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On transitive Cayley graphs of groups and semigroups

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Abstract

We investigate Cayley graphs of semigroups and show that they sometimes enjoy properties analogous to those of the Cayley graphs of groups.

1. Introduction

Let G be a semigroup, and let S be a nonempty subset of G. The *Cayley graph* $\mathrm{Cay}(G, S)$ of G relative to S is defined as the graph with vertex set G and edge set $E(S)$ consisting of those ordered pairs (x, y) such that $sx = y$ for some $s \in S$ (see [7]).

The aim of this paper is to describe all semigroups with Cayley graphs satisfying certain transitivity properties similar to those possessed by all Cayley graphs of groups. Let us first define a few properties which hold true for the Cayley graphs of groups.

Let $D(V, E)$ be a graph with vertex set V and edge set $E \subseteq V \times V$. A mapping $\phi : V \to V$ is called an *endomorphism* of the graph D if $(u^\phi, v^\phi) \in E$ for all $(u, v) \in E$. An *automorphism* is an endomorphism that is one-to-one and onto.

A graph $D(V, E)$ is said to be *vertex-transitive* if, for any two vertices $x, y \in V$, there exists an automorphism $\phi \in \mathrm{Aut}(D)$ such that $x^\phi = y$ (see [1]). More generally, a subset A of $\mathrm{End}(D)$ is said to be *vertex-transitive* on D, and D is said to be A*-vertex-transitive* if, for any two vertices $x, y \in V$, there exists an endomorphism $\phi \in A$ such that $x^\phi = y$ (see [3, Section 11.1]).

All Cayley graphs of groups are vertex-transitive, since the group on which the Cayley graph is defined acts by right multiplication as a vertex-transitive group of automorphisms.

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Cayley graphs of groups have received much attention in the literature and many interesting results have been obtained (see [1, 2, 5, 8]).

Now let G be a semigroup and let $S \subseteq G$. Denote the automorphism group (endomorphism monoid) of $\mathrm{Cay}(G, S)$ by $\mathrm{Aut}_S(G)$ (respectively, $\mathrm{End}_S(G)$). Thus

$$\mathrm{Aut}_S(G) = \mathrm{Aut}(\mathrm{Cay}(G, S)) \quad \text{and} \quad \mathrm{End}_S(G) = \mathrm{End}(\mathrm{Cay}(G, S)).$$

It is easily seen that each element of G, acting by right multiplication, defines an endomorphism of the Cayley graph $\mathrm{Cay}(G, S)$. Denote by $\mathrm{R}(G)$ the set of all endomorphisms of $\mathrm{Cay}(G, S)$ defined by the right multiplications by elements of G.

An element $\phi \in \mathrm{End}_S(G)$ will be called a *colour-preserving endomorphism* if $sx = y$ implies $s(x^\phi) = y^\phi$, for every $x, y \in G$ and $s \in S$. If we regard an edge (x, sx), for $s \in S$, as having 'colour' s, so that the elements of S are thought of as colours associated with the edges of the Cayley graph, then every colour-preserving endomorphism maps each edge to an edge of the same colour. Denote by $\mathrm{ColEnd}_S(G)$ (and $\mathrm{ColAut}_S(G)$) the sets of all colour-preserving endomorphisms (respectively, automorphisms) of $\mathrm{Cay}(G, S)$. Evidently,

$$\mathrm{ColAut}_S(G) \subseteq \mathrm{Aut}_S(G),$$
$$\mathrm{R}(G) \subseteq \mathrm{ColEnd}_S(G) \subseteq \mathrm{End}_S(G),$$

and $\mathrm{ColAut}_S(G)$, $\mathrm{R}(G)$, $\mathrm{ColEnd}_S(G)$ are submonoids of $\mathrm{End}_S(G)$.

Let G be a semigroup, S a subset of G and let $A \subseteq B \subseteq \mathrm{End}_S(G)$. If A is vertex-transitive on $\mathrm{Cay}(G, S)$, then the same holds true for B, too.

It is well known and easy to verify that, for every group G and every subset S of G, all of $\mathrm{End}_S(G)$, $\mathrm{ColEnd}_S(G)$, $\mathrm{Aut}_S(G)$, and $\mathrm{R}(G)$ are vertex-transitive on the Cayley graph $\mathrm{Cay}(G, S)$. Moreover, in this case it is not difficult to show that $\mathrm{R}(G) = \mathrm{ColEnd}_S(G)$ for every set S generating G.

2. Main theorems

Our first main theorem describes all semigroups G and all subsets S of G, satisfying a certain finiteness condition, such that the Cayley graph $\mathrm{Cay}(G, S)$ is $\mathrm{ColAut}_S(G)$-vertex-transitive. A semigroup is called a *right zero band* if it satisfies the identity $xy = y$.

Theorem 2.1. *Let G be a semigroup, and let S be a subset of G which generates a subsemigroup $\langle S\rangle$ such that all principal left ideals of $\langle S\rangle$ are finite. Then, the Cayley graph $\mathrm{Cay}(G, S)$ is $\mathrm{ColAut}_S(G)$-vertex-transitive if and only if the following conditions hold:*

(i) *$sG = G$, for all $s \in S$;*
(ii) *$\langle S\rangle$ is isomorphic to a direct product of a right zero band and a group;*
(iii) *$|\langle S\rangle g|$ is independent of the choice of $g \in G$.*

Next, we reduce the problem of describing vertex-transitive Cayley graphs of finite semigroups to the case of completely simple semigroups. Our result applies to all semigroups satisfying the same finiteness condition as in Theorem 2.1. Recall that a semigroup is *completely simple* if it has no proper ideals and has an idempotent minimal with respect to the partial order $e \leq f \Leftrightarrow e = ef = fe$. Completely simple semigroups can be represented as Rees matrix semigroups over groups (see Section 3).

Theorem 2.2. *Let G be a semigroup, and let S be a subset of G such that all principal left ideals of the subsemigroup $\langle S\rangle$ are finite. Then, the Cayley graph $\mathrm{Cay}(G, S)$ is $\mathrm{Aut}_S(G)$-vertex-transitive if and only if the following conditions hold:*

(i) $SG = G$;
(ii) $\langle S\rangle$ *is a completely simple semigroup;*
(iii) *the Cayley graph* $\mathrm{Cay}(\langle S\rangle, S)$ *is* $\mathrm{Aut}_S(\langle S\rangle)$*-vertex-transitive;*
(iv) $|\langle S\rangle g|$ *is independent of the choice of* $g \in G$.

