $G = \{g^k, \ k = 1, \ldots, m\}$ is isomorphic to group $Z_m$. Let

$$E_i = \ker(g - \omega_i) \cap D_Q, \quad i = 1, \ldots, m,$$

where $\omega_i = \exp(2 \pi \sqrt{-1} \frac{I}{m})$ is the $m$-th root of 1. Obviously $E_i = \{x(t) \in D_Q, \ \omega_i x(t) = S x(t + \frac{T}{m})\}$ is a $G$-invariant subspace. From the spectral theory of normal operator, we know that $E_i$'s are mutually orthogonal and $D_Q = E_1 \oplus \cdots \oplus E_m$. Since $A$ is a closed linear operator on $E$ which commutes with $g$, i.e. $g A = Ag$, then $E_i$ are invariant subspaces of $A$ for all $i$. Set $A_i = A|E_i$, then we have decomposition $A = A_1 \oplus \cdots \oplus A_m$. The same reasoning implies that $B = B_1 \oplus \cdots \oplus B_m$, $P_0 = P_0^{(1)} \oplus \cdots \oplus P_0^{(m)}$. Here $A_i$ is the operator $-J \frac{d}{dt}$ on $L^2([0, T/m], \mathbb{C}^{2n})$ with domain $E_i$, similar for $A_i - B_i$.

Applying Lemma 3.6 we have the next decomposition theorem.

**Theorem 5.1.**

$$\det((A - B)(A + P_0)^{-1}) = \det((A_1 - B_1)(A_1 + P_0^{(1)})^{-1}) \cdots \det((A_m - B_m)(A_m + P_0^{(m)})^{-1}). \quad (5.1)$$

**Remark 5.2.** If $B \in \mathfrak{B} = C([0, T], GL(2n, \mathbb{C}))$ and commutes with $g$, then (5.1) is also true.

Let $\nu_i = 2 \pi \sqrt{-1} \frac{I}{T}$, then $e^{\nu_i \frac{T}{m}} = \omega_i$. By Proposition 3.9, Proposition 3.8 and Theorem 3.5, we have

$$\det((A_i - B_i)(A_i + P_0^{(i)})^{-1})$$

$$= \det((A_{\omega_i} - B_{\omega_i})(A_{\omega_i} + P_{\omega_i})^{-1})$$

$$= \det((A_{\omega_i} - B_{\omega_i})(A_{\omega_i} + P_{\omega_i})^{-1})$$

$$= \det((A - \nu_i J - B)(A - \nu_i J + P_{\nu_i,0})^{-1}) \quad (5.2)$$

$$= \det((A - \nu_i J - B)(A + P_0)^{-1})(\det((A - \nu_i J + P_{\nu_i,0})(A + P_0)^{-1}))^{-1}, \quad (5.3)$$
where the operators on the right-hand side of (5.2)–(5.3) are acting on the Hilbert space $L^2([0, T/m], C^{2n})$, and without confusion, we also use the notations $A$, $B$. And hence, the following corollary is obtained immediately.

**Corollary 5.3.** $\det((A - B)(A + P_0)^{-1})$ is zero if and only if there exists some $i$, such that $\det((A - \nu_i J - B)(A + P_0)^{-1})$ is zero.

Recall that $P_B$ is the orthogonal projection from $E$ onto $\ker (A - B)$, then $\det((A - B + P_B)(A + P_0)^{-1}) \neq 0$. Denote by $s(B)$ the sign of $\det((A - B + P_B)(A + P_0)^{-1})$. We have

**Lemma 5.4.** For $\varepsilon > 0$ small enough, $s(B) = s(B - \varepsilon I_{2n})$.

**Proof.** Please note that for $\varepsilon > 0$, $\ker (A - B + \varepsilon P_B) = 0$, this implies

$$\text{sign}(\det((A - B + \varepsilon P_B)(A + P_0)^{-1})) = \text{sign}(\det((A - B + P_B)(A + P_0)^{-1})), \quad (5.4)$$

for $\varepsilon > 0$. On the other hand, for a fixed $\varepsilon > 0$ which is small enough, $\ker (A - B + \varepsilon P_B + \eta \varepsilon (I - P_B)) = 0$, for $\eta \in [0, 1]$. Thus

$$\text{sign}(\det((A - B + \varepsilon P_B + \eta \varepsilon (I - P_B))(A + P_0)^{-1})) = \text{sign}(\det((A - B + \varepsilon P_B)(A + P_0)^{-1})), \quad \eta \in [0, 1]. \quad (5.5)$$

The desired result is proved. □

With the above preparation, we have the following corollary.

