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Hyers–Ulam stability of a system of first order linear differential equations with constant coefficients ☆

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Abstract

In this paper, using matrix method, we prove the Hyers–Ulam stability of a system of first order linear differential equations with constant coefficients.

Keywords: Hyers–Ulam stability; System of linear differential equations; Matrix method

1. Introduction

Let X be a normed space over a scalar field $\mathbb{K}$, let I be an open interval, and let $a_0, a_1, \ldots, a_{n-1}$ be fixed elements of $\mathbb{K}$. Assume that for a fixed continuous function $g : I \to X$ and for any n times continuously differentiable function $y : I \to X$ satisfying the inequality

$$\left\| y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_1 y'(t) + a_0 y(t) + g(t) \right\| \leqslant \varepsilon$$

for all $t \in I$ and for a given $\varepsilon > 0$, there exists a function $y_0 : I \to X$ satisfying

$$y_0^{(n)}(t) + a_{n-1} y_0^{(n-1)}(t) + \cdots + a_1 y_0'(t) + a_0 y_0(t) + g(t) = 0$$

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and $\|y(t) - y_0(t)\| \leqslant K(\varepsilon)$ for any $t \in I$, where $K(\varepsilon)$ is an expression of ε with $\lim_{\varepsilon\to 0} K(\varepsilon) = 0$. Then, we say that the above differential equation has the Hyers–Ulam stability. For more detailed definition of the Hyers–Ulam stability, we may refer to [3–6, 12,14].

Alsina and Ger were the first authors who investigated the Hyers–Ulam stability of differential equations: They proved in [1] that if a differentiable function $y : I \to \mathbb{R}$ satisfies the differential inequality $|y'(t) - y(t)| \leqslant \varepsilon$, where I is an open subinterval of $\mathbb{R}$, then there exists a differentiable function $y_0 : I \to \mathbb{R}$ satisfying $y_0'(t) = y_0(t)$ and $|y(t) - y_0(t)| \leqslant 3\varepsilon$ for any $t \in I$.

This result of Alsina and Ger has been generalized by Takahasi et al. They proved in [13] that the Hyers–Ulam stability holds true for the Banach space valued differential equation $y'(t) = \lambda y(t)$. Indeed, the Hyers–Ulam stability has been proved for the first order linear differential equations in more general settings (see [7–11]).

Suppose we are given a system of first order linear differential equations with complex coefficients of the following form:

$$\begin{cases} y_1'(t) = a_{11}y_1(t) + a_{12}y_2(t) + \cdots + a_{1n}y_n(t) + b_1(t), \\ y_2'(t) = a_{21}y_1(t) + a_{22}y_2(t) + \cdots + a_{2n}y_n(t) + b_2(t), \\ \quad\vdots \qquad\quad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\quad \vdots \\ y_n'(t) = a_{n1}y_1(t) + a_{n2}y_2(t) + \cdots + a_{nn}y_n(t) + b_n(t). \end{cases} \tag{1}$$

This system may be written in a simple matrix notation

$$\vec{y}'(t) = \mathbf{A}\vec{y}(t) + \vec{b}(t)$$

if we set

$$\vec{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \qquad \vec{b}(t) = \begin{pmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{pmatrix}.$$

In this paper, we will make use of matrix method to prove the Hyers–Ulam stability of the system (1) of linear differential equations. More precisely, we prove that if a continuously differentiable vector function $\vec{y} : \mathbb{R} \to \mathbb{C}^n$ satisfies

$$\left\|\vec{y}'(t) - \mathbf{A}\vec{y}(t) - \vec{b}(t)\right\|_n \leqslant \varepsilon$$

for all $t \in \mathbb{R}$, where $\|\cdot\|_n$ is a norm on $\mathbb{C}^n$, then there exists a differentiable vector function $\vec{y}_0 : \mathbb{R} \to \mathbb{C}^n$ and a constant $K > 0$ such that

$$\vec{y}_0'(t) = \mathbf{A}\vec{y}_0(t) + \vec{b}(t) \quad \text{and} \quad \left\|\vec{y}(t) - \vec{y}_0(t)\right\|_n \leqslant K\varepsilon$$

for all $t \in \mathbb{R}$.

2. Preliminaries

Throughout this paper, let $(\mathbb{C}^n, \|\cdot\|_n)$ be a complex normed space and let $\mathbb{C}^{n\times n}$ be a vector space consisting of all $(n \times n)$ complex matrices. We notice that for $t \in \mathbb{R}$ and

$\mathbf{A} \in \mathbb{C}^{n\times n}$ we use the notation $\mathbf{A}t$ (instead of $t\mathbf{A}$) for the scalar multiplication of t and $\mathbf{A}$. We choose a norm $\|\cdot\|_{n\times n}$ on $\mathbb{C}^{n\times n}$ which is compatible with $\|\cdot\|_n$, i.e., both norms obey

$$\|\mathbf{AB}\|_{n\times n} \leqslant \|\mathbf{A}\|_{n\times n}\|\mathbf{B}\|_{n\times n}, \qquad \|\mathbf{A}\vec{x}\|_n \leqslant \|\mathbf{A}\|_{n\times n}\|\vec{x}\|_n \tag{2}$$

for all $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{n\times n}$ and $\vec{x} \in \mathbb{C}^n$.