In the next section we summarize the background theory on semigroups needed to prove these theorems. In Section 4 we give a series of examples of Cayley graphs of semigroups which demonstrate the necessity of the finiteness conditions, and the precision of some of the conclusions, of these theorems. Some general results about Cayley graphs of semigroups are proved in Sections 5 and 6, and the main theorems are proved in Section 7. Finally, Section 8 contains some concluding comments on these results.

3. Preliminaries on semigroups

We use standard concepts and notation of semigroup theory following [3, 4] and [6], and include concise background information in this section.

If $S \subseteq G$, then the subsemigroup generated by S in G is denoted by $\langle S\rangle$. An element s of a semigroup G is said to be *periodic* if there exist positive integers m, n such that $s^{m+n} = s^m$. A subset S of G is *periodic* if every element of S is periodic. In particular, if all principal left ideals of a semigroup are finite, then the semigroup is periodic.

A *band* is a semigroup entirely consisting of idempotents. A band is called a *left zero* (*right zero*, *rectangular*) *band* if it satisfies the identity $xy = x$ (respectively, $xy = y$, $xyx = x$). In fact, every rectangular band satisfies the identity $xyz = xz$, as well.

A semigroup is said to be *right* (*left*) *simple* if it has no proper *right* (*left*) *ideals*. A semigroup is *left* (*right*) *cancellative* if $xy = xz$ (respectively, $yx = zx$) implies $y = z$, for all $x, y, z \in S$. A semigroup is called a *right* (*left*) *group* if it is right (left) simple and left (right) cancellative.

We summarize these terms in the following table:

Concept	Property
Band	$x^2 = x$
Left zero band	$x^2 = x, xy = x$
Right zero band	$x^2 = x, xy = y$
Rectangular band	$x^2 = x, xyx = x$
Semilattice	$x^2 = x, xy = yx$
Right simple semigroup	No proper right ideals
Left cancellative semigroup	$xy = xz \Rightarrow y = z$
Right group	Right simple and left cancellative

A few known facts required for our proofs are collected in the following lemma (see [3], Theorem 1.27).

Lemma 3.1. *For any periodic semigroup G, the following are equivalent:*

(i) *G is right (left) simple;*
(ii) *G is a right (left) group;*
(iii) *G is isomorphic to the direct product of a right (left) zero band and a group;*
(iv) *G is a union of several of its left (right) ideals and each of these ideals is a group.*

If G is a semigroup, then G^1 (or G^0) stands for G with identity (respectively, zero) adjoined.

Suppose that H is a group, I and Λ are nonempty sets, and $P = [p_{\lambda i}]$ is a $(\Lambda \times I)$-matrix with entries $p_{\lambda i} \in H$ for all $\lambda \in \Lambda$, $i \in I$. The *Rees matrix semigroup* $M(H; I, \Lambda; P)$ over H with *sandwich-matrix* P consists of all triples $(h; i, \lambda)$, where $i \in I$, $\lambda \in \Lambda$, and $h \in H$, with multiplication defined by the rule

$$(h_1; i_1, \lambda_1)(h_2; i_2, \lambda_2) = (h_1 p_{\lambda_1 i_2} h_2; i_1, \lambda_2).$$

Now suppose that $Q = [q_{\lambda i}]$ is a $(\Lambda \times I)$-matrix with entries $q_{\lambda i}$ in the group H^0 with zero adjoined. Then the *Rees matrix semigroup* $M^0(H; I, \Lambda; Q)$ over H^0 with *sandwich-matrix* Q consists of zero 0 and all triples $(h; i, \lambda)$, for $i \in I$, $\lambda \in \Lambda$, and $h \in H^0$, where all triples $(0, i, \lambda)$ are identified with 0, and multiplication is defined by the rule

$$(h_1; i_1, \lambda_1)(h_2; i_2, \lambda_2) = (h_1 q_{\lambda_1 i_2} h_2; i_1, \lambda_2).$$

It is well known that every completely simple semigroup is isomorphic to a Rees matrix semigroup $M(H; I, \Lambda; P)$ over a group H (see [6], Theorem 3.3.1). A semigroup with zero is called *completely* 0-*simple* if it has no proper nonzero ideals and has a minimal nonzero idempotent. Every completely 0-simple semigroup is isomorphic to a Rees matrix semigroup $M^0(H; I, \Lambda; P)$ over a group H with zero adjoined. Conversely, every semigroup $M(H; I, \Lambda; P)$ is completely simple, and a semigroup $M^0(H; I, \Lambda; P)$ is completely 0-simple if and only if each row and column of P contains at least one nonzero entry (see [6], Theorem 3.2.3).

Let H be a group, $G = M(H; I, \Lambda; P)$, and let $i \in I$, $\lambda \in \Lambda$. Then we put

$$G_{*\lambda} = \{(h; i, \lambda) \mid h \in H, i \in I\},$$
$$G_{i*} = \{(h; i, \lambda) \mid h \in H, \lambda \in \Lambda\},$$
$$G_{i\lambda} = \{(h; i, \lambda) \mid h \in H\}.$$

In the case where $G = M^0(H; I, \Lambda; P)$ we include zero in all of these sets, i.e. put

$$G_{*\lambda} = \{0\} \cup \{(h; i, \lambda) \mid h \in H, i \in I\},$$
$$G_{i*} = \{0\} \cup \{(h; i, \lambda) \mid h \in H, \lambda \in \Lambda\},$$
$$G_{i\lambda} = \{0\} \cup \{(h; i, \lambda) \mid h \in H\}.$$

We need a few basic facts which follow immediately. For convenience we collect them in a separate lemma.

