Abstract

A priori bounds are determined for certain energy expressions for a class of semi-linear parabolic and hyperbolic initial-boundary value problems when a combination of the values of the solution initially and at a later time is prescribed.

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1. Introduction

There is a vast literature on non-well-posed problems in partial differential equations (e.g., see [1] and the references therein) and various methods have been used to stabilize the solution in such problems. One of these techniques is to perturb the differential equation as in [2,3], whereas Showalter [4] introduced the idea of perturbing the initial condition.

One of the classic non-well-posed problems is the final value problem for the forward heat equation or equivalently the initial value problem for the backward heat equation.
In fact, one may consider the initial-boundary value problem

\begin{align}
  u_t + \Delta u &= 0 & \text{in } \Omega \times (0, T), \\
  u &= f(x, t) & \text{on } \partial \Omega \times [0, T], \\
  u(x, 0) + z u(x, T) &= g(x) & \text{in } \Omega,
\end{align}

where \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^N \) with boundary \( \partial \Omega, x = (x_1, \ldots, x_N) \), \( \Delta \) is the \( N \)-dimensional Laplace operator, the subscript \( t \) denotes partial differentiation, and \( z \) is a constant parameter. This problem is not well-posed when \( z = 0 \) and the well-posedness of the solution of (1.1) when \( z \neq 0 \) was proved by Clark and Oppenheimer in [5].

The idea of specifying a combination of the values of the solution initially and at a later time \( T \) was used by Payne and Schaefer in some non-standard boundary value problems for hyperbolic equations in [6]. There, energy bounds were determined for ranges of values of the parameters as well as some uniqueness results, pointwise bounds, and non-existence results when the parameters were not in the range. Similar results were obtained for other non-standard problems in a variety of contexts in [7–9]. In each of these papers, the problems are not well-posed when the parameters are zero. Except for the viscous flow problem, the results obtained were for linear problems.

Here we shall consider a class of problems of the form

\begin{align}
  M u - Lu &= f(u) & \text{in } \Omega \times (0, T), \\
  B u &= 0 & \text{on } \partial \Omega \times [0, T], \\
  z u(x, 0) + u(x, T) &= g(x) & \text{in } \Omega,
\end{align}

where \( M \) is a first- or second-order time derivative operator, \( L \) is an elliptic operator, \( B \) is a Dirichlet, Neumann, or Robin boundary operator, and \( f \) is suitably conditioned. We determine \( L^2 \) bounds for the solution, assumed to exist, by means of differential inequalities in Section 2 when \( M \) is first order and consider the case \( M \) is second order (with an additional initial condition) in Section 3.

\section{2. \( L^2 \) bounds, parabolic case}

For definiteness and simplicity, we consider the problem

\begin{align}
  u_t - \Delta u &= f(u) & \text{in } \Omega \times (0, T), \\
  u &= 0 & \text{on } \partial \Omega \times [0, T], \\
  z u(x, 0) + u(x, T) &= g(x) & \text{in } \Omega,
\end{align}

where \( z \) is a non-zero constant and \( f \) is assumed to satisfy the condition

\[ s f(s) \leq \gamma s^2 + k \]

for constants \( \gamma > 0 \) and \( k \geq 0 \). The symbols \( \Delta, \Omega, \) and \( x \) are as in Section 1. We are interested in determining an a priori bound on the energy expression

\[ E(t) = \int_{\Omega} u^2(x, t) \, dx, \]

where \( u \) is a solution of (2.1).
We first multiply the differential equation in (2.1) by \( u \) and then integrate over space and time. This leads to

\[
\int_0^t \int_{\Omega} \frac{1}{2} (u^2)_\eta \, dx \, d\eta + \int_0^t \int_{\Omega} |\nabla u|^2 \, dx \, d\eta - \int_0^t \int_{\partial \Omega} \frac{\partial u}{\partial n} \, ds \, d\eta = \int_0^t \int_{\Omega} u f (u) \, dx \, d\eta,
\]

where \( \nabla \) is the gradient operator and \( \partial / \partial n \) is the outward normal derivative operator on the boundary. By the boundary condition in (2.1) and (2.2), we have for \( 0 \leq t \leq T \),

\[
E(t) \leq 2 \int_0^t \int_{\Omega} (\gamma u^2 + k) \, dx \, d\eta + E(0) \leq 2 \gamma \int_0^t E(\eta) \, d\eta + 2k|\Omega|T + E(0),
\]