**Corollary 5.5.** Under the above assumption,

1. if $m$ is odd, $s(B) = s(B_m)$;
2. if $m$ is even, $s(B) = s(B_m)s(B_{m/2})$.

where $s(B_j) = \text{sign}(\det((A_j - B_j)(A_j + P_0^{(j)})^{-1}))$.

**Proof.** Suppose $A - B$ is non-degenerate, otherwise use $B - \varepsilon I_{2n}$ ($\varepsilon > 0$ small enough) instead of $B$. Notice that for $1 \leq i \leq m - 1$, $\omega_i = \exp(2\pi \sqrt{-1} \frac{i}{m})$, thus $\omega_i = \bar{\omega}_{m-i}$. By Proposition 3.9,

$$\det((A_i - B_i)(A_i + P_0^{(i)})^{-1}) = \det((A_{m-i} - B_{m-i})(A_{m-i} + P_0^{(m-i)})^{-1}). \quad (5.6)$$

If $m$ is odd, by Theorem 5.1,

$$\det((A - B)(A + P_0)^{-1})$$

$$= \det((A_m - B_m)(A_m + P_0^{(m)})^{-1}) \cdot \prod_{i=1}^{m-1} (\det((A_i - B_i)(A_i + P_0^{(i)})^{-1}))^2. \quad (5.7)$$
If \( m \) is even,

\[
\det((A - B)(A + P_0)^{-1}) = \det((A - B)(A + P_{0_{\frac{m}{2}}})^{-1}) \cdot \det((A_m - B_m)(A_m + P_{0_{(m)}})^{-1}) \cdot \prod_{i=1}^{\frac{m}{2}-1} (\det((A_i - B_i)(A_i + P_{0_{(i)}})^{-1}))^2.
\]

(5.8)

The desired result is easily obtained. \( \square \)

Since we have proved the Hill formula, we could get the decomposition formula by the usual determinant of linear Poincaré map. To illustrate this, we only consider the case \( \ker(Q - I_{2n}) = 0 \), and this is equivalent to \( \ker(S - \omega_i I_{2n}) = 0 \), for \( i = 1, \ldots, m \). Moreover, suppose \( z(t) \) is a \( Q \)-periodic solution satisfying the cyclic symmetry, and \( B(t) \) is a path of real symmetric matrices. \( Bg = gB \) implies

\[
S^T B(t) S = B(t + T/m).
\]

(5.9)

Direct calculation shows that

\[
\gamma_z\left(t + \frac{T}{m}\right) = S^T \gamma_z(t) S \gamma_z\left(\frac{T}{m}\right),
\]

(5.10)

therefore

\[
\gamma_z\left(\frac{kT}{m}\right) = (S^k)^T \left( S \gamma_z\left(\frac{T}{m}\right) \right)^k.
\]

(5.11)

Since \( S^m = Q \), for some \( Q \in O(2n) \), we have

\[
Q \gamma_z(T) = \left( S \gamma_z\left(\frac{T}{m}\right) \right)^m.
\]

(5.12)

In the case that \( z(t) \) is a periodic solution, that is, \( Q = I_{2n} \), (5.12) implies that the linear or spectral stability of the \( \gamma_z(T) \) is the same as that of \( S \gamma_z\left(\frac{T}{m}\right) \), and this means that Definition 1.1 is reasonable.