Lemma 3.2. *Let H be a group, and let $G = M(H; I, \Lambda; P)$ be a completely simple semigroup or let $G = M^0(H; I, \Lambda; P)$ be a completely 0-simple semigroup. Then, for all $i, j \in I$, $\lambda, \mu \in \Lambda$, and $s = (h; i, \lambda) \in G$,*

(i) *the set $G_{*\lambda}$ is a minimal nonzero left ideal of G;*
(ii) *the set G_{i*} is a minimal nonzero right ideal of G;*
(iii) *$Gs = G_{*\mu}s = G_{*\lambda}$;*
(iv) *$sG = sG_{j*} = G_{i*}$;*
(v) *$s \in Gs \cap sG = G_{i\lambda}$;*
(vi) *the set $G_{i\lambda}$ is a left ideal of G_{i*} and a right ideal of $G_{*\lambda}$;*
(vii) *if $p_{\lambda i} = 0$, then $G_{i\lambda}^2 = 0$;*
(vii) *if $p_{\lambda i} \neq 0$, then $G_{i\lambda}$ is a maximal subgroup of G isomorphic to H;*
(vii) *each maximal subgroup of G coincides with $G_{j\mu}$, for some $j \in I$, $\mu \in \Lambda$;*
(ix) *$M(H; I, \Lambda; P)$ is a right (left) group if and only if $|I| = 1$ (respectively, $|\Lambda| = 1$);*
(x) *if $G = M(H; I, \Lambda; P)$, then each $G_{*\lambda}$ is a left group, and each G_{i*} is a right group.*

Let I and J be ideals of a semigroup G such that $J \subseteq I$. The Rees quotient semigroup I/J is the semigroup with zero obtained from I by identifying with 0 all elements of the ideal J. If I has zero and $J = \{0\}$, then $I/J = I$. The Rees quotient semigroup I/J is called a *factor* of G. In the case where $J = \emptyset$, we assume that $I/J = I$. Take any element g in G, put $I = G^1gG^1$ and denote by J the set of all elements which generate principal ideals properly contained in I. Then J is also an ideal of G, and I/J is called a *principal factor* of G.

Recall that a *null* semigroup or a semigroup with *zero multiplication* is a semigroup G with zero such that $G^2 = 0$. Each principal factor of a semigroup G is either simple, or 0-simple, or a semigroup with zero multiplication ([6], Proposition 3.1.5). In addition, all periodic simple or 0-simple semigroups are completely simple or completely 0-simple, respectively ([6], Theorem 3.2.11). Obviously, every factor of a periodic semigroup is periodic too. For convenience of further reference we combine these facts in the following lemma.

Lemma 3.3. *Every principal factor of a periodic semigroup is completely simple or completely 0-simple, or a null semigroup.*

Lemma 3.4. *Let G be a periodic completely simple or completely 0-simple semigroup, and let L be a subsemigroup of G. If L does not contain 0, then L is completely simple.*

Proof. We consider only completely 0-simple semigroups G, because the case of a completely simple semigroup is similar and even easier. By the comment preceding Lemma 3.2 we may assume that $G = M^0(H; I, \Lambda; P)$, where H is a group and P is a $\Lambda \times I$-matrix over H^0. Take any element x in L, and suppose that $x \in G_{i\lambda}$ for some $i \in I, \lambda \in \Lambda$. Since $0 \notin L$, we see that $x^2 \neq 0$, and so it follows from Lemma 3.2 that $p_{\lambda i} \neq 0$. The same Lemma 3.2 tells us that $G_{i\lambda}$ is a group. Thus each element of L lies in a subgroup of G. Since G is periodic, we see that every element of L generates a subgroup of L. Therefore L is a union of groups. For any $x = (h; i, \lambda), y = (g; j, \mu) \in L$, we get $0 \neq xyx \in L \cap G_{i\lambda}$. Hence $L \cap G_{i\lambda}$ is a subgroup of L. Clearly, y belongs to the ideal

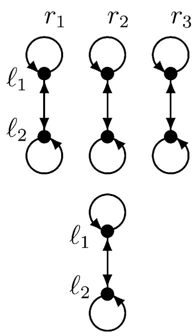

Fig. 1. $\mathrm{Cay}(G, \{(\ell_1, r_1), (\ell_2, r_2)\})$.

generated by x in L. This means that L is simple. Clifford's theorem shows that every simple semigroup, which is a union of groups, is completely simple (see [6], Theorem 4.1.3). Therefore L is completely simple. □

If G is a semigroup, $S \subseteq G$, and $g \in G$, then Sg is called a *right coset* of S.

4. Examples

Theorem 2.2 does not generalize to a semigroup G with a subset S which generates a subsemigroup $\langle S\rangle$ with infinite principal left ideals, as the following example shows.

Example 1. Let $G = \{\dots, e_{-1}, e_0, e_1, e_2, \dots\}$ be a semigroup with multiplication defined by $e_i e_j = e_{\max\{i,j\}}$, and let $S = G$. Then the Cayley graph $\mathrm{Cay}(G, S)$ has edges (e_i, e_j) for all $i \leq j$, and therefore it is $\mathrm{Aut}_S(G)$-vertex-transitive. However, $\langle S\rangle = S = G$ is a semilattice, and so it is not completely simple. In this case all principal left ideals of $\langle S\rangle$ are infinite.

The next example of a vertex-transitive Cayley graph shows that condition (iv) in Theorem 2.2 cannot be replaced by

$$|\langle S\rangle| = |\langle S\rangle g| \qquad \text{for all } g \in G.$$

Example 2. Let $L = \{\ell_1, \ell_2\}$ be a left zero band, $R = \{r_1, r_2, r_3\}$ a right zero band, and let G be the union of $L \times R$ and L with multiplication defined by

$$(\ell, r)\ell' = \ell'(\ell, r) = \ell,$$

for any $\ell, \ell' \in L$ and $r \in R$. Let $S = \{(\ell_1, r_1), (\ell_2, r_2)\}$. The Cayley graph $\mathrm{Cay}(G, S)$ is shown in Fig. 1. It is vertex-transitive. We see that $|\langle S\rangle| = 4$, whereas $|\langle S\rangle\ell_1| = 2$.

Condition (iv) of Theorem 2.2 does not transfer to $\mathrm{End}_S(G)$-vertex-transitive graphs, as shown by the following example.

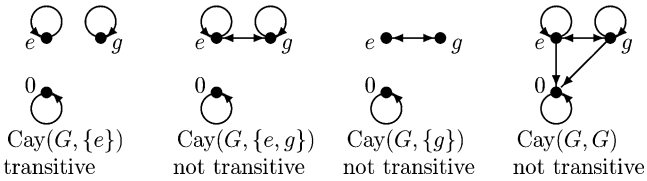

Fig. 2. $G = \{0\} \cup \{e, g\}$.