Furthermore, we assume that $\|\vec{c}\|_n < \infty$ and $\|\mathbf{C}\|_{n\times n} < \infty$ for all $\vec{c} \in \mathbb{C}^n$ and $\mathbf{C} \in \mathbb{C}^{n\times n}$. For any $(u_1, \ldots, u_n)^{\mathrm{tr}}, (w_1, \ldots, w_n)^{\mathrm{tr}} \in \mathbb{C}^n$ with $|u_i| \leqslant w_i$ ($i \in \{1, \ldots, n\}$), we assume that

$$|u_i| \leqslant \left\|(u_1, \ldots, u_n)^{\mathrm{tr}}\right\|_n \leqslant \left\|(w_1, \ldots, w_n)^{\mathrm{tr}}\right\|_n, \tag{3}$$

where $(w_1, \ldots, w_n)^{\mathrm{tr}}$ is the transpose of the vector $(w_1, \ldots, w_n)$, i.e., $(w_1, \ldots, w_n)^{\mathrm{tr}}$ is a column vector.

Let $\mathbf{A}$ be a matrix in $\mathbb{C}^{n\times n}$, $n \geqslant 2$, and assume that $\mathbf{A}$ has d distinct eigenvalues λ_μ with algebraic multiplicity n_μ and geometric multiplicity m_μ, where $\mu \in \{1, \ldots, d\}$. Assume that Λ^-, Λ^0, Λ^+ are the disjoint subsets of $\{1, \ldots, d\}$ such that $\Lambda^- \cup \Lambda^0 \cup \Lambda^+ = \{1, \ldots, d\}$ and the real part of λ_μ, say $\Re(\lambda_\mu)$, satisfies

$$\Re(\lambda_\mu) \begin{cases} < 0 & \text{for } \mu \in \Lambda^-, \\ = 0 & \text{for } \mu \in \Lambda^0, \\ > 0 & \text{for } \mu \in \Lambda^+. \end{cases}$$

We choose a nonsingular matrix $\mathbf{N} \in \mathbb{C}^{n\times n}$ with $\mathbf{N}^{-1}\mathbf{AN} = \mathbf{J}$, where $\mathbf{J}$ is the Jordan form matrix of the form

$$\mathbf{J} = \begin{pmatrix} \mathbf{J}_{11} & & & & \\ & \ddots & & O & \\ & & \mathbf{J}_{\mu\nu} & & \\ & O & & \ddots & \\ & & & & \mathbf{J}_{dm_d} \end{pmatrix} \tag{4}$$

and the Jordan block $\mathbf{J}_{\mu\nu}$ is a $(p_{\mu\nu} \times p_{\mu\nu})$ matrix of the form

$$\mathbf{J}_{\mu\nu} = \begin{pmatrix} \lambda_\mu & 1 & 0 & \cdots & 0 \\ 0 & \lambda_\mu & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \cdots & \lambda_\mu \end{pmatrix}$$

for each $\mu \in \{1, \ldots, d\}$ and $\nu \in \{1, \ldots, m_\mu\}$. We notice that $p_{\mu 1} + \cdots + p_{\mu m_\mu} = n_\mu$ for any $\mu \in \{1, \ldots, d\}$.

For example, if the ith diagonal entry of $\mathbf{J}$ in (4) is one of the diagonal entries of $\mathbf{J}_{\mu\nu}$, then i satisfies

$$\begin{aligned} &(n_1 + \cdots + n_{\mu-1}) + (p_{\mu 1} + \cdots + p_{\mu(\nu-1)}) \\ &\quad < i \leqslant (n_1 + \cdots + n_{\mu-1}) + (p_{\mu 1} + \cdots + p_{\mu\nu}). \end{aligned} \tag{5}$$

Furthermore, it holds that

$$e^{\mathbf{A}t} = \mathbf{N}e^{\mathbf{J}t}\mathbf{N}^{-1} \tag{6}$$

if we set

$$e^{\mathbf{J}t} = \begin{pmatrix} e^{\mathbf{J}_{11}t} & & & & \\ & \ddots & & O & \\ & & e^{\mathbf{J}_{\mu\nu}t} & & \\ & O & & \ddots & \\ & & & & e^{\mathbf{J}_{dm_d}t} \end{pmatrix}.$$

Let us denote by $\mathbf{O}_{\alpha\beta}$ the zero square matrix with $p_{\alpha\beta}$ rows (columns). We replace each $e^{\mathbf{J}_{\alpha\beta}t}$ in the above $e^{\mathbf{J}t}$ by $\mathbf{O}_{\alpha\beta}$, except $e^{\mathbf{J}_{\mu\nu}t}$, and we denote the resulting matrix by $\langle e^{\mathbf{J}_{\mu\nu}t}\rangle$, i.e.,

$$\left\langle e^{\mathbf{J}_{\mu\nu}t}\right\rangle = \begin{pmatrix} \mathbf{O} & & & & & \\ & \ddots & & & O & \\ & & \mathbf{O} & & & \\ & & & e^{\mathbf{J}_{\mu\nu}t} & & \\ & & & & \mathbf{O} & \\ & O & & & & \ddots \\ & & & & & & \mathbf{O} \end{pmatrix}, \tag{7}$$

where we simply write $\mathbf{O}$ instead of $\mathbf{O}_{\alpha\beta}$ as a matter of convenience. We notice that

$$e^{\mathbf{J}_{\mu\nu}t} = e^{\lambda_\mu t} \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{p_{\mu\nu}-2}}{(p_{\mu\nu}-2)!} & \frac{t^{p_{\mu\nu}-1}}{(p_{\mu\nu}-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{p_{\mu\nu}-3}}{(p_{\mu\nu}-3)!} & \frac{t^{p_{\mu\nu}-2}}{(p_{\mu\nu}-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & t \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \tag{8}$$

and

$$e^{\mathbf{J}t} = \sum_{\mu=1}^{d} \sum_{\nu=1}^{m_\mu} \left\langle e^{\mathbf{J}_{\mu\nu}t}\right\rangle. \tag{9}$$