By Theorem 1.3,

\[
\det((A - \nu_i J - B)(A + P_0)^{-1}) = C(S)\omega_i^{-n} \det(S \gamma(T/m) - \omega_i I_{2n}),
\]

(5.13)

and in this case \( \ker(A - \nu_i J) = 0 \)

\[
\det((A - \nu_i J)(A + P_0)^{-1}) = C(S)\omega_i^{-n} \det(S - \omega_i I_{2n}).
\]

(5.14)

From (5.3), (5.13), (5.14) we have

\[
\det((A_i - B_i)(A_i + P_{0i})^{-1}) = \det(S \gamma(T/m) - \omega_i I_{2n}) \cdot (\det(S - \omega_i I_{2n}))^{-1}.
\]

(5.15)
Taking product on both sides of (5.16), we get
\[
\prod_{i=1}^{m} \det((A_i - B_i)(A_i + P_i^0)^{-1}) = \prod_{i=1}^{m} \det(S_{\gamma}(T/m) - \omega_i I_{2n}) \cdot \prod_{i=1}^{m} (\det(S - \omega_i I_{2n}))^{-1}
\]
\[= \det((S_{\gamma}(T/m))^m - I_{2n}) \cdot (\det(S^m - I_{2n}))^{-1}
\]
\[= \det(Q_{\gamma}(T) - I_{2n}) \cdot C(Q)
\]
\[= \det((A - B)(A + P_0)^{-1}),
\]
where the third equality is from (5.12) and (4.7), and the last equality is from (1.9). Hence we proved the decomposition formula by the Hill formula.

6. Relation with the index theory

The Maslov-type index is a very useful tool in studying the multiplicity and stability of periodic solution in Hamiltonian systems [8]. We will see that, the Fredholm determinant has a very closed relationship with it. Recall that, the Maslov-type index could be defined by the intersection number of path with the singular set in Sp(2n) [8], the spectral flow or the relative Morse index [1,6,13]. For reader’s convenience, we simply illustrate the relative Morse index by index theory of Fredholm operators, the details could be found in the above references. Let $E$ be a Hilbert space. For a closed subspace $U$ of $E$, $P_U$ denotes the orthogonal projection from $E$ to $U$ and $U^\perp$ denotes the orthogonal complement of $U$. Let $U$ and $W$ be two closed subspaces of $E$. Denote it by $U \sim W$ if $P_U - P_W$ is a compact operator. In this case, $U, W$ are called commensurable subspaces of $E$, and both $W \cap U^\perp$ and $W^\perp \cap U$ are of finite dimension. The relative dimension of $W$ with respect to $U$ is defined by

\[\dim(W, U) = \dim(W \cap U^\perp) - \dim(W^\perp \cap U).\] (6.1)

Note that if both $W$ and $U$ are of finite dimension, dim$(W, U) = \dim W - \dim U$. Let $O(E)$ be the set of closed subspaces of $E$. Define a metric on $O(E)$ by

\[d(U, W) = \|P_U - P_W\|
\]
for $U, W \in O(E)$. It is easy to check that $(O(E), d)$ is a complete metric space. Set

\[Y(U) = \{W \mid W \in O(E) \text{ and } W \sim U\}.
\]

It is not difficult to verify that if $U_1 \sim U_2$ and $U_2 \sim U_3$, then $U_1 \sim U_3$. The following proposition comes from [1].

**Proposition 6.1.** Suppose $U$ and $W$ are two commensurable subspaces of $E$. Then

(i) $\dim(W, U) = -\dim(U, W)$.

(ii) $P_U|_W : W \to U$ is a Fredholm operator with index $\text{ind}(P_U|_W) = \dim(W, U)$.
(iii) If in addition \( W \sim W' \), then \( \dim(U, W') = \dim(U, W) + \dim(W, W') \).

(iv) If \( U \in O(E) \) then \( Y(U) \) is a closed subset of \( O(E) \). Moreover, \( \dim(\cdot, V) : Y(U) \to \mathbb{Z} \) is a continuous function.

