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Quantum α-determinant cyclic modules of $\mathcal{U}_q(\mathfrak{gl}_n)$

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Abstract

As a particular one parameter deformation of the quantum determinant, we introduce a quantum α-determinant $\det_q^{(\alpha)}$ and study the $\mathcal{U}_q(\mathfrak{gl}_n)$-cyclic module generated by it: We show that the multiplicity of each irreducible representation in this cyclic module is determined by a certain polynomial called the q-content discriminant. A part of the present result is a quantum counterpart for the result of Matsumoto and Wakayama [S. Matsumoto, M. Wakayama, Alpha-determinant cyclic modules of $\mathfrak{gl}_n(\mathbb{C})$, J. Lie Theory 16 (2006) 393–405], however, a new distinguished feature arises in our situation. Specifically, we determine the degeneration of the multiplicities for '*classical*' singular points and give a general conjecture for singular points involving *semi-classical* and *quantum* singularities. Moreover, we introduce a quantum α-permanent $\mathrm{per}_q^{(\alpha)}$ and establish another conjecture which describes a 'reciprocity' between the multiplicities of the irreducible summands of the cyclic modules generated respectively by $\det_q^{(\alpha)}$ and $\mathrm{per}_q^{(\alpha)}$.

Keywords: α-Determinant; Quantum group; Iwahori–Hecke algebra; q-Young symmetrizer; Cyclic module; Irreducible decomposition; Elementary divisors; Content polynomial; Kostka number; Partition function

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Contents

1. Introduction

Let $\mathcal{A}(\mathrm{Mat}_n)$ be the associative $\mathbb{C}$-algebra consisting of polynomial functions on the set Mat_n of n by n matrices. We denote by x_{ij} the standard coordinate function on Mat_n with respect to the matrix unit E_{ij}. The right translation of the general linear group GL_n on $\mathcal{A}(\mathrm{Mat}_n)$ induces the representation $\rho_{\mathfrak{gl}_n}$ of the enveloping algebra $\mathcal{U}(\mathfrak{gl}_n)$ of $\mathfrak{gl}_n = \mathfrak{gl}_n(\mathbb{C})$ as

$$\rho_{\mathfrak{gl}_n}(e_{ij}) = \sum_{k=1}^{n} x_{ki} \frac{\partial}{\partial x_{kj}}.$$

Here $\{e_{ij}\}_{1\leqslant i,j\leqslant n}$ is the standard basis of $\mathfrak{gl}_n$ so that $[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{li} e_{kj}$. It is a very basic fact that the determinant

$$\det X = \sum_{w\in\mathfrak{S}_n} (-1)^{\ell(w)} x_{w(1)1} \cdots x_{w(n)n} \in \mathcal{A}(\mathrm{Mat}_n)$$

of $X = \sum_{i,j} x_{ij} E_{ij} \in \mathrm{Mat}_n(\mathcal{A}(\mathrm{Mat}_n))$ is an invariant of the action of GL_n. In other words, we see that $\rho_{\mathfrak{gl}_n}(\mathcal{U}(\mathfrak{gl}_n)) \cdot \det X = \mathbb{C} \cdot \det X$. We also have another distinguished element called the permanent $\mathrm{per}\, X$ in $\mathcal{A}(\mathrm{Mat}_n)$ defined by

$$\mathrm{per}\, X = \sum_{w\in\mathfrak{S}_n} x_{w(1)1} \cdots x_{w(n)n}.$$

Though $\operatorname{per} X$ is not an invariant of the action, the cyclic module $\rho_{\mathfrak{gl}_n}(\mathcal{U}(\mathfrak{gl}_n)) \cdot \operatorname{per} X$ gives an irreducible representation of GL_n on the space of n-symmetric tensors of $\mathbb{C}^n$.

For a complex parameter α, the α-*determinant* is defined by

$$\det^{(\alpha)} X = \sum_{w \in \mathfrak{S}_n} \alpha^{n-\nu_n(w)} x_{w(1)1} x_{w(2)2} \cdots x_{w(n)n} \in \mathcal{A}(\mathrm{Mat}_n),$$

where $\nu_n(w)$ is the number of cycles in $w \in \mathfrak{S}_n$ [8]. We notice that $\det^{(-1)} X = \det X$ and $\det^{(1)} X = \operatorname{per} X$. Thus the α-determinant interpolates the determinant and permanent. In the representation-theoretic point of view, we can also understand that the $\mathcal{U}(\mathfrak{gl}_n)$-cyclic module generated by the α-determinant interpolates the two irreducible representations; the skew-symmetric tensor representation $\rho_{\mathfrak{gl}_n}(\mathcal{U}(\mathfrak{gl}_n)) \cdot \det X$ and symmetric tensor representation $\rho_{\mathfrak{gl}_n}(\mathcal{U}(\mathfrak{gl}_n)) \cdot \operatorname{per} X$.