(2.4)

where \( |\Omega| \) denotes the volume of \( \Omega \). Now let

\[
P(t) = \int_0^t E(\eta) \, d\eta.
\]

Then (2.4) can be written

\[
P'(t) \leq 2\gamma P(t) + 2k|\Omega|T + E(0)
\]

(2.5)

and solving (2.5), we find

\[
P(t) \leq \frac{1}{2\gamma} [2k|\Omega|T + E(0)](e^{2\gamma t} - 1).
\]

On substituting for \( P'(t) \) and \( 2\gamma P(t) \) in (2.5), we then have

\[
E(t) \leq [2k|\Omega|T + E(0)]e^{2\gamma t}
\]

(2.6)

for \( 0 \leq t \leq T \).

To obtain an a priori bound, we need to bound \( E(0) \). For this, we evaluate (2.6) at \( t = T \) and use (2.1), i.e.,

\[
\int_{\Omega} [g(x) - xu(x, 0)]^2 \, dx \leq [2k|\Omega|T + \int_{\Omega} u^2(x, 0) \, dx] e^{2\gamma T}.
\]

We now suppress the \( x \) dependence and collect terms so that

\[
\left( \alpha^2 - e^{2\gamma T} \right) E(0) \leq 2\alpha \int_{\Omega} g u(0) \, dx + 2k|\Omega|T e^{2\gamma T} - \int_{\Omega} g^2 \, dx.
\]

Using a weighted arithmetic–geometric mean inequality with \( \varepsilon > 0 \), we have

\[
\left( \alpha^2 - |\alpha|\varepsilon - e^{2\gamma T} \right) E(0) \leq 2k|\Omega|T e^{2\gamma T} + \left( \frac{|\alpha|}{\varepsilon} - 1 \right) \int_{\Omega} g^2 \, dx.
\]
It follows that if we choose
\[ 0 < \varepsilon < \frac{x^2 - 1}{|x|}, \quad \gamma < \frac{1}{2T} \ln[x^2 - |x|\varepsilon], \] (2.7)
then
\[ E(0) \leq C_1 e^{2\gamma T} + C_2 \int_{\Omega} g^2 \, dx \]
for computable constants \( C_1 \) and \( C_2 \) and, hence,
\[ E(t) \leq \left[ 2k|\Omega|T + C_1 e^{2\gamma T} + C_2 \int_{\Omega} g^2(x) \, dx \right] e^{2\gamma T} \] (2.8)
for \( 0 \leq t \leq T \). We summarize this result in the following statement.

**Theorem 1.** If \( u \) is a solution of (2.1) where \( |x| > 1 \) and \( f \) satisfies (2.2), then \( E(t) \) satisfies (2.8) for \( 0 \leq t \leq T \).

We first note that the result in Theorem 1 follows in the case of Neumann \( \partial u/\partial n = 0 \) or Robin \( \partial u/\partial n + \beta u = 0, \beta > 0 \) conditions on the boundary as a result of the Green identity
\[ \int_{\Omega} [u \Delta u + |\nabla u|^2] \, dx = \int_{\partial\Omega} u \frac{\partial u}{\partial n} \, ds. \]
Moreover, the result can be extended to the more general elliptic operator of the form
\[ Lu = \sum_{i,j=1}^{N} \partial \left[ \alpha_{ij}(x) \frac{\partial u}{\partial x_i} \right] / \partial x_j \]
under Dirichlet or generalized Neumann and Robin conditions by using a generalized version of the Green identity to arrive at the inequality (2.4).