If \( A_1 \) is a bounded self-adjoint Fredholm operator on \( E \), there is a unique \( A_1 \)-invariant orthogonal splitting

\[
E = E_+(A_1) \oplus E_-(A_1) \oplus E_0(A_1)
\]

with \( E_+(A_1) \), \( E_-(A_1) \) and \( E_0(A_1) \) being respectively the subspaces on which \( A_1 \) is positive definite, negative definite and null. For a pair of bounded self-adjoint Fredholm operators \( A_1 \) and \( \tilde{A} \), it will be denoted by \( A_1 \sim \tilde{A} \) if \( E_-(A_1) \sim E_-(\tilde{A}) \). In this case, the relative Morse index \( I(A_1, \tilde{A}) \) is defined by

\[
I(A_1, \tilde{A}) = \dim(E_-(\tilde{A}), E_-(A_1)).
\]  

Please note that if \( A_1 \) is bounded self-adjoint Fredholm operator and \( A_1 - \tilde{A} \) is compact, then \( A_1 \sim \tilde{A} \).

We come back to the Hamiltonian systems, for a dense set \( M \), its \( 1/2 \) inner product is defined by,

\[
\langle x, y \rangle = ((|A| + id)x, y), \quad \text{for } x, y \in M,
\]

and let \( E \) be the Hilbert space spanned by the eigenvector of \( A \) under the \( 1/2 \) norm. Let \( \tilde{A} \) be the operator defined by

\[
\langle \tilde{A}x, y \rangle = (Ax, y), \quad \forall x, y \in H,
\]

then \( \tilde{A} = (id + |A|)^{-1}A \). Similarly, we define \( \tilde{B} = (id + |A|)^{-1}B \) on \( E \). Throughout this section, \( B(t) \) is always assumed to be a continuous path of real symmetric matrices. Obviously, \( \tilde{A} \) is a bounded self-adjoint Fredholm operator and \( \tilde{B} \) is a compact self-adjoint operator on \( E \), thus the relative Morse index \( I(\tilde{A}, \tilde{A} - \tilde{B}) \) is well defined. For the unbounded Fredholm self-adjoint operators \( A, A - B \), we define

\[
I(A, A - B) = I(\tilde{A}, \tilde{A} - \tilde{B}).
\]

Recall that \( s(B) \) is the sign of \( \det((A - B + P_B)(A + P_0)^{-1}) \). We have the next theorem

**Theorem 6.2.**

\[
s(B) = (-1)^b,
\]

where \( b = I(\tilde{A}, \tilde{A} - \tilde{B}) \).
Proof. Without loss of generality, assume $A - B$ is non-degenerate, otherwise use $B - \varepsilon I$ instead of $B$. Set $B_n = P_n BP_n$, then $\tilde{B}_n$ converges to $\tilde{B}$ in the operator norm, and $E_-(\tilde{A} - \tilde{B}_n)$ converges to $E_-(\tilde{A} - \tilde{B})$, by the continuous of the relative Morse index, we have

$$I(\tilde{A}, \tilde{A} - \tilde{B}) = I(\tilde{A}, \tilde{A} - \tilde{B}_n),$$

(6.7)

for $n$ large enough. Let $V_n = \text{Ran}(P_n)$, then $V_n$ is finite-dimensional and $E = V_n \oplus V_n^\perp$. Then $\tilde{A} = \tilde{A}|_{V_n} \oplus \tilde{A}|_{V_n^\perp}$ and $\tilde{B}_n = \tilde{B}_n|_{V_n}$, by the splitting property of relative Morse index, we have

$$I(\tilde{A}, \tilde{A} - \tilde{B}_n) = \dim E_-(\tilde{A}|_{V_n} - \tilde{B}_n) - \dim E_-(\tilde{A}|_{V_n})$$

$$= \dim E_-(A|_{V_n} - B_n) - \dim E_-(A|_{V_n}).$$

(6.8)

On the other hand,

$$\det((A - B)(A + P_0)^{-1}) = \lim_{n \to \infty} \det((A|_{V_n} - B_n)(A|_{V_n} + P_0)^{-1}),$$

(6.9)

so for $n$ large enough, $s(B)$ is equal to sign of $\det((A|_{V_n} - B_n)(A|_{V_n} + P_0)^{-1})$ which is the usual determinant of finite-dimensional matrix. Obviously, $\det((A|_{V_n} - B_n)(A|_{V_n} + P_0)^{-1})$ is positive (negative) if and only if the difference of total multiplicities of the negative eigenvalues of $A|_{V_n} - B_n$ and $A|_{V_n} + P_0$ is even (odd), and this is a sufficient and necessary condition of that $b$ is even (odd), this finishes the proof. \qed

From Eq. (6.5) and Theorem 6.2, we get the proof of Theorem 1.5. Let $B_1, B_2$ be two paths of Hermitian matrices, by Theorem 1.5, we have the following corollary.