Therefore it is natural to study the structure of the interpolating module $V_n^{(\alpha)} := \rho_{\mathfrak{gl}_n}(\mathcal{U}(\mathfrak{gl}_n)) \cdot \det^{(\alpha)} X$. This is done in [6]. The results are summarized as follows (see Section 4.2). The module $V_n^{(\alpha)}$ is isomorphic to the tensor module $(\mathbb{C}^n)^{\otimes n} \cong \bigoplus_\lambda E_\lambda^{\oplus f^\lambda}$ for all but finite exceptional values of α. Here E_λ denotes the irreducible highest weight module of $\mathcal{U}(\mathfrak{gl}_n)$ of highest weight λ and f^λ the multiplicity of E_λ. The isotypic component $E_\lambda^{\oplus f^\lambda}$ in $V_n^{(\alpha)}$ of highest weight λ disappears when α is a root of the (modified) *content polynomial* $c_\lambda(x)$ [5]. Further, if we consider the cyclic module $\rho_{\mathfrak{gl}_2}(\mathcal{U}(\mathfrak{gl}_2)) \cdot (\det^{(\alpha)} X)^k$ for positive integers k, we see that the disappearance of a subrepresentation (from the cyclic module in a general position) is described by the *Jacobi polynomial* with special parameters determined by k and the corresponding subrepresentation [3]. More precisely, for each irreducible representation with highest weight λ appearing in $\rho_{\mathfrak{gl}_2}(\mathcal{U}(\mathfrak{gl}_2)) \cdot (\det^{(\alpha)} X)^k$, there is a polynomial $F_k^\lambda(x)$ such that the λ-isotypic component in $\rho_{\mathfrak{gl}_2}(\mathcal{U}(\mathfrak{gl}_2)) \cdot (\det^{(\alpha)} X)^k$ is killed when α is a root of $F_k^\lambda(x)$, and the polynomial $F_k^\lambda(x)$ is given as the Jacobi polynomial whose parameters are explicitly determined by k and λ. Thus, we expect to find new families of polynomials systematically by considering the cyclic modules $\rho_{\mathfrak{gl}_n}(\mathcal{U}(\mathfrak{gl}_n)) \cdot (\det^{(\alpha)} X)^k$ for $n \geqslant 3$ as polynomials whose roots describe the degeneration of the module $\rho_{\mathfrak{gl}_n}(\mathcal{U}(\mathfrak{gl}_n)) \cdot (\det^{(\alpha)} X)^k$.

The point of the story is also that only one element $\det^{(\alpha)} X$ generates various irreducible representations with emphasizing that special polynomials describe the degeneration of the module $\rho_{\mathfrak{gl}_n}(\mathcal{U}(\mathfrak{gl}_n)) \cdot \det^{(\alpha)} X$. This is quite a *contrast* to the standard representation theory in which we construct the irreducible $\mathcal{U}(\mathfrak{gl}_n)$-modules by utilizing various minor determinants as highest weight vectors and we get various special functions as matrix coefficients of them.

The situation allows us to propose a strategy for discovering new special polynomials as polynomials which controls the structure of cyclic $\mathcal{U}(\mathfrak{gl}_n)$-modules generated by the α-determinants. Since there are several special functions such as the Jacobi (big/little) q-polynomials which we obtain as matrix coefficients of irreducible representations of *quantum groups*, it is natural to quantize the situation described above to get wider class of special polynomials.

We take the natural representation ρ of the quantum enveloping algebra $\mathcal{U}_q(\mathfrak{gl}_n)$ on the quantum matrix algebra $\mathcal{A}_q(\mathrm{Mat}_n)$. We introduce a natural quantization $\det_q^{(\alpha)} X \in \mathcal{A}_q(\mathrm{Mat}_n)$ of the α-determinant $\det^{(\alpha)}$ as a particular deformation of the quantum determinant $\det_q X$ in [2] by the formula

$$\det_q^{(\alpha)} X := \sum_{w \in \mathfrak{S}_n} \alpha^{n-\nu_n(w)} q^{\ell(w)} x_{w(1)1} \cdots x_{w(n)n} \in \mathcal{A}_q(\mathrm{Mat}_n).$$

We call this element $\det_q^{(\alpha)}$ *quantum α-determinant*. We notice that the quantum α-determinant $\det_q^{(-1)}$ for $\alpha = -1$ is nothing but the quantum determinant $\det_q$.

In the present paper, as a beginning of the study, we treat the irreducible decomposition of the cyclic module $\boldsymbol{V}_{n,q}^{(\alpha)} := \rho(\mathcal{U}_q(\mathfrak{gl}_n)) \cdot \det_q^{(\alpha)}$ and show that there are finite number of values called *singular values* such that the structure of $\boldsymbol{V}_{n,q}^{(\alpha)}$ changes drastically if α is one of such values, and the singular values are actually described as roots of some polynomials called *q-content discriminants*.

We now briefly sketch the contents of the paper below.

The basic conventions on the quantum matrix algebra $\mathcal{A}_q(\mathrm{Mat}_n)$, the quantum enveloping algebra $\mathcal{U}_q(\mathfrak{gl}_n)$ (as a quantum group) and the Iwahori–Hecke algebra $\mathcal{H}_q(\mathfrak{S}_n)$ are collected briefly in Section 2.

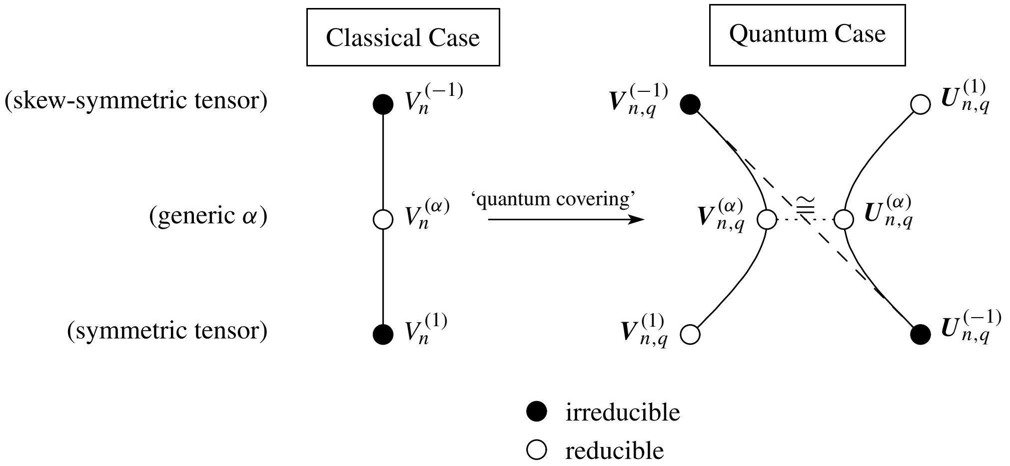
In Section 3, we investigate several basic properties of the cyclic module $\boldsymbol{V}_{n,q}^{(\alpha)}$. As a starting point, we show in Proposition 3.4 that $\boldsymbol{V}_{n,q}^{(\alpha)}$ is equivalent to the tensor product module $(\mathbb{C}^n)^{\otimes n}$ for all but finite α. Namely, for almost all values α, the multiplicity $m_q^\lambda(\alpha)$ of the highest weight module $\boldsymbol{E}_{n,q}^\lambda$ corresponding to a highest weight (= partition) λ in $\boldsymbol{V}_{n,q}^{(\alpha)}$ is equal to the number f^λ of standard tableaux with shape λ. If α is one of those finite exceptions, that is, if $m_q^\lambda(\alpha) < f^\lambda$ holds, we call it a singular point (or value) as we mentioned above. To describe highest weight vectors of the irreducible factors of the decomposition of $\boldsymbol{V}_{n,q}^{(\alpha)}$ in terms of the quantum α-determinants, we employ the q-Young symmetrizers studied in [1].