In addition to the linear case \( f(u) = u \), where the bound (2.8) simplifies to
\[ E(t) \leq C_2 \int_{\Omega} g^2(x) \, dx \, e^{2T} \] (2.9)
for \( 0 \leq t \leq T \), the result in Theorem 1 follows in the cases \( f(u) = e^{-u} \) and \( f(u) = u \sin u \). If, in fact, one assumes that \( uf(u) \leq 0 \), then one obtains (2.9) without the exponential factor on the right-hand side. The restriction that \( |x| > 1 \) is clear from (2.7).

Finally, we remark that a uniqueness result may be deduced for the problem (2.1) when a solution exists if \( f \) satisfies (2.2) and has a bounded derivative.
3. $L^2$ bounds, hyperbolic case

We now consider as an example of a class of hyperbolic problems, the following initial-boundary value problem:

$$\begin{align*}
    u_{tt} - \Delta u &= f(u) \quad \text{in } \Omega \times (0, T), \\
    u &= 0 \quad \text{on } \partial \Omega \times [0, T], \\
    xu(x, 0) + u(x, T) &= g(x) \quad \text{in } \Omega, \\
    \beta u_t(x, 0) + u_t(x, T) &= h(x) \quad \text{in } \Omega,
\end{align*}$$

(3.1)

where $\alpha$ and $\beta$ are non-zero constants and $f$ is such that

$$|F(u(x, t_2)) - F(u(x, t_1))| \leq K, \quad F(u) = \int_0^u f(s) \, ds$$

(3.2)

for all $x \in \bar{\Omega}$ and $0 \leq t_1 \leq t_2 \leq T$. Here we define the energy expression

$$E(t) = \int_\Omega [(u_t)^2 + |\nabla u|^2] \, dx$$

(3.3)

and seek an explicit bound in terms of data.

We begin by multiplying the differential equation in (3.1) by $u_t$, integrating over space and time, and integrating by parts to obtain

$$\int_0^t \int_\Omega \left[ \frac{1}{2} (u_\eta)^2 + \frac{1}{2} (|\nabla u|^2)_\eta \right] \, dx \, d\eta = \int_0^t \int_\Omega f(u) u_\eta \, dx \, d\eta.$$

It follows by (3.2) that

$$E(t) \leq 2K |\Omega| + E(0)$$

(3.4)

for $0 \leq t \leq T$. Now evaluating (3.4) at $t = T$ and proceeding as in the previous section, we have on suppressing the $x$ dependence

$$\int_\Omega \left\{ [h - \beta u_t(0)]^2 + |\nabla g - \alpha \nabla u(0)|^2 \right\} \, dx \leq 2K |\Omega| + \int_\Omega \left\{ [u_t(0)]^2 + |\nabla u(0)|^2 \right\} \, dx.$$

Collecting terms and using the weighted arithmetic–geometric mean inequality with $\sigma_1, \sigma_2 > 0$, we obtain

$$\begin{align*}
    (\beta^2 - |\beta| \sigma_1 - 1) \int_\Omega [u_t(0)]^2 \, dx + (\alpha^2 - |\alpha| \sigma_2 - 1) \int_\Omega |\nabla u(0)|^2 \, dx \\
    \leq 2K |\Omega| + \left( \frac{|\beta|}{\sigma_1} - 1 \right) \int_\Omega h^2 \, dx + \left( \frac{|\alpha|}{\sigma_2} - 1 \right) \int_\Omega |\nabla g|^2 \, dx.
\end{align*}$$

If we choose

$$0 < \sigma_1 < \frac{\beta^2 - 1}{|\beta|}, \quad 0 < \sigma_2 < \frac{\alpha^2 - 1}{|\alpha|},$$

then

$$E(t) \leq 2K |\Omega| + \int_0^t \left( \frac{|\beta|}{\sigma_1} - 1 \right) \int_\Omega h^2 \, dx + \left( \frac{|\alpha|}{\sigma_2} - 1 \right) \int_\Omega |\nabla g|^2 \, dx.$$
then it follows that there are computable constants $K_1$, $K_2$ and $K_3$ such that

$$E(0) \leq K_1 + K_2 \int_{\Omega} h^2 \, dx + K_3 \int_{\Omega} |\nabla g|^2 \, dx$$

and, hence,

$$E(t) \leq 2K |\Omega| + K_1 + K_2 \int_{\Omega} h^2(x) \, dx + K_3 \int_{\Omega} |\nabla g(x)|^2 \, dx$$

for $0 \leq t \leq T$ provided that $|\alpha| > 1$ and $|\beta| > 1$.