Example 3. Let $G = \{0, e, g\}$ be the the group $\{e, g\}$ with zero 0 adjoined. The only $\mathrm{End}_S(G)$-vertex-transitive Cayley graph of G is $\mathrm{Cay}(G, \{e\})$ (see Fig. 2). In this graph $|\langle S\rangle g| = 2$ and $|\langle S\rangle 0| = 1$.

The last two examples show that it is difficult to reduce further the question of $\mathrm{Aut}_S(G)$-vertex-transitivity to the right subgroups $\langle S\rangle_{*\lambda}$ of the completely simple semigroup $\langle S\rangle$ in Theorem 2.2.

Example 4. Let $G_i = \{e_i, g_i, g_i^2, g_i^3\}$ be a cyclic group of order 4, for $i \in \{1, 2\}$, and let $G = G_1 \cup G_2$ be the union of these two groups with multiplication defined by

$$g_i^a g_j^b = g_i^{a+b},$$

for all integers a, b and all $i, j \in \{1, 2\}$. Thus G_1 and G_2 are left ideals of G, and G is a completely simple semigroup. This semigroup G is isomorphic to the Rees matrix semigroup $M(Z_4; I, \Lambda; P)$, where $|I| = 2$, $|\Lambda| = 1$, Z_4 is a cyclic group of order 4, and $P = [e \quad e]$. Equivalently, it is isomorphic to the direct product of the cyclic group of order 4 and the left zero band of order two. Let $S = \{g_1^2, g_2\}$. Then $\langle S\rangle = G$, and we see that the Cayley graph $\mathrm{Cay}(G, S)$ is not vertex-transitive, because e_1 belongs to a cycle of length 2, and e_2 does not (see Fig. 3). However, $\mathrm{Cay}(G_1, e_1Se_1)$ and $\mathrm{Cay}(G_1, e_2Se_2)$ are isomorphic.

Example 5. Let $Z_2 = \{0, 1\}$ be the group of order two. For simplicity, we denote the elements of $Z_2 \times Z_2$ by pairs $\{00, 01, 10, 11\}$. Let G be the Rees matrix semigroup $M(Z_2 \times Z_2; I, \Lambda; P)$, where $I = \{1, 2\}$, $\Lambda = \{1, 2\}$, and $P = \begin{bmatrix} 11 & 00 \\ 00 & 00 \end{bmatrix}$. Let $S = \{(1; 1, 2), (1; 2, 1)\}$. Then $\langle S\rangle = G$, and the left groups $G_{*\lambda}$, where $\lambda \in \{1, 2\}$, are the connected components of $\mathrm{Cay}(G, S)$ and the left ideals of G. The left group G_{*2} induces a subgraph C which is a connected component of $\mathrm{Cay}(G, S)$. However, C is not equal to any Cayley graph of the right group G_{*2}. The whole Cayley graph $\mathrm{Cay}(G, S)$ is vertex-transitive. However, its subgraph induced by the left group G_{*2} is not equal to any Cayley graph of this right group (see Fig. 4).

5. General properties of Cayley graphs

Let $D = (V, E)$ be a graph. The in-degree (out-degree) of a vertex $v \in V$ is the number of vertices $u \in V$ such that $(u, v) \in E$ (respectively, $(v, u) \in E$). If D is finite and vertex-transitive, then the in-degree is the same for each vertex, and is equal to its out-degree; in

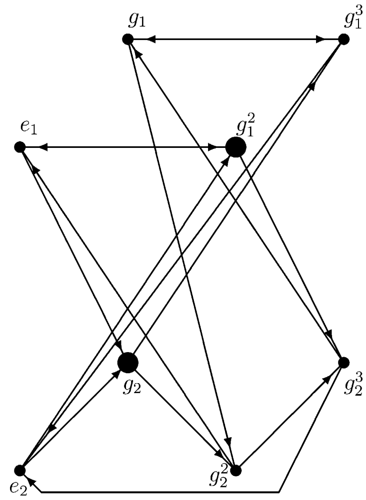

Fig. 3. Cay$(G, \{g_1^2, g_2\})$.

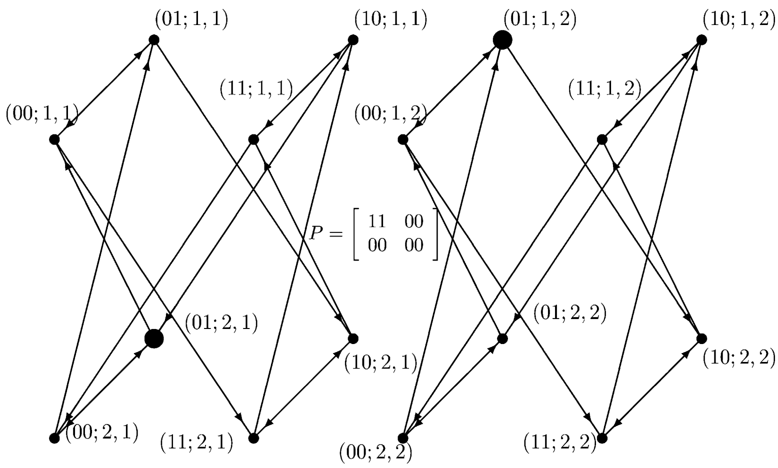

Fig. 4. Cay$(G, \{(01; 1, 2), (01; 2, 1)\})$.

this case the common value is called the *degree* or *valency* of D. The *underlying undirected graph* of D has the same set of vertices V and it has an undirected edge $\{u, v\}$ for each directed edge (u, v) of D. The graph D is said to be *connected* if its underlying undirected graph is connected. If, for each pair of vertices u, v of D, there exists a directed path from u to v, then D is said to be *strongly connected*.