For all $\mu \in \{1, \ldots, d\}$ and $\nu \in \{1, \ldots, m_\mu\}$, we define a $(p_{\mu\nu} \times p_{\mu\nu})$ matrix $\mathbf{T}_{\mu\nu} = (t_{ij})$ by

$$t_{ij} = \begin{cases} |\Re(\lambda_\mu)|^{-(j-i+1)} & \text{for } j \geqslant i, \\ 0 & \text{for } j < i \end{cases}$$

and we replace any $\mathbf{J}_{\alpha\beta}$ in the $\mathbf{J}$ of (4) by $\mathbf{O}_{\alpha\beta}$ for each pair $(\alpha, \beta) \neq (\mu, \nu)$ and replace the $\mathbf{J}_{\mu\nu}$ by $\mathbf{T}_{\mu\nu}$, where $\mathbf{O}_{\alpha\beta}$ is the $(p_{\alpha\beta} \times p_{\alpha\beta})$ matrix whose all entries are zeros. The resulting matrix will be denoted by $\langle \mathbf{T}_{\mu\nu}\rangle$, i.e.,

$$\langle \mathbf{T}_{\mu\nu}\rangle = \begin{pmatrix} \mathbf{O} & & & & & \\ & \ddots & & & O & \\ & & \mathbf{O} & & & \\ & & & \mathbf{T}_{\mu\nu} & & \\ & & & & \mathbf{O} & \\ & O & & & & \ddots \\ & & & & & & \mathbf{O} \end{pmatrix}. \tag{10}$$

Given any positive numbers $\varepsilon_{\mu\nu}^{(k)}$, $k \in \{0, \dots, p_{\mu\nu} - 1\}$, let us define a $(p_{\mu\nu} \times p_{\mu\nu})$ matrix $\mathbf{S}_{\mu\nu} = (s_{ij})$ by

$$s_{ij} = \begin{cases} \varepsilon_{\mu\nu}^{(j-i)} & \text{for } j \geqslant i, \\ 0 & \text{for } j < i, \end{cases}$$

and we replace each $\mathbf{J}_{\alpha\beta}$ in the $\mathbf{J}$ of (4) by $\mathbf{O}_{\alpha\beta}$ for every pair $(\alpha, \beta) \neq (\mu, \nu)$ and $\mathbf{J}_{\mu\nu}$ by $\mathbf{S}_{\mu\nu}$. We will use the notation $\langle \mathbf{S}_{\mu\nu} \rangle$ for the resulting matrix:

$$\langle \mathbf{S}_{\mu\nu} \rangle = \begin{pmatrix} \mathbf{O} & & & & & & \\ & \ddots & & & & O & \\ & & \mathbf{O} & & & & \\ & & & \mathbf{S}_{\mu\nu} & & & \\ & & & & \mathbf{O} & & \\ & O & & & & \ddots & \\ & & & & & & \mathbf{O} \end{pmatrix}. \tag{11}$$

Using (10) and (11), we define

$$\langle \mathbf{B}_{\mu\nu} \rangle = \begin{cases} \langle \mathbf{S}_{\mu\nu} \rangle & \text{for } \mu \in \Lambda^0, \\ \langle \mathbf{T}_{\mu\nu} \rangle & \text{for } \mu \in \Lambda^- \cup \Lambda^+, \end{cases}$$

for any $\mu \in \{1, \dots, d\}$ and $\nu \in \{1, \dots, m_\mu\}$, and further define

$$\mathbf{B} = \sum_{\mu=1}^{d} \sum_{\nu=1}^{m_\mu} \langle \mathbf{B}_{\mu\nu} \rangle. \tag{12}$$

3. Main result

The following lemma can be easily proved by using the mathematical induction. Hence, we omit the proof.

Lemma 1. *Let m be a nonnegative integer.*

(a) *If $\lambda < 0$, then it holds that*

$$\int_{-\infty}^{0} s^m e^{-\lambda s}\, ds = -\frac{m!}{\lambda^{m+1}} \quad \textit{and} \quad \int_{-\infty}^{t} (t-s)^m e^{\lambda(t-s)}\, ds = \frac{(-1)^{m+1} m!}{\lambda^{m+1}}$$

for any $t \in \mathbb{R}$.

(b) *If $\lambda > 0$, then it holds that*

$$\int_{0}^{\infty} s^m e^{-\lambda s}\, ds = \frac{m!}{\lambda^{m+1}} \quad \textit{and} \quad \int_{t}^{\infty} (t-s)^m e^{\lambda(t-s)}\, ds = \frac{(-1)^m m!}{\lambda^{m+1}}$$

for any $t \in \mathbb{R}$.

Using the notations and assumptions given in the preceding section, we will prove the Hyers–Ulam stability of the system (1) of first order linear differential equations with constant coefficients.

Theorem 2. *In addition to the assumptions in Section* 2*, let* $\vec{b}:\mathbb{R}\to\mathbb{C}^n$ *and* $\vec{y}:\mathbb{R}\to\mathbb{C}^n$ *be a continuous vector function and a continuously differentiable vector function, respectively. Assume that a continuous vector function* $\vec{v}:\mathbb{R}\to\mathbb{C}^n$ *defined by*

$$\vec{v}(t)=\vec{y}'(t)-\mathbf{A}\vec{y}(t)-\vec{b}(t) \tag{13}$$

satisfies

$$\left\|\vec{v}(t)\right\|_n\leqslant\varepsilon \tag{14}$$

for all $t\in\mathbb{R}$ *and for some* $\varepsilon>0$*. Suppose that for each* $\mu\in\Lambda^0$*, there exist positive numbers* $\varepsilon_{\mu\nu}^{(k)}$ *such that*