The degeneration of the cyclic module $\boldsymbol{V}_{n,q}^{(\alpha)}$ is discussed in Section 4. For each λ, a $f^\lambda \times f^\lambda$ matrix called a q-content transition matrix and a polynomial in α called a q-content discriminant are introduced. The zeros of the q-content discriminants are the singular values. In contrast with the classical theory developed in [6], the q-Young symmetrizers cannot give us enough information about the zeros of q-content discriminants, and the explicit description of the zeros of these polynomials seems to be a far reaching problem. This is the most difficult point in the present study which we have never encountered in the classical situation [6].

In the classical theory, the content transition matrix is a scalar one whose scalar is given by the so-called content polynomial (see [5]). It follows hence that only $\pm\frac{1}{k}$ $(1 \leqslant k < n)$ are the singular points and the degeneration $m_1^\lambda(\pm\frac{1}{k}) < f^\lambda$ implies the vanishing $m_1^\lambda(\pm\frac{1}{k}) = 0$. In the quantization, the singular points $-1, -\frac{1}{2}, \dots, -\frac{1}{n-1}$ remains singular, whereas the points $1, \frac{1}{2}, \dots, \frac{1}{n-1}$ themselves are no longer singular but are q-deformed, and even new values (depending on q) other than $2n-2$ singular values above come up as extra singular points. We call the first member of singular points *classical*, the second one *semi-classical* and the last one *quantum*, respectively (see Section 4.3). We devote ourselves to investigate such singular points in the latter half of the section and give one conjecture concerning the multiplicity degeneration for singular values (Conjecture A, see also Theorem 4.11). Indeed, for our quantum case, it occurs that $0 < m_q^\lambda(\alpha) < f^\lambda$ for a quantum singular point α. We furthermore translate this conjecture into the framework of the bimodule of $\mathcal{U}_q(\mathfrak{gl}_n)$ and $\mathcal{H}_q(\mathfrak{S}_n)$ in terms of Schur–Weyl duality [1,2]. We also provide explicit calculations of some q-content discriminants for readers' help. It would be also interesting to study the relation between the multiplicity $m_q^\lambda(\alpha)$ and the multiplicity of α as a root of the corresponding q-content discriminant. We treat this subject in the future study.

In the last section, we introduce a notion of quantum α-permanent $\mathrm{per}_q^{(\alpha)} = \det_{q^{-1}}^{(-\alpha)}$ and discuss shortly the $\mathcal{U}_q(\mathfrak{gl}_n)$-cyclic module $\boldsymbol{U}_{n,q}^{(\alpha)}$ generated by $\mathrm{per}_q^{(\alpha)}$ through examples. In the classical case, as we mentioned above, the $\mathcal{U}(\mathfrak{gl}_n)$-cyclic module generated by the α-determinant interpolates the two irreducible representations $\rho_{\mathfrak{gl}_n}(\mathcal{U}(\mathfrak{gl}_n)) \cdot \det^{(-1)} X$ and $\rho_{\mathfrak{gl}_n}(\mathcal{U}(\mathfrak{gl}_n)) \cdot \det^{(1)} X$.

Whereas, in the quantum situation, the cyclic module $\boldsymbol{V}_{n,q}^{(\alpha)}$ is irreducible (skew-symmetric tensor representation) at $\alpha = -1$ but not irreducible at $\alpha = 1$, as well as the cyclic module $\boldsymbol{U}_{n,q}^{(\alpha)}$ is irreducible (symmetric tensor representation) at $\alpha = -1$ but not irreducible at $\alpha = 1$.



By looking at this 'quantum covering structure' of interpolating property as well as examining examples (Examples 3.11, 3.12 and 5.2, 5.3, respectively), we infer an existence of 'reciprocity' between cyclic modules generated by $\det_q^{(\alpha)}$ and $\mathrm{per}_q^{(\alpha)}$. Based on this observation, we establish another conjecture (Conjecture B) which describes a 'reciprocity' between the multiplicities of the irreducible summands of the cyclic modules $\boldsymbol{V}_{n,q}^{(\alpha)}$ and $\boldsymbol{U}_{n,q}^{(\alpha)}$. In the very final section of the paper, we introduce the partition functions as respective generating functions

$$\vartheta_q^{\det}(t,\alpha) := \sum_{\lambda:\ \text{partitions}} \frac{m_q^\lambda(\alpha)}{f^\lambda} t^{|\lambda|} = \sum_{n=0}^{\infty}\sum_{\lambda\vdash n} \frac{m_q^\lambda(\alpha)}{f^\lambda} t^n,$$