In fact, a bound of the form (3.5) can be obtained in the case that $|\alpha| < 1$ and $|\beta| < 1$ ($\alpha \neq 0$, $\beta \neq 0$) if we use the equation

$$E(t) = E(T) - 2 \int_t^T \left( \int_{\Omega} F(u) \, dx \right) \, d\eta,$$  

$0 \leq t \leq T$

(3.6)

and evaluate at $t = 0$, i.e.,

$$\int_{\Omega} \{[u_t(0)]^2 + |\nabla u(0)|^2\} \, dx \leq \int_{\Omega} \{[u_t(T)]^2 + |\nabla u(T)|^2\} \, dx + 2K |\Omega|.$$

By the initial conditions in (3.1) and the weighted arithmetic–geometric mean inequality, it follows that

$$\left(\frac{1}{\beta^2} - 1 - \frac{\sigma_3}{\beta^2}\right) \int_{\Omega} [u_t(T)]^2 \, dx + \left(\frac{1}{\alpha^2} - 1 - \frac{\sigma_4}{\alpha^2}\right) \int_{\Omega} |\nabla u(T)|^2 \, dx \leq 2K |\Omega| + \frac{1}{\beta^2} \left(\frac{1}{\sigma_3} - 1\right) \int_{\Omega} h^2 \, dx + \frac{1}{\alpha^2} \left(\frac{1}{\sigma_4} - 1\right) \int_{\Omega} |\nabla g|^2 \, dx,$$

where we choose

$$0 < \sigma_3 < 1 - \beta^2, \quad 0 < \sigma_4 < 1 - \alpha^2,$$

since $|\alpha| < 1$ and $|\beta| < 1$. Consequently, we have computable constants $K_4$, $K_5$, and $K_6$ such that

$$E(T) \leq K_4 + K_5 \int_{\Omega} h^2 \, dx + K_6 \int_{\Omega} |\nabla g|^2 \, dx$$

and

$$E(t) \leq 2K |\Omega| + K_4 + K_5 \int_{\Omega} h^2(x) \, dx + K_6 \int_{\Omega} |\nabla g(x)|^2 \, dx$$

(3.7)

for $0 \leq t \leq T$ by (3.6).

We formalize this result in the following theorem.
Theorem 2. If \( u \) is a solution of (3.1) where \( f \) satisfies (3.2) and \( \alpha \) and \( \beta \) are non-zero constants satisfying either \( |\alpha| > 1 \) and \( |\beta| > 1 \) or \( |\alpha| < 1 \) and \( |\beta| < 1 \), then \( E(t) \) given by (3.3) satisfies the a priori bound

\[
E(t) \leq 2K|\Omega| + C_1 + C_2 \int_{\Omega} h^2(x) \, dx + C_3 \int_{\Omega} |\nabla g(x)|^2 \, dx
\]

for \( 0 \leq t \leq T \) and computable constants \( C_1, C_2, \) and \( C_3 \) which depend on \( \alpha, \beta, K, \) and \( |\Omega| \).

As in Section 2, the result in Theorem 2 can be extended to other boundary conditions and more general elliptic operators than the Laplace operator as well as the function \( f(s) = e^{-s} \) satisfies the condition (3.2). Finally, we note that it was shown in [6] that when \( f(s) = 0 \), a solution may either not exist or not be unique when \( |\alpha| > 1 \) and \( |\beta| < 1 \) or \( |\alpha| < 1 \) and \( |\beta| > 1 \).

References