Lemma 5.1. *Let G be a semigroup with a subset S, let $g \in G$ and let C_g be the set of all vertices v of the Cayley graph* Cay(G, S) *such that there exists a directed path from g to v. Then C_g is equal to the right coset $\langle S \rangle g$.*

Proof. Take any $v \in C_g$. There exists a (directed) path $g = v_1, v_2, \ldots, v_n = v$, where $n > 1$. By the definition of $\mathrm{Cay}(G, S)$ we get $v_{i+1} = s_i v_i$, for $i = 1, \ldots, n-1$ and some $s_i \in S$. Hence $v = s_{n-1} \cdots s_2 s_1 g$, and so $v \in \langle S \rangle g$.

Conversely, pick any $v \in \langle S \rangle g$. Since $\langle S \rangle$ is generated by S, there exist $s_1, \ldots, s_{n-1} \in S$ such that $v = s_{n-1} \cdots s_2 s_1 g$, where $n > 1$. Setting $v_1 = g$ and $v_{i+1} = s_i v_i$ for $i = 1, \ldots, n-1$, we see that $v_n = v$ and

$$(g, s_1 g), (s_1 g, s_2 s_1 g), \ldots, (s_{n-2} \cdots s_1 g, v)$$

are the edges of $\mathrm{Cay}(G, S)$. Therefore $g = v_1, v_2, \ldots, v_n = v$ is a directed path from g to v in $\mathrm{Cay}(G, S)$. Thus $C_g = \langle S \rangle g$. □

Lemma 5.2. *Let G be a semigroup with a subset S such that $\langle S \rangle = M(H; I, \Lambda; P)$ is completely simple, $SG = G$, and let $T = \langle S \rangle_{*\lambda}$ for some $\lambda \in \Lambda$. Then every connected component of the Cayley graph $\mathrm{Cay}(G, S)$ is strongly connected and, for each $v \in G$, the component containing v is equal to $Tv = \langle S \rangle v$.*

Proof. Consider any edge (u, v) of $\mathrm{Cay}(G, S)$. There exists $r \in S$ such that $v = ru$. Since $SG = G$, we can find elements $s' \in S, u' \in G$ such that $u = s'u'$. Hence $v = rs'u'$. Now $s' \in B = \langle S \rangle_{*\mu}$ for some $\mu \in \Lambda$, and by Lemma 3.2, B is a left ideal of $\langle S \rangle$ and is left simple. Therefore $rs' \in B$ and $B = Brs'$. Hence $s' \in Brs'$, and so $s' = r'rs'$ for some $r' \in B$. Since $v = ru = rs'u'$, we get $r'v = r'rs'u' = s'u' = u$. Thus $u \in \langle S \rangle v$. By Lemma 5.1 there exists a directed path from v to u. Since (u, v) was chosen arbitrarily, it follows that every connected component of $\mathrm{Cay}(G, S)$ is strongly connected.

Let C be the set of vertices of a connected component of $\mathrm{Cay}(G, S)$, and let $v \in C$. Given that $SG = G$, there exist $s \in S, g \in G$ such that $sg = v$. Let $\nu \in \Lambda$ be such that $s \in \langle S \rangle_{*\nu}$. Again Lemma 3.2 tells us that $\langle S \rangle_{*\nu}$ is left simple, and so $s \in \langle S \rangle_{*\nu} s \subseteq \langle S \rangle s$; whence $v = sg \in \langle S \rangle v$. Since C is strongly connected, it follows from Lemma 5.1 that $C = \langle S \rangle v$. By Lemma 3.2, $\langle S \rangle s = Ts$. Therefore $\langle S \rangle v = Tv$, which completes the proof. □

Corollary 5.3. *Let G be a semigroup with subset S such that $SG = G$, let $\langle S \rangle$ be isomorphic to a direct product of a right zero band Z and a group K, and let H be a maximal subgroup of $\langle S \rangle$. Then*

(i) *each right coset Hg coincides with the right coset $\langle S \rangle g$;*

(ii) *the right $\langle S \rangle$-cosets are the connected components of* $\mathrm{Cay}(G, S)$.

Proof. Denote by e the identity of K and let P be the $(|Z| \times 1)$-matrix with all entries equal to e. Then it is easily seen that $\langle S \rangle = M(K; \{1\}, Z; P)$. It follows from Lemma 3.2 that $H = \langle S \rangle_{*z_0}$ and $H = \{(k, z_0) \mid k \in K\}$ for some $z_0 \in Z$. Hence the assertion follows from Lemma 5.2. □

Lemma 5.4. *Let G be a semigroup with a periodic element s such that $sG = G$, and let I be a subset of G such that $sI \subseteq I$. Then $x \notin I$ implies $sx \notin I$.*

Proof. Since S is periodic, $s^{m+n} = s^m$, for some positive integers m, n. Take any element $x \notin I$ and suppose in contrast that $sx \in I$. Since $sG = G$, for each positive integer i there exists $x_i \in I$ such that $x = s^i x_i$. Every x_i is in $G \backslash I$, because $sI \subseteq I$. If $n = 1$, then we have $x = s^m x_m = s^{m+1} x_m = s(s^m x_m) = sx \in I$. If $n > 1$, we get $x = s^{m+n-1} x_{m+n-1} = s^{n-1}(s^m x_{m+n-1}) = s^{n-1}(s^{m+n} x_{m+n-1}) = s^{n-1}(sx) \in I$. These contradictions prove that $sx \notin I$. □

Lemma 5.5. *Let G be a semigroup with a periodic subset S such that $sG = G$ for all $s \in S$, and let I be a subset of G such that $SI \subseteq I$. Then I is a union of connected components of* Cay(G, S). *In particular, the subsemigroup $\langle S \rangle$ generated by S in G is a union of connected components of* Cay(G, S).

Proof. Let $x \in I$. Take any edge (x, y) in Cay(G, S). Then $y = sx$, for some $s \in S$. Since $sI \subseteq I$, we get $y \in I$ by Lemma 5.4. Similarly, suppose that (y, x) is an edge of Cay(G, S). Then $x = sy$ for some $s \in S$. If y was not in I, then by Lemma 5.4, $x = sy$ would not be in I either, which would be a contradiction. Hence $y \in I$. Thus all vertices of I are adjacent to vertices of I only. Hence I is a union of connected components of Cay(G, S). The final assertion follows, because $s\langle S \rangle \subseteq \langle S \rangle$, for each $s \in S$. □

6. Transitivity properties of Cayley graphs

In this section we prove several preparatory lemmas for the proof of the main theorems.