$$\int_{-\infty}^{\infty}|t-s|^k\left|\left[\mathbf{N}^{-1}\vec{v}(s)\right]_i\right|ds\leqslant k!\varepsilon_{\mu\nu}^{(k)}\varepsilon\left\|\mathbf{N}^{-1}\right\|_{n\times n} \tag{15}$$

for all $t\in\mathbb{R}$*, all* $\nu\in\{1,\dots,m_\mu\}$*, any integers* i *satisfying* (5)*, and for any* $k\in\{0,\dots,p_{\mu\nu}-1\}$*, where* $\mathbf{N}$ *is a nonsingular matrix such that* $\mathbf{N}^{-1}\mathbf{A}\mathbf{N}$ *is a Jordan form matrix* (*see* (4)) *and* $[\mathbf{N}^{-1}\vec{v}(s)]_i$ *denotes the* i*th component of* $\mathbf{N}^{-1}\vec{v}(s)$*. Then there exists a differentiable vector function* $\vec{y}_0:\mathbb{R}\to\mathbb{C}^n$ *such that*

$$\vec{y}_0'(t)=\mathbf{A}\vec{y}_0(t)+\vec{b}(t) \tag{16}$$

and

$$\left\|\vec{y}(t)-\vec{y}_0(t)\right\|_n\leqslant\varepsilon\|\mathbf{N}\|_{n\times n}\left\|\mathbf{N}^{-1}\right\|_{n\times n}\|\mathbf{B}\vec{e}\|_n$$

for all $t\in\mathbb{R}$*, where* $\vec{e}=(1,1,\dots,1)^{\mathrm{tr}}\in\mathbb{C}^n$*.*

4. Proof of Theorem 2

By (13), we have

$$\vec{y}'(t)=\mathbf{A}\vec{y}(t)+\vec{b}(t)+\vec{v}(t) \tag{17}$$

for all $t\in\mathbb{R}$. We notice that $\mathbf{N}\in\mathbb{C}^{n\times n}$ is a nonsingular matrix for which $\mathbf{J}=\mathbf{N}^{-1}\mathbf{A}\mathbf{N}$ is a Jordan matrix of the form (4). In this proof, we use the notations $e^{\mathbf{A}t}$, $\langle e^{\mathbf{J}_{\mu\nu}t}\rangle$ and $e^{\mathbf{J}_{\mu\nu}t}$ and refer to (6), (7) and (8) for their details.

In view of [15, §16] (cf. [2, Chapter 31]), the solution of (17) is

$$\vec{y}(t)=e^{\mathbf{A}t}\,\vec{y}(0)+e^{\mathbf{A}t}\int_0^t e^{-\mathbf{A}s}\vec{b}(s)\,ds+e^{\mathbf{A}t}\int_0^t e^{-\mathbf{A}s}\vec{v}(s)\,ds \tag{18}$$

for all $t\in\mathbb{R}$.

Set $\vec{w}(t) = \mathbf{N}^{-1}\vec{v}(t) = (w_1(t), \dots, w_n(t))^{\mathrm{tr}}$ and for every fixed $\mu \in \{1, \dots, d\}$ and $\nu \in \{1, \dots, m_\mu\}$, let us define a column vector $\langle \vec{w}_{\mu\nu}(t)\rangle$ by

$$\text{the } i\text{th component of } \langle \vec{w}_{\mu\nu}(t)\rangle = \begin{cases} w_i(t) & \text{if } i \text{ satisfies (5)}, \\ 0 & \text{otherwise.} \end{cases}$$

Then, we have

$$\vec{w}(t) = \sum_{\mu=1}^{d}\sum_{\nu=1}^{m_\mu}\langle \vec{w}_{\mu\nu}(t)\rangle.$$

It now follows from (6), (7), (9) and the last equality that

$$\begin{aligned} e^{-\mathbf{A}s}\vec{v}(s) &= \left(\mathbf{N}e^{-\mathbf{J}s}\right)\left(\mathbf{N}^{-1}\vec{v}(s)\right) = \mathbf{N}\left(e^{-\mathbf{J}s}\vec{w}(s)\right) \\ &= \mathbf{N}\sum_{\mu=1}^{d}\sum_{\nu=1}^{m_\mu}\langle e^{-\mathbf{J}_{\mu\nu}s}\rangle\sum_{\rho=1}^{d}\sum_{\sigma=1}^{m_\rho}\langle \vec{w}_{\rho\sigma}(s)\rangle \\ &= \mathbf{N}\sum_{\mu=1}^{d}\sum_{\nu=1}^{m_\mu}\langle e^{-\mathbf{J}_{\mu\nu}s}\rangle\langle \vec{w}_{\mu\nu}(s)\rangle \end{aligned}$$

and

$$\int_0^t e^{-\mathbf{A}s}\vec{v}(s)\,ds = \mathbf{N}\sum_{\mu=1}^{d}\sum_{\nu=1}^{m_\mu}\int_0^t \langle e^{-\mathbf{J}_{\mu\nu}s}\rangle\langle \vec{w}_{\mu\nu}(s)\rangle\,ds, \tag{19}$$

where $\int_0^t \langle e^{-\mathbf{J}_{\mu\nu}s}\rangle\langle \vec{w}_{\mu\nu}(s)\rangle\,ds$ is a column vector with n components.