Corollary 6.3. $\det((A - B_1 + P_{B_1})(A - B_2 + P_{B_2})^{-1})$ is positive (negative) if and only if $I(\tilde{A} - \tilde{B}_1, \tilde{A} - \tilde{B}_2)$ is even (odd).

Proof. By the product formula,

$$\det((A - B_1 + P_{B_1})(A - B_2 + P_{B_2})^{-1})$$

$$= \det((A - B_1 + P_{B_1})(A + P_0)^{-1}) \det((A + P_0)(A - B_2 + P_{B_2})^{-1}),$$

therefore, the sign of $\det((A - B_1 + P_{B_1})(A - B_2 + P_{B_2})^{-1})$ is equal to $s(B_1)/S(B_2)$. On the other hand,

$$I(\tilde{A} - \tilde{B}_1, \tilde{A} - \tilde{B}_2) = I(\tilde{A} - \tilde{B}_1, \tilde{A}) + I(\tilde{A}, \tilde{A} - \tilde{B}_2) = I(\tilde{A}, \tilde{A} - \tilde{B}_2) - I(\tilde{A}, \tilde{A} - \tilde{B}_1),$$

and the desired result is obtained. \qed

Recall that for $\omega \in U$, $S \in \text{Sp}(2n) \cap O(2n)$, $A_\omega$, $B_\omega$ and $P_0^\omega$ are the corresponding operators under the $\omega S$ boundary condition. $I(\tilde{A}_\omega, \tilde{A}_\omega - \tilde{B}_\omega)$ is the relative Morse index under the $\omega S$ boundary condition, then we have

Corollary 6.4. $\det((A - vJ - B)(A - vJ + P_{v,0})^{-1})$ is positive (negative) if and only if $I(\tilde{A}_\omega, \tilde{A}_\omega - \tilde{B}_\omega)$ is even (odd).
Remark 6.5. Suppose \( \ker(S + I_{2n}) = 0 \), then \( \det(S + I_{2n}) > 0 \) since \( S \in O(2n) \). By some similar computation to (5.14), we have

\[
\det \left( \left( A - B - \frac{\sqrt{-1} \pi}{T} J \right) \left( A - \frac{\sqrt{-1} \pi}{T} J \right)^{-1} \right) = \det \left( \left( A - B - \frac{\sqrt{-1} \pi}{T} J \right) \left( A + P_0 \right)^{-1} \right) C(S)(-1)^n \det(S + I_{2n}).
\]

That \( A - B - \frac{\sqrt{-1} \pi}{T} J \) is non-degenerate under the \( S \)-boundary condition is equivalent to that \( A - B \) is non-degenerate under the \( -S \) boundary condition. By 2) of Corollary 1.4, under the non-degenerate condition, \( \det((A - B - \frac{\sqrt{-1} \pi}{T} J)(A - \frac{\sqrt{-1} \pi}{T} J)^{-1}) < 0 \) implies that \( z \) is spectrally unstable. From Corollary 6.4, this implies that, if \( I(\tilde{A}, \tilde{A} - B) \) is odd, then \( z \) is spectrally unstable. Similarly, by Corollary 6.3 and 1) of Corollary 1.4, if \( z \) is non-degenerate for the \( S \) boundary condition, then it is spectrally unstable if the relative Morse index \( I(A, A - B) \) is odd.

This criteria to judge the instability by using the relative Morse index (Maslov-type index) had been gotten by a totally different way in [6]. For the Lagrangian system, it had been proved in [2].

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