Lemma 6.1. *Let G be a semigroup, and let S be a subset of G.*

(i) *If* $\mathrm{End}_S(G)$ *is vertex-transitive on* Cay(G, S), *then $SG = G$.*
(ii) *If* $\mathrm{ColEnd}_S(G)$ *is vertex-transitive on* Cay(G, S), *then $sG = G$ for each $s \in S$.*

Proof. Pick any $g \in G$ and $s \in S$. There exists ϕ in $\mathrm{End}_S(G)$ such that $(sg)^\phi = g$. Since (g, sg) is an edge, also $(g^\phi, (sg)^\phi)$ is an edge. Hence $(sg)^\phi = s'(g)^\phi$ for some $s' \in S$. Thus $g = s'(g)^\phi \in SG$, and so $SG = G$, i.e. (i) holds.

Furthermore, if $\mathrm{ColEnd}_S(G)$ is vertex-transitive on Cay(G, S), then we may assume that $\phi \in \mathrm{ColEnd}_S(G)$, whence $s' = s$, and so $sG = G$, i.e. (ii) holds. □

Lemma 6.2. *Let G be a semigroup with a periodic subset S such that* $\mathrm{ColEnd}_S(G)$ *is vertex-transitive on* Cay(G, S). *Then $\langle S \rangle$ is right simple.*

Proof. For each $s \in S$, $sG = G$ by Lemma 6.1. Take any $g \in \langle S \rangle$ and $s \in S$. Since $sG = G$, there exists $g' \in G$ such that $g = sg'$. By transitivity, $g' = (g)^\phi$ for some $\phi \in \mathrm{ColEnd}_S(G)$. Hence $g = s(g)^\phi$, and it follows from Lemma 5.4 (with $I = \langle S \rangle$) that $(g)^\phi$ lies in $\langle S \rangle$. This means that $g \in s\langle S \rangle$. Therefore $s\langle S \rangle = \langle S \rangle$ for all $s \in S$. By induction we deduce that $t\langle S \rangle = \langle S \rangle$ for all $t \in \langle S \rangle$. Thus $\langle S \rangle$ has no proper right ideals, i.e. it is right simple. □

Lemma 6.3. *Let G be a semigroup, and let S be a subset of G such that all principal left ideals of the subsemigroup $\langle S \rangle$ are finite, and suppose that the Cayley graph* Cay(G, S) *is*

$\mathrm{Aut}_S(G)$*-vertex-transitive. Then* $\langle S\rangle$ *is a completely simple semigroup, and each element of* S *generates* G *as an ideal.*

Proof. Let $T = \langle S\rangle$. Clearly, the condition that all the principal left ideals of T are finite implies that T is periodic. Take any element $g \in G$. By Lemma 5.1, Tg is equal to the set C_g of all vertices v of $\mathrm{Cay}(G, S)$ such that there exists a directed path from g to v. By Lemma 6.1, $SG = G$, and so there exists $s \in S$, $h \in G$, such that $sh = g$. Since the principal left ideal Ts is finite, we see that the set $C_g = (Ts)h$ is finite. Since $\mathrm{Cay}(G, S)$ is $\mathrm{Aut}_S(G)$-vertex-transitive, we get $|C_g| = |C_v|$ for all $v \in G$. If $v \in C_g$, then evidently $C_v \subseteq C_g$. It follows that $C_v = C_g$, so $v \in C_v$, so also $g \in C_g$. Therefore C_g is strongly connected. Further, if C_g and C_v have a common vertex for some $v \in G$, then it is clear that the union $C_g \cup C_v$ is also strongly connected. Thus C_g is a connected component of $\mathrm{Cay}(G, S)$.

Pick any $s, t \in T$. We have proved that $s \in C_s = Ts$. Since ts belongs to the connected component $Ts = C_s = C_{ts} = Tts$, we get $s \in Tts$. Hence $s \in T^1tT^1$, and so $T = T^1tT^1$ (where, recall, T^1 denotes T with identity adjoined). Therefore T is simple, and it follows from Lemma 3.3 that T is completely simple, because it is periodic.

For any $s \in S$, we get $T = T^1sT^1$. Let $g \in G$. Since $SG = G$, we have $g = tg'$ for some $t \in S$, $g' \in G$. Then since $T = T^1sT^1$, we have $t = s_1ss_2$, where $s_1, s_2 \in T$, so $g = s_1ss_2g' \in G^1sG^1$. Hence $G = G^1sG^1$. □

Corollary 6.4. *Let* G *be a semigroup, and let* S *be a periodic subset of* G*. If the Cayley graph* $\mathrm{Cay}(G, S)$ *is* $\mathrm{ColEnd}_S(G)$*-vertex-transitive, then the following conditions hold:*

(i) $sG = G$ *for all* $s \in S$,
(ii) $\langle S\rangle$ *is a direct product of a right zero band and a group,*
(iii) *the right cosets* $\langle S\rangle g$ *are precisely the connected components of* $\mathrm{Cay}(G, S)$.

Proof. By Lemma 6.1, (i) holds. By Lemma 6.2, $\langle S\rangle$ is right simple, and Lemma 3.1 yields (ii). Then by Corollary 5.3 the connected components of $\mathrm{Cay}(G, S)$ are the right cosets $\langle S\rangle g$, for all $g \in G$. □

7. Proofs of the main theorems

Proof of Theorem 2.1. The ‘only if’ part. Suppose that $\mathrm{ColAut}_S(G)$ is vertex-transitive. Since $\mathrm{ColAut}(G, S) \subseteq \mathrm{ColEnd}(G, S)$, Corollary 6.4 applies, whence (i) and (ii) follow. Clearly, the connected components of $\mathrm{Cay}(G, S)$ have the same cardinality because of vertex-transitivity, and so (iii) follows from Corollary 6.4(iii).