Let us define

$$\langle \vec{c}_{\mu\nu}\rangle = \begin{cases} -\int_{-\infty}^{0}\langle e^{-\mathbf{J}_{\mu\nu}s}\rangle\langle \vec{w}_{\mu\nu}(s)\rangle\,ds & \text{for } \mu \in \Lambda^-, \\ (0, 0, \dots, 0)^{\mathrm{tr}} & \text{for } \mu \in \Lambda^0, \\ \int_0^{\infty}\langle e^{-\mathbf{J}_{\mu\nu}s}\rangle\langle \vec{w}_{\mu\nu}(s)\rangle\,ds & \text{for } \mu \in \Lambda^+ \end{cases}$$

and

$$\langle \vec{r}_{\mu\nu}(t)\rangle = \begin{cases} \int_{-\infty}^{t}\langle e^{-\mathbf{J}_{\mu\nu}s}\rangle\langle \vec{w}_{\mu\nu}(s)\rangle\,ds & \text{for } \mu \in \Lambda^-, \\ \int_0^{t}\langle e^{-\mathbf{J}_{\mu\nu}s}\rangle\langle \vec{w}_{\mu\nu}(s)\rangle\,ds & \text{for } \mu \in \Lambda^0, \\ -\int_t^{\infty}\langle e^{-\mathbf{J}_{\mu\nu}s}\rangle\langle \vec{w}_{\mu\nu}(s)\rangle\,ds & \text{for } \mu \in \Lambda^+. \end{cases} \tag{20}$$

From now on, we set

$$i_{\mu\nu} = (n_1 + \cdots + n_{\mu-1}) + (p_{\mu 1} + \cdots + p_{\mu(\nu-1)})$$

for any fixed $\mu \in \{1, \dots, d\}$ and $\nu \in \{1, \dots, m_\mu\}$, where $p_{\mu 0} = 0$ for any $\mu \in \{1, \dots, d\}$. In view of (7) and (8), the ith component of $\langle e^{-\mathbf{J}_{\mu\nu}s}\rangle\langle \vec{w}_{\mu\nu}(s)\rangle$ is given below:

$$\left[\langle e^{-\mathbf{J}_{\mu\nu}s}\rangle\langle \vec{w}_{\mu\nu}(s)\rangle\right]_i = \begin{cases} e^{-\lambda_\mu s}\sum_{k=i}^{i_{\mu\nu}+p_{\mu\nu}}\frac{(-s)^{k-i}}{(k-i)!}w_k(s) & \text{if } i \text{ satisfies (5)}, \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

Hence, Lemma 1 implies that $\int_{-\infty}^{0}[\langle e^{-\mathbf{J}_{\mu\nu}s}\rangle\langle\vec{w}_{\mu\nu}(s)\rangle]_i\,ds$ or $\int_0^{\infty}[\langle e^{-\mathbf{J}_{\mu\nu}s}\rangle\langle\vec{w}_{\mu\nu}(s)\rangle]_i\,ds$ is bounded for $\mu\in\Lambda^-$ or $\mu\in\Lambda^+$, respectively, because it follows from (2), (3) and (14) that

$$\left|w_k(s)\right|\leqslant\left\|\mathbf{N}^{-1}\vec{v}(s)\right\|_n\leqslant\varepsilon\left\|\mathbf{N}^{-1}\right\|_{n\times n}.$$

Thus, $\langle\vec{c}_{\mu\nu}\rangle$ is a constant vector.

By (19) and (20), we obtain

$$\int_0^t e^{-\mathbf{A}s}\vec{v}(s)\,ds=\mathbf{N}\sum_{\mu=1}^{d}\sum_{\nu=1}^{m_\mu}\bigl(\langle\vec{c}_{\mu\nu}\rangle+\langle\vec{r}_{\mu\nu}(t)\rangle\bigr). \tag{22}$$

If we set

$$\vec{c}=\sum_{\mu=1}^{d}\sum_{\nu=1}^{m_\mu}\langle\vec{c}_{\mu\nu}\rangle\quad\text{and}\quad\vec{r}(t)=\sum_{\mu=1}^{d}\sum_{\nu=1}^{m_\mu}\langle\vec{r}_{\mu\nu}(t)\rangle, \tag{23}$$

then it follows from (6) and (22) that

$$e^{\mathbf{A}t}\int_0^t e^{-\mathbf{A}s}\vec{v}(s)\,ds=\mathbf{N}e^{\mathbf{J}t}\bigl(\vec{c}+\vec{r}(t)\bigr). \tag{24}$$

We now define

$$\vec{y}_0(t)=e^{\mathbf{A}t}\bigl\{\vec{y}(0)+\mathbf{N}\vec{c}\bigr\}+e^{\mathbf{A}t}\int_0^t e^{-\mathbf{A}s}\vec{b}(s)\,ds \tag{25}$$

for all $t\in\mathbb{R}$. By applying (18), (25), (24), (6), (2), (9), (23), and (20) in order, we get