$$\vartheta_q^{\mathrm{per}}(t,\alpha) := \sum_{\lambda:\ \text{partitions}} \frac{m_q^\lambda(\alpha)_{\mathrm{per}}}{f^\lambda} t^{|\lambda|} = \sum_{n=0}^{\infty}\sum_{\lambda\vdash n} \frac{m_q^\lambda(\alpha)_{\mathrm{per}}}{f^\lambda} t^n$$

of the multiplicities and restate a weaker version of the two conjectures above in terms of the partition functions (here $m_q^\lambda(\alpha)_{\mathrm{per}}$ denotes the multiplicity of $\boldsymbol{E}_{n,q}^\lambda$ in $\boldsymbol{U}_{n,q}^{(\alpha)}$). If α is not singular, then $\vartheta_q^{\det}(t,\alpha)$ is identical with the generating function $\prod_{n=1}^\infty (1-t^n)^{-1}$ of the number of partitions.

Since one can obtain neither the zeros of q-content discriminants nor the discriminants themselves explicitly, in order to understand a deeper structure of the q-content discriminants as polynomials, it might be an inevitable task to characterize the q-content transition matrices in a suitable way, e.g. like in a R-matrix formalism. Although the situation is quite mysterious, in Proposition 3.8, we describe a symmetric structure of some matrix (see Lemma 4.2) closely related to the q-content transition matrices.

Conventions

Throughout the paper, $\mathbb{C}$ is the complex number field and $\mathbb{Z}$ is the ring of rational integers. The symbol q denotes a nonzero complex number and we always fix a branch for the square root $q^{1/2}$. We also assume that q is not a root of unity to assure the complete reducibility of the finite dimensional representations of $\mathcal{U}_q(\mathfrak{gl}_n)$ and $\mathcal{H}_q(\mathfrak{S}_n)$ as well as to utilize the highest weight theory of $\mathcal{U}_q(\mathfrak{gl}_n)$. (We will further suppose that q is 'generic' to make the discussion simple. See the end of Section 3.)

We denote by $\mathfrak{S}_n$ the symmetric group of degree n. The simple transposition $(k, k+1)$ is denoted by s_k. For $w \in \mathfrak{S}_n$, $\ell(w)$ is the inversion number of w and $\nu_n(w)$ is the number of cycles in w. Notice that $\nu_n(\cdot)$ is a class function on $\mathfrak{S}_n$, but $\ell(\cdot)$ is not.

For a partition (or a Young diagram) $\lambda = (\lambda_1, \dots, \lambda_k)$ of n, $\mathfrak{S}_\lambda := \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_k}$ is the Young subgroup of $\mathfrak{S}_n$. We also denote by $\mathrm{STab}(\lambda)$ the set of all standard tableaux with shape λ, and put $f^\lambda = |\mathrm{STab}(\lambda)|$.

2. Preliminaries on representations of quantum groups

We briefly recall the basic notion of the quantum enveloping algebra $\mathcal{U}_q(\mathfrak{gl}_n)$, the quantum matrix algebra $\mathcal{A}_q(\mathrm{Mat}_n)$, and the Iwahori–Hecke algebra $\mathcal{H}_q(\mathfrak{S}_n)$ to fix the conventions. The conventions on quantum groups are the same as in Noumi–Yamada–Mimachi [7] and Jimbo [2], but the conventions for the Iwahori–Hecke algebra are slightly different from those in Gyoja [1] because of the compatibility with the conventions on quantum algebras.

2.1. *Quantum enveloping algebra*

By definition, $\mathcal{L}_n$ is a $\mathbb{Z}$-module $\mathcal{L}_n := \mathbb{Z}\varepsilon_1 + \cdots + \mathbb{Z}\varepsilon_n$ generated by the symbols $\varepsilon_1, \dots, \varepsilon_n$. We fix a bilinear form $\langle\cdot,\cdot\rangle$ on $\mathcal{L}_n$ defined by $\langle\varepsilon_i, \varepsilon_j\rangle = \delta_{ij}$. Each element in the lattice $\mathcal{L}_n$ is called an *integral weight*. The quantum enveloping algebra $\mathcal{U}_q(\mathfrak{gl}_n)$ is a $\mathbb{C}$-associative algebra generated by the symbols e_i, f_i $(1 \leqslant i \leqslant n-1)$ and q^λ $(\lambda \in \frac{1}{2}\mathcal{L}_n)$ satisfying certain fundamental relations (see [7]). The algebra $\mathcal{U}_q(\mathfrak{gl}_n)$ has a Hopf algebra structure with the coproducts

$$\begin{aligned}
\Delta\big(q^\lambda\big) &= q^\lambda \otimes q^\lambda, \\
\Delta(e_k) &= e_k \otimes q^{-(\varepsilon_k - \varepsilon_{k+1})/2} + q^{(\varepsilon_k - \varepsilon_{k+1})/2} \otimes e_k, \\
\Delta(f_k) &= f_k \otimes q^{-(\varepsilon_k - \varepsilon_{k+1})/2} + q^{(\varepsilon_k - \varepsilon_{k+1})/2} \otimes f_k.
\end{aligned}$$