The ‘if’ part. Suppose that conditions (i), (ii) and (iii) of Theorem 2.1 hold, and in particular that $\langle S\rangle$ is isomorphic to the direct product $K \times Z$ of a group K and a right zero band Z. Note that then $K \times Z$ is isomorphic to the Rees matrix semigroup $M(K; I, \Lambda; P)$, where $I = \{i\}$, $|\Lambda| = |Z|$, and all entries of the $(\lambda \times I)$-matrix P are equal to the identity of K. Therefore we can apply Lemma 3.2 taking into account that, for $i \in I$ and any $\lambda \in \Lambda$, $k \in K$,

$$\langle S\rangle_{i*} = \langle S\rangle \qquad \text{and} \qquad \langle S\rangle_{*\lambda} = \langle S\rangle_{i\lambda}.$$

Denote by H a maximal subgroup of $\langle S\rangle$. By Lemma 3.2, there exist $i \in I$ and $\mu \in \Lambda$ (and the corresponding element $z \in Z$) such that

$$H = \langle S\rangle_{i\mu} = \{(k, z) \mid k \in K\}. \tag{7.1}$$

Each principal left ideal of $\langle S\rangle$ is equal to the minimal ideal $\langle S\rangle_{*\lambda} = \langle S\rangle_{i\lambda}$, for some $\lambda \in \Lambda$. Hence all these principal left ideals are isomorphic to K by Lemma 3.2. By assumption, therefore, K and H are finite.

By Corollary 5.3, each coset $\langle S\rangle g$ coincides with Hg, and the connected components of $\mathrm{Cay}(G, S)$ are the right H-cosets. Note that H is one of these cosets, since it is a group. Therefore by (iii) all these cosets have the same cardinality as H. In particular, for $h, h' \in H$ and $g \in G$, $hg = h'g$ implies $h = h'$.

Choose elements $g, g' \in G$. We shall define a mapping $\varphi : G \to G$ such that $g^{\varphi} = g'$, and show that φ is a colour preserving automorphism of $\mathrm{Cay}(G, S)$. It follows from Lemma 5.2 that g belongs to the connected component Hg. Therefore there exists $h \in H$ such that $hg = g$. Similarly, $h'g' = g'$ for some $h' \in H$. Denote by e the identity of H. Then $hg = (eh)g = e(hg) = eg$, and by our observation in the previous paragraph, $h = e$. Similarly, $h' = e$. Consider two cases.

Case 1. $Hg \neq Hg'$. Then, for $x \in G$, we define

$$x^{\varphi} = \begin{cases} tg' & \text{if } x = tg \text{ for some } t \in H; \\ tg & \text{if } x = tg' \text{ for some } t \in H; \\ x & \text{if } x \notin Hg \cup Hg'. \end{cases}$$

Then φ is well defined and is a bijection, because $|Hg| = |Hg'| = |H|$. Clearly, $g^{\varphi} = g'$.

Take any $x \in G$, $s \in S$. If $x = tg \in Hg$, then $sx \in Hg$, since H is a left ideal of $\langle S\rangle$ by (1). Therefore the edge (x, sx) in mapped by φ to the edge $(tg', stg') = (x^{\varphi}, s(x^{\varphi}))$. An analogous property holds if $x \in Hg'$. Also, φ leaves invariant all edges involving vertices of $G \backslash (Hg \cup Hg')$. Thus $(sx)^{\phi} = s(x^{\phi})$, i.e. φ is a colour-preserving automorphism of $\mathrm{Cay}(G, S)$.

Case 2. $Hg = Hg'$. For $x \in G$, we define

$$x^{\varphi} = \begin{cases} hg' & \text{if } x = hg \in Hg; \\ x & \text{if } x \notin Hg. \end{cases}$$

Since $|H|$ is finite and $|Hg'| = |Hg| = |H|$, it follows that φ is a bijection (in particular, it is onto). Take any $s \in S$, $x \in G$. If $x = hg \in Hg$, then $sh \in H$ and $sx \in Hg$, because H is a left ideal of $\langle S\rangle$ by (1). Therefore $(sx)^{\phi} = shg'$, and so $(sx)^{\phi} = s(x^{\phi})$. On the other hand, if $x \notin Hg$, then $s(Hg) = (sH)g \subseteq Hg$, since H is a left ideal of $\langle S\rangle$, and it follows from Lemma 5.4 that $sx \notin Hg$. Therefore $(sx)^{\phi} = sx = s(x^{\phi})$. This means that $\varphi \in \mathrm{ColAut}_S(G)$.

Thus we have verified that $\mathrm{Cay}(G, S)$ is $\mathrm{ColAut}_S(G)$-vertex-transitive. □

If $H \subseteq G$ and $SH \subseteq H$, then by $\mathrm{Cay}(H, S)$ we denote the subgraph of $\mathrm{Cay}(G, S)$ *induced by* H, i.e. the graph with vertex set H and edges (h, sh) for all $h \in H$, $s \in S$. Note that if $S \subseteq H$, then $\mathrm{Cay}(H, S)$ is a Cayley graph of H.

Proof of Theorem 2.2. The 'only if' part. Suppose that $\mathrm{Cay}(G, S)$ is $\mathrm{Aut}_S(G)$-vertex-transitive. Then $SG = G$ by Lemma 6.1, and $\langle S\rangle = M(H; I, \Lambda; P)$ is a completely simple semigroup by Lemma 6.3. Thus conditions (i) and (ii) hold.

Take any $\lambda \in \Lambda$ and put $T = \langle S\rangle_{*\lambda}$. By Lemma 5.2, for each g in G, the connected component of $\mathrm{Cay}(G, S)$ containing G is equal to $Tg = \langle S\rangle g$.

Now T is a left ideal of $\langle S\rangle$ by Lemma 3.2, and so, for $s \in S$, we see that $STs \subseteq Ts$, and the subgraph induced on Ts is $\mathrm{Cay}(Ts, S)$. Since $\mathrm{Cay}(Ts, S)$ is a connected component of $\mathrm{Cay}(G, S)$, its automorphism group is vertex-transitive. The Cayley graph $\mathrm{Cay}(\langle S\rangle, S)$ is a union of the connected components $\mathrm{Cay}(Ts, S)$ of $\mathrm{Cay}(G, S)$, for $s \in S$, and so it is also $\mathrm{Aut}_S(\langle S\rangle)$-vertex-transitive. Thus (iii) holds.

Since all connected components of an $\mathrm{Aut}_S(G)$-vertex-transitive graph are isomorphic, we see that $|Tg| = |\langle S\rangle g|$ is independent of g, i.e. condition (iv) holds.