$$\begin{aligned}
\left\|\vec{y}(t)-\vec{y}_0(t)\right\|_n&=\left\|\mathbf{N}e^{\mathbf{J}t}\vec{r}(t)\right\|_n\leqslant\|\mathbf{N}\|_{n\times n}\left\|e^{\mathbf{J}t}\vec{r}(t)\right\|_n\\
&=\|\mathbf{N}\|_{n\times n}\left\|\sum_{\mu=1}^{d}\sum_{\nu=1}^{m_\mu}\langle e^{\mathbf{J}_{\mu\nu}t}\rangle\sum_{\rho=1}^{d}\sum_{\sigma=1}^{m_\rho}\langle\vec{r}_{\rho\sigma}(t)\rangle\right\|_n\\
&=\|\mathbf{N}\|_{n\times n}\left\|\sum_{\mu=1}^{d}\sum_{\nu=1}^{m_\mu}\langle e^{\mathbf{J}_{\mu\nu}t}\rangle\langle\vec{r}_{\mu\nu}(t)\rangle\right\|_n\\
&=\|\mathbf{N}\|_{n\times n}\left\|\sum_{\mu\in\Lambda^-}\sum_{\nu=1}^{m_\mu}\langle e^{\mathbf{J}_{\mu\nu}t}\rangle\int_{-\infty}^{t}\langle e^{-\mathbf{J}_{\mu\nu}s}\rangle\langle\vec{w}_{\mu\nu}(s)\rangle\,ds\right.\\
&\qquad+\sum_{\mu\in\Lambda^0}\sum_{\nu=1}^{m_\mu}\langle e^{\mathbf{J}_{\mu\nu}t}\rangle\int_0^t\langle e^{-\mathbf{J}_{\mu\nu}s}\rangle\langle\vec{w}_{\mu\nu}(s)\rangle\,ds\\
&\qquad\left.-\sum_{\mu\in\Lambda^+}\sum_{\nu=1}^{m_\mu}\langle e^{\mathbf{J}_{\mu\nu}t}\rangle\int_t^{\infty}\langle e^{-\mathbf{J}_{\mu\nu}s}\rangle\langle\vec{w}_{\mu\nu}(s)\rangle\,ds\right\|_n.
\end{aligned}$$

A somewhat tedious calculation, together with (7) and (8), leads to the conclusion that $\langle e^{\mathbf{J}_{\mu\nu}t}\rangle\langle e^{-\mathbf{J}_{\mu\nu}s}\rangle = \langle e^{\mathbf{J}_{\mu\nu}(t-s)}\rangle$ holds for all $s, t \in \mathbb{R}$. Therefore, we further obtain

$$\begin{aligned}\left\|\vec{y}(t) - \vec{y}_0(t)\right\|_n = \|\mathbf{N}\|_{n\times n}\Bigg\| &\sum_{\mu\in\Lambda^-}\sum_{\nu=1}^{m_\mu}\int_{-\infty}^{t}\left\langle e^{\mathbf{J}_{\mu\nu}(t-s)}\right\rangle\left\langle\vec{w}_{\mu\nu}(s)\right\rangle ds \\ &+ \sum_{\mu\in\Lambda^0}\sum_{\nu=1}^{m_\mu}\int_{0}^{t}\left\langle e^{\mathbf{J}_{\mu\nu}(t-s)}\right\rangle\left\langle\vec{w}_{\mu\nu}(s)\right\rangle ds \\ &- \sum_{\mu\in\Lambda^+}\sum_{\nu=1}^{m_\mu}\int_{t}^{\infty}\left\langle e^{\mathbf{J}_{\mu\nu}(t-s)}\right\rangle\left\langle\vec{w}_{\mu\nu}(s)\right\rangle ds\Bigg\|_n \end{aligned} \tag{26}$$

for all $t \in \mathbb{R}$.

Let $\mu \in \Lambda^-$ be given. In view of (7) and (8) (or Ref. (21)), the ith component of $\langle e^{\mathbf{J}_{\mu\nu}(t-s)}\rangle\langle\vec{w}_{\mu\nu}(s)\rangle$ is

$$\left[\left\langle e^{\mathbf{J}_{\mu\nu}(t-s)}\right\rangle\left\langle\vec{w}_{\mu\nu}(s)\right\rangle\right]_i = \begin{cases} e^{\lambda_\mu(t-s)}\sum_{k=i}^{i_{\mu\nu}+p_{\mu\nu}}\frac{(t-s)^{k-i}}{(k-i)!}w_k(s) & \text{if } i \text{ satisfies (5)}, \\ 0 & \text{otherwise.}\end{cases} \tag{27}$$

Hence, the ith component α_i of $\int_{-\infty}^{t}\langle e^{\mathbf{J}_{\mu\nu}(t-s)}\rangle\langle\vec{w}_{\mu\nu}(s)\rangle\,ds$ is

$$\alpha_i = \begin{cases} \sum_{k=i}^{i_{\mu\nu}+p_{\mu\nu}}\frac{1}{(k-i)!}\int_{-\infty}^{t}(t-s)^{k-i}e^{\lambda_\mu(t-s)}w_k(s)\,ds & \text{if } i \text{ satisfies (5)}, \\ 0 & \text{otherwise.}\end{cases}$$

Now, by using (2), (3), (14) and Lemma 1(a), we have

$$\begin{aligned}|\alpha_i| &\leqslant \sum_{k=i}^{i_{\mu\nu}+p_{\mu\nu}}\frac{1}{(k-i)!}\int_{-\infty}^{t}(t-s)^{k-i}e^{\Re(\lambda_\mu)(t-s)}\left\|\vec{w}(s)\right\|_n ds \\ &\leqslant \varepsilon\left\|\mathbf{N}^{-1}\right\|_{n\times n}\sum_{k=i}^{i_{\mu\nu}+p_{\mu\nu}}\left|\Re(\lambda_\mu)\right|^{-k+i-1}\end{aligned}$$

for each i satisfying (5), and $|\alpha_i| \leqslant 0$ if (5) is not satisfied with i.

On account of (10), the ith component $[\langle\mathbf{T}_{\mu\nu}\rangle\vec{e}]_i$ of $\langle\mathbf{T}_{\mu\nu}\rangle\vec{e}$ is

$$\left[\langle\mathbf{T}_{\mu\nu}\rangle\vec{e}\right]_i = \begin{cases} \sum_{k=i}^{i_{\mu\nu}+p_{\mu\nu}}|\Re(\lambda_\mu)|^{-k+i-1} & \text{if } i \text{ satisfies (5)}, \\ 0 & \text{otherwise,}\end{cases}$$

where $\vec{e} = (1, 1, \ldots, 1)^{\mathrm{tr}}$. Hence, we get

$$|\alpha_i| \leqslant \varepsilon\left\|\mathbf{N}^{-1}\right\|_{n\times n}\left[\langle\mathbf{T}_{\mu\nu}\rangle\vec{e}\right]_i \tag{28}$$

for all $i \in \{1, \ldots, n\}$.