The *vector representation* $\rho_{\mathbb{C}^n}$ of $\mathcal{U}_q(\mathfrak{gl}_n)$ on $\mathbb{C}^n$ is defined by

$$\rho_{\mathbb{C}^n}\big(q^\lambda\big) \cdot \boldsymbol{e}_j = q^{\langle\lambda, \varepsilon_j\rangle}\boldsymbol{e}_j, \qquad \rho_{\mathbb{C}^n}(e_k) \cdot \boldsymbol{e}_j = \delta_{j,k+1}\boldsymbol{e}_k, \qquad \rho_{\mathbb{C}^n}(f_k) \cdot \boldsymbol{e}_j = \delta_{jk}\boldsymbol{e}_{k+1},$$

where $\{\boldsymbol{e}_j\}_{1 \leqslant j \leqslant n}$ is the standard basis of $\mathbb{C}^n$. By the coproduct on $\mathcal{U}_q(\mathfrak{gl}_n)$, the tensor product representation $\rho_{\mathbb{C}^n}^{\otimes n}$ of $\mathcal{U}_q(\mathfrak{gl}_n)$ on $(\mathbb{C}^n)^{\otimes n}$ is given by

$$\begin{aligned}
\rho_{\mathbb{C}^n}^{\otimes n}\big(q^\lambda\big) \cdot \boldsymbol{e}_{j_1} \otimes \cdots \otimes \boldsymbol{e}_{j_n} &= q^{\langle\lambda, \varepsilon_{j_1} + \cdots + \varepsilon_{j_n}\rangle}\boldsymbol{e}_{j_1} \otimes \cdots \otimes \boldsymbol{e}_{j_n}, \\
\rho_{\mathbb{C}^n}^{\otimes n}(e_k) \cdot \boldsymbol{e}_{j_1} \otimes \cdots \otimes \boldsymbol{e}_{j_n} &= \sum_{l=1}^{n} \delta_{j_l,k+1} q_k^l(j_1, \dots, j_n)\boldsymbol{e}_{j_1} \otimes \cdots \otimes \boldsymbol{e}_{j_{l-1}} \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_{j_{l+1}} \otimes \cdots \otimes \boldsymbol{e}_{j_n}, \\
\rho_{\mathbb{C}^n}^{\otimes n}(f_k) \cdot \boldsymbol{e}_{j_1} \otimes \cdots \otimes \boldsymbol{e}_{j_n} &= \sum_{l=1}^{n} \delta_{j_l,k} q_k^l(j_1, \dots, j_n)\boldsymbol{e}_{j_1} \otimes \cdots \otimes \boldsymbol{e}_{j_{l-1}} \otimes \boldsymbol{e}_{k+1} \otimes \boldsymbol{e}_{j_{l+1}} \otimes \cdots \otimes \boldsymbol{e}_{j_n},
\end{aligned} \tag{2.1}$$

where we put $q_k^l(j_1, \dots, j_n) := q^{\langle(\varepsilon_k - \varepsilon_{k+1})/2, \varepsilon_{j_1} + \cdots + \varepsilon_{j_{l-1}} - \varepsilon_{j_{l+1}} - \cdots - \varepsilon_{j_n}\rangle}$ for simplicity.

Each finite dimensional irreducible $\mathcal{U}_q(\mathfrak{gl}_n)$-module is a highest weight module, and it is parametrized by a *dominant* integral weight, that is, an integral weight $\lambda_1\varepsilon_1+\cdots+\lambda_n\varepsilon_n$ with the property $\lambda_1\geqslant\cdots\geqslant\lambda_n\geqslant 0$. We identify a dominant integral weight λ with a partition (or a Young diagram) $(\lambda_1,\dots,\lambda_n)$. We often denote by the same symbol λ to indicate both the weight $\lambda_1\varepsilon_1+\cdots+\lambda_n\varepsilon_n$ and the partition $(\lambda_1,\dots,\lambda_n)$. The highest weight $\mathcal{U}_q(\mathfrak{gl}_n)$-module corresponding to λ is denoted by $\boldsymbol{E}_{n,q}^{\lambda}$.

2.2. Quantum matrix algebra

The quantum matrix algebra $\mathcal{A}_q(\mathrm{Mat}_n)$ is a $\mathbb{C}$-associative algebra generated by n^2 letters x_{ij} $(1\leqslant i,j\leqslant n)$ obeying the following fundamental relations

$$\begin{aligned}
&x_{ik}x_{jk}=qx_{jk}x_{ik}, \qquad x_{ki}x_{kj}=qx_{kj}x_{ki} \quad (i<j),\\
&x_{il}x_{jk}=x_{jk}x_{il}, \qquad x_{ik}x_{jl}-x_{jl}x_{ik}=\bigl(q-q^{-1}\bigr)x_{il}x_{jk} \quad (i<j,k<l).
\end{aligned}$$

The algebra $\mathcal{A}_q(\mathrm{Mat}_n)$ becomes a bialgebra having the coproduct

$$\Delta(x_{ij})=\sum_{k=1}^{n}x_{ik}\otimes x_{kj}.$$

The algebra $\mathcal{A}_q(\mathrm{Mat}_n)$ becomes a left $\mathcal{U}_q(\mathfrak{gl}_n)$-module by

$$\rho\bigl(q^{\lambda}\bigr)\cdot x_{ij}=q^{\langle\lambda,\varepsilon_j\rangle}x_{ij}, \qquad \rho(e_k)\cdot x_{ij}=\delta_{j,k+1}x_{ik}, \qquad \rho(f_k)\cdot x_{ij}=\delta_{jk}x_{i,k+1}.$$

Using the coproduct of $\mathcal{U}_q(\mathfrak{gl}_n)$, (via the tensor product representation) we have

$$\begin{aligned}
&\rho\bigl(q^{\lambda}\bigr)\cdot x_{i_1j_1}\cdots x_{i_nj_n}=q^{\langle\lambda,\varepsilon_{j_1}+\cdots+\varepsilon_{j_n}\rangle}x_{i_1j_1}\cdots x_{i_nj_n},\\
&\rho(e_k)\cdot x_{i_1j_1}\cdots x_{i_lj_l}\cdots x_{i_nj_n}=\sum_{l=1}^{n}\delta_{j_l,k+1}\cdot q_k^l(j_1,\dots,j_n)\cdot x_{i_1j_1}\cdots x_{i_lk}\cdots x_{i_nj_n},\\
&\rho(f_k)\cdot x_{i_1j_1}\cdots x_{i_lj_l}\cdots x_{i_nj_n}=\sum_{l=1}^{n}\delta_{j_l,k}\cdot q_k^l(j_1,\dots,j_n)\cdot x_{i_1j_1}\cdots x_{i_l,k+1}\cdots x_{i_nj_n}. \qquad (2.2)
\end{aligned}$$

Notice that the $\mathcal{U}_q(\mathfrak{gl}_n)$-submodule $\bigoplus_{j=1}^{n}\mathbb{C}\cdot x_{ij}$ $(i=1,\dots,n)$ is equivalent to the vector representation $\mathbb{C}^n$.