The 'if' part. Condition (ii) says that $\langle S\rangle$ is completely simple, and so by the remark preceding Lemma 3.2 it is isomorphic to a Rees matrix semigroup $M = M(H; I, \Lambda; P)$ over a group H. Take any $\lambda \in \Lambda$ and put $T = M_{*\lambda}$. Lemma 5.2 tells us that each connected component of $\mathrm{Cay}(G, S)$ is equal to Tg, for some $g \in G$, and that $Tg = \langle S\rangle g$ and contains g. Note that $t_1 g = t_2 g$ implies $t_1 = t_2$ by condition (iv) and since T is finite.

Let Ts and Th be distinct connected components of $\mathrm{Cay}(G, S)$. For $x \in G$, define

$$x^{\varphi} = \begin{cases} th & \text{if } x = tg \text{ for some } t \in T \\ tg & \text{if } x = th \text{ for some } t \in T \\ x & \text{if } x \notin Tg \cup Th. \end{cases}$$

Then φ is well-defined, and is a bijection by Condition (iv). As in the proof of 'Case 1' in the proof of Theorem 2.1, we see that $\varphi \in \mathrm{Aut}_S(G)$.

Since, for $s \in S$, the graph $\mathrm{Cay}(Ts, S)$ is a connected component of $\mathrm{Cay}(\langle S\rangle, S)$, we see that the automorphism group of $\mathrm{Cay}(Ts, S)$ is vertex-transitive by condition (iii). It follows that $\mathrm{Cay}(G, S)$ is $\mathrm{Aut}_S(G)$-vertex-transitive. This completes the proof. □

8. Final comments

We prove here a corollary to Theorem 2.2 where conditions (i) to (iv) of Theorem 2.2 collapse to a single simple condition for a certain class of finite simple semigroups.

Corollary 8.1. *Let G be a finite rectangular band, and let S be a subset of G. Then the Cayley graph $\mathrm{Cay}(G, S)$ is $\mathrm{Aut}_S(G)$-vertex-transitive if and only if $S \cap gG \neq \emptyset$ for all $g \in G$.*

Proof. It is well known that every rectangular band G is isomorphic to a direct product $G = L \times R$ of a left zero band L and a right zero band R.

The 'if' part. Suppose that $S \cap gG \neq \emptyset$ for all $g \in G$. Then, for each $\ell \in L$, there exists $r_\ell \in R$ such that $(\ell, r_\ell) \in S$. It follows that $SG = G$, and that S generates the rectangular band $L \times \{r_\ell \mid \ell \in L\}$. Thus conditions (i) and (ii) of Theorem 2.2 hold (note that all rectangular bands are completely simple).

Since $\langle S\rangle s$ is a left zero band, $\mathrm{Cay}(\langle S\rangle s, S)$ is a complete graph, and so (iii) holds. If $g = (\ell, r) \in G$, then $\langle S\rangle g = L \times \{r\}$, whence $|\langle S\rangle g| = |L|$, and so (iv) is satisfied. It follows that $\mathrm{Cay}(G, S)$ is $\mathrm{Aut}_S(G)$-vertex-transitive.

The 'only if' part. Suppose that $\mathrm{Cay}(G, S)$ is $\mathrm{Aut}_S(G)$-vertex-transitive. Since $G = R \times L$, it follows that for every $g \in G$

$$G \backslash gG = (G \backslash gG)G$$

is a right ideal of G. Hence condition (i) of Theorem 2.2 implies that $S \cap gG \neq gG = \emptyset$ for all $g \in G$, which completes the proof. □

Remark 8.2. In fact we can remove the word 'finite' from the hypothesis of Corollary 8.1, and the assertion remains valid, but it no longer follows from Theorem 2.2.

Given a family of graphs $D_i = (V_i, E_i)$, where $i \in I$, their *union* is the graph $D = \cup_{i\in I} D_i$ defined by

$$D = (\cup_{i\in I} V_i, \cup_{i\in I} E_i).$$

Note that in this definition we do not assume that the V_i are pairwise disjoint. In fact in the next lemma they will be all equal.

Lemma 8.3. *Let G be a semigroup, and let S be a subset of G. Then*

$$\mathrm{Cay}(G, S) = \cup_{s\in S} \mathrm{Cay}(G, \{s\}).$$

If $\mathrm{Cay}(G, S)$ is $\mathrm{ColAut}_S(G)$-vertex-transitive then, for each $s \in S$, the Cayley graph $\mathrm{Cay}(G, \{s\})$ is $\mathrm{ColAut}_{\{s\}}(G)$-vertex-transitive, too.

Proof. It is easily verified that $\mathrm{ColAut}_S(G) = \cap_{s\in S}\mathrm{ColAut}_{\{s\}}(G)$, and hence these assertions follow. □

Question 1. Is it true that if G is a semigroup with a subset S such that $\mathrm{Cay}(G, \{s\})$ is $\mathrm{Aut}_{\{s\}}(G)$-vertex-transitive, for every $s \in S$, then the whole Cayley graph $\mathrm{Cay}(G, S)$ is $\mathrm{ColAut}_S(G)$-vertex-transitive, too?

References

[1] L. Babai, Automorphism groups, isomorphism, reconstruction, in: Handbook of Combinatorics, Elsevier, Amsterdam, 1995, pp. 1447–1540.

[2] N. Biggs, Algebraic Graph Theory, Cambridge University Press, Cambridge, 1993.

[3] A.H. Clifford, G.B. Preston, The Algebraic Theory of Semigroups, vol. I, II, American Mathematical Society, Providence, RI, 1964.

[4] P.A. Grillet, Semigroups, Dekker, New York, 1995.

[5] M.-C. Heydemann, Cayley graphs and interconnection networks, in: Graph Symmetry: Algebraic Methods and Applications (Montreal, Canada, July 1–12, 1996), Kluwer, Dordrecht, 1997, pp. 167–224.

[6] J.M. Howie, Fundamentals of Semigroup Theory, Clarendon Press, Oxford, 1995.

[7] A.V. Kelarev, On undirected Cayley graphs, Aust. J. Combin. 25 (2002) 73–78.

[8] C.E. Praeger, Finite transitive permutation groups and finite vertex-transitive graphs, in: Graph Symmetry: Algebraic Methods and Applications (Montreal, Canada, July 1–12, 1996), Kluwer, Dordrecht, 1997, pp. 277–318.