If $\mu \in \Lambda^+$ is arbitrarily given and if we denote by β_i the ith component of

$$\int_{t}^{\infty}\left\langle e^{\mathbf{J}_{\mu\nu}(t-s)}\right\rangle\left\langle\vec{w}_{\mu\nu}(s)\right\rangle ds,$$

then we obtain by a similar argument that

$$|\beta_i| \leqslant \varepsilon \left\|\mathbf{N}^{-1}\right\|_{n\times n} \left[\langle \mathbf{T}_{\mu\nu}\rangle \vec{e}\right]_i \tag{29}$$

for any $i \in \{1, \dots, n\}$.

Now, let $\mu \in \Lambda^0$ be given, and let γ_i be the ith component of

$$\int_0^t \left\langle e^{\mathbf{J}_{\mu\nu}(t-s)}\right\rangle \left\langle \vec{w}_{\mu\nu}(s)\right\rangle ds.$$

In view of (27), we get

$$\gamma_i = \begin{cases} \sum_{k=i}^{i_{\mu\nu}+p_{\mu\nu}} \frac{1}{(k-i)!} \int_0^t (t-s)^{k-i} e^{\lambda_\mu(t-s)} w_k(s)\, ds & \text{if } i \text{ satisfies (5)}, \\ 0 & \text{otherwise}. \end{cases}$$

Using (2), (3) and (15), we have

$$|\gamma_i| \leqslant \sum_{k=i}^{i_{\mu\nu}+p_{\mu\nu}} \frac{1}{(k-i)!} \left| \int_0^t |t-s|^{k-i} \left|w_k(s)\right| ds \right| \leqslant \varepsilon \left\|\mathbf{N}^{-1}\right\|_{n\times n} \sum_{k=i}^{i_{\mu\nu}+p_{\mu\nu}} \varepsilon_{\mu\nu}^{(k-i)}$$

for each i satisfying (5), and $|\gamma_i| \leqslant 0$ if (5) is not satisfied with i.

According to (11), the ith component of $\langle \mathbf{S}_{\mu\nu}\rangle \vec{e}$ is

$$\left[\langle \mathbf{S}_{\mu\nu}\rangle \vec{e}\right]_i = \begin{cases} \sum_{k=i}^{i_{\mu\nu}+p_{\mu\nu}} \varepsilon_{\mu\nu}^{(k-i)} & \text{if } i \text{ satisfies (5)}, \\ 0 & \text{otherwise} \end{cases}$$

and hence

$$|\gamma_i| \leqslant \varepsilon \left\|\mathbf{N}^{-1}\right\|_{n\times n} \left[\langle \mathbf{S}_{\mu\nu}\rangle \vec{e}\right]_i \tag{30}$$

for each $i \in \{1, \dots, n\}$.

Consequently, it follows from (3), (26), (28), (29), (30) and (12) that

$$\begin{aligned} \left\|\vec{y}(t) - \vec{y}_0(t)\right\|_n &\leqslant \|\mathbf{N}\|_{n\times n} \left\| \sum_{\mu\in\Lambda^-\cup\Lambda^+} \sum_{\nu=1}^{m_\mu} \varepsilon \left\|\mathbf{N}^{-1}\right\|_{n\times n} \langle \mathbf{T}_{\mu\nu}\rangle \vec{e} \right. \\ &\qquad \left. + \sum_{\mu\in\Lambda^0} \sum_{\nu=1}^{m_\mu} \varepsilon \left\|\mathbf{N}^{-1}\right\|_{n\times n} \langle \mathbf{S}_{\mu\nu}\rangle \vec{e} \right\|_n \\ &= \varepsilon \|\mathbf{N}\|_{n\times n} \left\|\mathbf{N}^{-1}\right\|_{n\times n} \|\mathbf{B}\vec{e}\|_n \end{aligned}$$

for all $t \in \mathbb{R}$.

According to [2, Chapter 31] (cf. [15, §16]), $\vec{y}_0(t)$ is a solution of the system (1) of linear differential equations (see (25)), or equivalently, $\vec{y}_0(t)$ satisfies the differential equation (16) for all $t \in \mathbb{R}$. □

5. Some examples

Some of the most important matrix norms are induced by p-norms. For $1 \leqslant p \leqslant \infty$, the norm induced by the p-norm,

$$\|\mathbf{A}\|_p = \sup_{\vec{x} \neq \vec{0}} \frac{\|\mathbf{A}\vec{x}\|_p}{\|\vec{x}\|_p} \quad \left(\mathbf{A} \in \mathbb{C}^{n \times n}\right),$$

is called the matrix p-norm. For example, we get

$$\|\mathbf{A}\|_1 = \max_{1 \leqslant j \leqslant n} \sum_{i=1}^{n} |a_{ij}|, \quad \|\mathbf{A}\|_\infty = \max_{1 \leqslant i \leqslant n} \sum_{j=1}^{n} |a_{ij}|.$$

It is well known that the matrix p-norm, together with the p-norm, satisfies both conditions in (2).