2.3. Iwahori–Hecke algebra and Schur–Weyl type duality

The Iwahori–Hecke algebra $\mathcal{H}_q(\mathfrak{S}_n)$ is an associative $\mathbb{C}$-algebra generated by the symbols h_i $(1\leqslant i\leqslant n-1)$ with the fundamental relations

$$\begin{aligned}
&h_ih_{i+1}h_i=h_{i+1}h_ih_{i+1} \quad (1\leqslant i\leqslant n-2),\\
&h_ih_j=h_jh_i \quad \bigl(|i-j|\geqslant 2\bigr),\\
&(h_i+q)\bigl(h_i-q^{-1}\bigr)=0 \quad (1\leqslant i\leqslant n-1).
\end{aligned}$$

If $w = s_{i_1} \cdots s_{i_l}$ is the shortest expression of $w \in \mathfrak{S}_n$, then we put $h_w = h_{i_1} \cdots h_{i_l}$. The elements h_w for $w \in \mathfrak{S}_n$ form a basis of $\mathcal{H}_q(\mathfrak{S}_n)$ as a vector space. The algebra $\mathcal{H}_q(\mathfrak{S}_n)$ acts on $(\mathbb{C}^n)^{\otimes n}$ by

$$\begin{aligned} &\boldsymbol{e}_{j_1} \otimes \cdots \otimes \boldsymbol{e}_{j_n} \cdot \pi(h_k) \\ &\quad = \begin{cases} \boldsymbol{e}_{j_1} \otimes \cdots \otimes \boldsymbol{e}_{j_{k+1}} \otimes \boldsymbol{e}_{j_k} \otimes \cdots \otimes \boldsymbol{e}_{j_n}, & j_k < j_{k+1}, \\ q^{-1} \boldsymbol{e}_{j_1} \otimes \cdots \otimes \boldsymbol{e}_{j_n}, & j_k = j_{k+1}, \\ \boldsymbol{e}_{j_1} \otimes \cdots \otimes \boldsymbol{e}_{j_{k+1}} \otimes \boldsymbol{e}_{j_k} \otimes \cdots \otimes \boldsymbol{e}_{j_n} - (q - q^{-1}) \boldsymbol{e}_{j_1} \otimes \cdots \otimes \boldsymbol{e}_{j_n}, & j_k > j_{k+1}. \end{cases} \end{aligned} \tag{2.3}$$

The subalgebra $\pi(\mathcal{H}_q(\mathfrak{S}_n))$ is the commutant of $\rho_{\mathbb{C}^n}^{\otimes n}(\mathcal{U}_q(\mathfrak{gl}_n))$ in $\mathrm{End}((\mathbb{C}^n)^{\otimes n})$, and vice versa (see, e.g. [2]). We have consequently the decomposition

$$\left(\mathbb{C}^n\right)^{\otimes n} \cong \bigoplus_{\lambda \vdash n} \left(\boldsymbol{E}_{n,q}^{\lambda}\right)^{\oplus f^{\lambda}}$$

as a $\mathcal{U}_q(\mathfrak{gl}_n)$-module. This fact is referred to as *Schur–Weyl duality*.

For a given Young diagram $\lambda \vdash n$, we define

$$e_+ = e_+(\lambda) := \sum_{w \in W_+(\lambda)} q^{-\ell(w)} h_w, \qquad e_- = e_-(\lambda) := \sum_{w \in W_-(\lambda)} (-q)^{\ell(w)} h_w,$$

where $W_{\pm}(\lambda)$ are certain subgroups of $\mathfrak{S}_n$ (see [1] for definition). These satisfy the equations

$$e_+^2 = \left(\sum_{w \in W_+(\lambda)} q^{-2\ell(w)} \right) e_+, \qquad e_-^2 = \left(\sum_{w \in W_-(\lambda)} (-q)^{2\ell(w)} \right) e_-.$$

Using $e_{\pm}$, we can define the q-Young symmetrizer $\mathbb{E}_q(T) \in \mathcal{H}_q(\mathfrak{S}_n)$ for each $T \in \mathrm{STab}(\lambda)$. We refer to Gyoja [1] for precise and detailed information on q-Young symmetrizers, and we only give several examples here.

Example 2.1. The q-Young symmetrizers for 3-box standard tableaux are given by

$$\begin{aligned} \mathbb{E}_q\left(\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\right) &= e_+\left(\begin{array}{|c|c|c|}\hline \ &\ &\ \\\hline\end{array}\right) = 1 + q^{-1} h_1 + q^{-1} h_2 + q^{-3} h_1 h_2 h_1 + q^{-2} h_1 h_2 + q^{-2} h_2 h_1, \\ \mathbb{E}_q\left(\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\right) &= h_2 e_-\left(\begin{array}{|c|c|}\hline \ &\ \\\hline \ \\\cline{1-1}\end{array}\right) h_2^{-1} e_+\left(\begin{array}{|c|c|}\hline \ &\ \\\hline \ \\\cline{1-1}\end{array}\right) = 1 + q^{-1} h_1 - \left(q - q^{-1}\right) h_2 - q^{-1} h_1 h_2 h_1 - h_1 h_2 \\ &\quad - \left(1 - q^{-2}\right) h_2 h_1, \\ \mathbb{E}_q\left(\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\right) &= e_-\left(\begin{array}{|c|c|}\hline \ &\ \\\hline \ \\\cline{1-1}\end{array}\right) h_2 e_+\left(\begin{array}{|c|c|}\hline \ &\ \\\hline \ \\\cline{1-1}\end{array}\right) h_2^{-1} = 1 - q h_1 - q^{-2} h_2 h_1 + q^{-1} h_1 h_2 h_1, \\ \mathbb{E}_q\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\right) &= e_-\left(\begin{array}{|c|}\hline \ \\\hline \ \\\hline \ \\\hline\end{array}\right) = 1 - q h_1 - q h_2 - q^3 h_1 h_2 h_1 + q^2 h_1 h_2 + q^2 h_2 h_1. \end{aligned}$$