Example 1. We consider a system of first order linear differential equations of the following form

$$\begin{cases} y_1'(t) = y_1(t) + 2y_2(t) + b_1(t), \\ y_2'(t) = 3y_1(t) + 2y_2(t) + b_2(t). \end{cases} \tag{31}$$

This system can be written in a matrix notation

$$\vec{y}'(t) = \mathbf{A}\vec{y}(t) + \vec{b}(t),$$

where

$$\vec{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}, \qquad \vec{b}(t) = \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix}.$$

Assume that a continuous vector function $\vec{b} : \mathbb{R} \to \mathbb{C}^2$ and a continuously differentiable vector function $\vec{y} : \mathbb{R} \to \mathbb{C}^2$ satisfy

$$\left\|\vec{y}'(t) - \mathbf{A}\vec{y}(t) - \vec{b}(t)\right\|_\infty \leqslant \varepsilon$$

for all $t \in \mathbb{R}$ and for some $\varepsilon \geqslant 0$.

Since $\mathbf{A}$ has two distinct eigenvalues -1 and 4, we can choose a nonsingular matrix $\mathbf{N}$ and a diagonal matrix $\mathbf{J}$

$$\mathbf{N} = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}, \qquad \mathbf{J} = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}$$

such that $\mathbf{J} = \mathbf{N}^{-1}\mathbf{A}\mathbf{N}$. Furthermore, since $d = 2$, $m_1 = m_2 = 1$ and $p_{11} = p_{21} = 1$, it follows from (12) that

$$\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{pmatrix}.$$

According to Theorem 2 and (25), there exists a differentiable vector function $\vec{y}_0 : \mathbb{R} \to \mathbb{C}^2$ of the form

$$\vec{y}_0(t) = e^{\mathbf{A}t}\vec{k} + e^{\mathbf{A}t} \int_0^t e^{-\mathbf{A}s}\vec{b}(s)\,ds$$

with

$$\left\|\vec{y}(t)-\vec{y}_0(t)\right\|_\infty \leqslant 4\varepsilon$$

for all $t \in \mathbb{R}$, where $\vec{k} \in \mathbb{C}^2$ is a constant.

If we set $\vec{b}(t)=\vec{0}$ in the above example, then there exist two constants k_1 and k_2 such that the functions $y_{01}, y_{02}:\mathbb{R}\to\mathbb{C}$ defined by

$$y_{01}(t)=\left(\frac{3}{5}k_1-\frac{2}{5}k_2\right)e^{-t}+\left(\frac{2}{5}k_1+\frac{2}{5}k_2\right)e^{4t},$$
$$y_{02}(t)=\left(-\frac{3}{5}k_1+\frac{2}{5}k_2\right)e^{-t}+\left(\frac{3}{5}k_1+\frac{3}{5}k_2\right)e^{4t}$$

satisfy the system (31) of differential equations as well as the inequality

$$\max\left\{\left|y_1(t)-y_{01}(t)\right|,\ \left|y_2(t)-y_{02}(t)\right|\right\} \leqslant 4\varepsilon$$

for each $t \in \mathbb{R}$.

Example 2. Assume that $y_1, y_2, y_3:\mathbb{R}\to\mathbb{C}$ are continuously differentiable functions and satisfy the inequality

$$\left\|\vec{y}'(t)-\mathbf{A}\vec{y}(t)-\vec{b}(t)\right\|_\infty \leqslant \varepsilon$$

for all $t \in \mathbb{R}$ and for some $\varepsilon \geqslant 0$, where

$$\vec{y}(t)=\begin{pmatrix} y_1(t)\\ y_2(t)\\ y_3(t)\end{pmatrix},\qquad \mathbf{A}=\begin{pmatrix} -2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{pmatrix},\qquad \vec{b}(t)=\begin{pmatrix} b_1(t)\\ b_2(t)\\ b_3(t)\end{pmatrix},$$

and $b_1, b_2, b_3:\mathbb{R}\to\mathbb{C}$ are continuous functions.

We can easily choose a nonsingular matrix $\mathbf{N}$ and a diagonal matrix $\mathbf{J}$

$$\mathbf{N}=\begin{pmatrix} -2 & 3 & 1\\ 1 & 0 & 2\\ 0 & 1 & -1\end{pmatrix},\qquad \mathbf{J}=\begin{pmatrix} -3 & 0 & 0\\ 0 & -3 & 0\\ 0 & 0 & 5\end{pmatrix},\qquad \mathbf{N}^{-1}=\begin{pmatrix} -\frac{1}{4} & \frac{1}{2} & \frac{3}{4}\\ \frac{1}{8} & \frac{1}{4} & \frac{5}{8}\\ \frac{1}{8} & \frac{1}{4} & -\frac{3}{8}\end{pmatrix}$$

such that $\mathbf{J}=\mathbf{N}^{-1}\mathbf{A}\mathbf{N}$. Furthermore, since $d=2$, $m_1=n_1=2$, $m_2=n_2=1$ and $p_{11}=p_{12}=p_{21}=1$, it follows from (12) that

$$\mathbf{B}=\begin{pmatrix} \frac{1}{3} & 0 & 0\\ 0 & \frac{1}{3} & 0\\ 0 & 0 & \frac{1}{5}\end{pmatrix}.$$

According to Theorem 2, there exists a differentiable vector function $\vec{y}_0:\mathbb{R}\to\mathbb{C}^3$ satisfying $\vec{y}_0'(t)=\mathbf{A}\vec{y}_0(t)+\vec{b}(t)$ as well as

$$\left\|\vec{y}(t)-\vec{y}_0(t)\right\|_\infty \leqslant 3\varepsilon$$

for all $t \in \mathbb{R}$. Moreover, due to (25), there exists a constant $\vec{k} \in \mathbb{C}^3$ such that $\vec{y}_0(t)$ has the following form

$$\vec{y}_0(t) = e^{\mathbf{A}t}\vec{k} + e^{\mathbf{A}t} \int_0^t e^{-\mathbf{A}s}\vec{b}(s)\,ds.$$

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