Example 2.2. For spaces of q-symmetric tensors and q-skew-symmetric tensors representations, we have

$$\mathbb{E}_q\left(\begin{array}{|c|c|c|c|}\hline 1&2&\cdot&n\\\hline\end{array}\right) = e_+((n)) = \sum_{w \in \mathfrak{S}_n} q^{-\ell(w)} h_w, \qquad \mathbb{E}_q\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline \cdot\\\hline n\\\hline\end{array}\right) = e_-((1, \ldots, 1)) = \sum_{w \in \mathfrak{S}_n} (-q)^{\ell(w)} h_w$$

in general.

3. Quantum α-determinant

We now introduce a *quantum* (*column*) α*-determinant* $\det_q^{(\alpha)}$ by

$$\det_q^{(\alpha)} := \sum_{w\in\mathfrak{S}_n} \alpha^{n-\nu_n(w)} q^{\ell(w)} x_{w(1)1}\cdots x_{w(n)n} \in \mathcal{A}_q(\mathrm{Mat}_n).$$

Since $(-1)^{n-\nu_n(w)} = (-1)^{\ell(w)}$, quantum (-1)-determinant $\det_q^{(-1)}$ agrees with the quantum determinant $\det_q$. We also remark that $\det_q^{(\alpha)}(X) = \det_q^{(\alpha)}({}^tX)$, which follows from the fact that $\nu_n(w) = \nu_n(w^{-1})$, $\ell(w) = \ell(w^{-1})$ and $x_{w(1)1}\cdots x_{w(n)n} = x_{1w^{-1}(1)}\cdots x_{nw^{-1}(n)}$ for any $w \in \mathfrak{S}_n$. For the sake of convenience, we write

$$D_q^{(\alpha)}(j_1,\dots,j_n) := \sum_{w\in\mathfrak{S}_n} \alpha^{n-\nu_n(w)} q^{\ell(w)} x_{w(1),j_1}\cdots x_{w(n),j_n},$$

for $1 \leqslant j_1,\dots,j_n \leqslant n$. We notice that $\det_q^{(\alpha)} = D_q^{(\alpha)}(1,2,\dots,n)$.

3.1. Quantum α-determinant cyclic modules $\boldsymbol{V}_{n,q}^{(\alpha)}$

We are interested in the $\mathcal{U}_q(\mathfrak{gl}_n)$-module $\boldsymbol{V}_{n,q}^{(\alpha)} := \rho(\mathcal{U}_q(\mathfrak{gl}_n)) \cdot \det_q^{(\alpha)}$. The basic fact is that every quantum α-determinant $D_q^{(\alpha)}(j_1,\dots,j_n)$ is contained in $\boldsymbol{V}_{n,q}^{(\alpha)}$ (Proposition 3.3). To prove this, we first notice the following

Lemma 3.1. *The equalities*

$$\rho(q^\lambda)\cdot D_q^{(\alpha)}(j_1,\dots,j_n) = q^{\langle\lambda,\varepsilon_{j_1}+\cdots+\varepsilon_{j_n}\rangle} D_q^{(\alpha)}(j_1,\dots,j_n),$$

$$\rho(e_k)\cdot D_q^{(\alpha)}(j_1,\dots,j_n) = \sum_{l=1}^n \delta_{j_l,k+1} q_k^l(j_1,\dots,j_n) D_q^{(\alpha)}(j_1,\dots,j_{l-1},k,j_{l+1},\dots,j_n),$$

$$\rho(f_k)\cdot D_q^{(\alpha)}(j_1,\dots,j_n) = \sum_{l=1}^n \delta_{j_l,k} q_k^l(j_1,\dots,j_n) D_q^{(\alpha)}(j_1,\dots,j_{l-1},k+1,j_{l+1},\dots,j_n)$$

hold.

Proof. By (2.2), the assertion is verified by a straightforward calculation. □

We give some instructive example which may indicate a highest weight vector of a irreducible representation (see Section 3.3).

Example 3.2. We have

$$\rho(e_1)\cdot D_q^{(\alpha)}(1,1,2) = q_1^3(1,1,2) D_q^{(\alpha)}(1,1,1) = q^{\langle(\varepsilon_1-\varepsilon_2)/2,\varepsilon_1+\varepsilon_1\rangle} D_q^{(\alpha)}(1,1,1)$$
$$= q D_q^{(\alpha)}(1,1,1),$$

$$\rho(e_1)\cdot D_q^{(\alpha)}(1,2,1)=q_1^2(1,2,1)D_q^{(\alpha)}(1,1,1)=q^{\langle(\varepsilon_1-\varepsilon_2)/2,\varepsilon_1-\varepsilon_1\rangle}D_q^{(\alpha)}(1,1,1)=D_q^{(\alpha)}(1,1,1),$$
$$\rho(e_1)\cdot D_q^{(\alpha)}(2,1,1)=q_1^1(2,1,1)D_q^{(\alpha)}(1,1,1)=q^{\langle(\varepsilon_1-\varepsilon_2)/2,-\varepsilon_1-\varepsilon_1\rangle}D_q^{(\alpha)}(1,1,1)$$
$$=q^{-1}D_q^{(\alpha)}(1,1,1),$$

and hence, we conclude that

$$\rho(e_1)\cdot\big(D_q^{(\alpha)}(1,1,2)-qD_q^{(\alpha)}(1,2,1)\big)=\rho(e_1)\cdot\big(D_q^{(\alpha)}(1,2,1)-qD_q^{(\alpha)}(2,1,1)\big)=0.$$

These two vectors are also killed by $\rho(e_2)$ trivially.

Proposition 3.3 *(Quantum analogue of [6, Lemma 2.2]). The equality*

$$\boldsymbol{V}_{n,q}^{(\alpha)}=\sum_{1\leqslant j_1,\dots,j_n\leqslant n}\mathbb{C}\cdot D_q^{(\alpha)}(j_1,\dots,j_n) \tag{3.1}$$

holds.

Proof. Let $\boldsymbol{L}_{n,q}^{(\alpha)}$ be the right-hand side of (3.1). By Lemma 3.1, $\boldsymbol{L}_{n,q}^{(\alpha)}$ is $\rho(\mathcal{U}_q(\mathfrak{gl}_n))$-invariant and $\boldsymbol{V}_{n,q}^{(\alpha)}\subset\boldsymbol{L}_{n,q}^{(\alpha)}$. To prove the opposite inclusion $\boldsymbol{V}_{n,q}^{(\alpha)}\supset\boldsymbol{L}_{n,q}^{(\alpha)}$, we introduce the linear map

$$\Phi_{n,q}^{(\alpha)}:\big(\mathbb{C}^n\big)^{\otimes n}\ni\boldsymbol{e}_{j_1}\otimes\cdots\otimes\boldsymbol{e}_{j_n}\mapsto D_q^{(\alpha)}(j_1,\dots,j_n)\in\boldsymbol{L}_{n,q}^{(\alpha)}\quad(1\leqslant j_1,\dots,j_n\leqslant n). \tag{3.2}$$

By Lemma 3.1 again and the formula (2.1), we find that $\Phi_{n,q}^{(\alpha)}$ defines a surjective $\mathcal{U}_q(\mathfrak{gl}_n)$-intertwiner such that $\Phi_{n,q}^{(\alpha)}(\boldsymbol{e}_1\otimes\cdots\otimes\boldsymbol{e}_n)=\det_q^{(\alpha)}$. Using the elementary fact that

$$\big(\mathbb{C}^n\big)^{\otimes n}=\rho_{\mathbb{C}^n}^{\otimes n}\big(\mathcal{U}_q(\mathfrak{gl}_n)\big)\cdot\boldsymbol{e}_1\otimes\cdots\otimes\boldsymbol{e}_n,$$

we have

$$\boldsymbol{V}_{n,q}^{(\alpha)}=\rho\big(\mathcal{U}_q(\mathfrak{gl}_n)\big)\cdot\det_q^{(\alpha)}=\Phi_{n,q}^{(\alpha)}\big(\rho_{\mathbb{C}^n}^{\otimes n}\big(\mathcal{U}_q(\mathfrak{gl}_n)\big)\cdot\boldsymbol{e}_1\otimes\cdots\otimes\boldsymbol{e}_n\big)=\Phi_{n,q}^{(\alpha)}\big(\big(\mathbb{C}^n\big)^{\otimes n}\big)\supset\boldsymbol{L}_{n,q}^{(\alpha)}.$$

This completes the proof. □

By Proposition 3.3 and the surjectivity of the intertwiner $\Phi_{n,q}^{(\alpha)}$ defined in (3.2), the cyclic $\mathcal{U}_q(\mathfrak{gl}_n)$-module $\boldsymbol{V}_{n,q}^{(\alpha)}$ is isomorphic to the tensor product module $(\mathbb{C}^n)^{\otimes n}$ if and only if the intertwiner $\Phi_{n,q}^{(\alpha)}$ is bijective, that is, the α-determinants $D_q^{(\alpha)}(j_1,\dots,j_n)$ are linearly independent. Namely, we have the following basic result.

Proposition 3.4. *Put*

$$\mathrm{Sing}_{n,q}:=\big\{\alpha\in\mathbb{C}\ \big|\ D_q^{(\alpha)}(j_1,\dots,j_n)\ \textit{are linearly dependent}\big\}.$$

If $\alpha\in\mathbb{C}\setminus\mathrm{Sing}_{n,q}$, then the irreducible decomposition of $\boldsymbol{V}_{n,q}^{(\alpha)}$ is given as

$$\boldsymbol{V}_{n,q}^{(\alpha)}\cong\bigoplus_{\lambda\vdash n}\big(\boldsymbol{E}_{n,q}^{\lambda}\big)^{\oplus f^\lambda}.$$

In other words, the multiplicity $m_q^\lambda(\alpha)$ *of the irreducible representation* $\boldsymbol{E}_{n,q}^\lambda$ *in* $\boldsymbol{V}_{n,q}^{(\alpha)}$ *is equal to* f^λ.

Let us look at the weight space decomposition of $\boldsymbol{V}_{n,q}^{(\alpha)}$. For each $\boldsymbol{j}=(j_1,\dots,j_n)\in\{1,\dots,n\}^n$, we associate an integral weight

$$\mathrm{wt}(\boldsymbol{j})=\sum_{i=1}^n \varepsilon_{j_i}=\sum_{i=1}^n \nu_i\varepsilon_i\in\mathcal{L}_n\quad\bigl(\nu_k=\bigl|\{l;\,j_l=k\}\bigr|\bigr).$$

Lemma 3.1 says that $D_q^{(\alpha)}(\boldsymbol{j})=D_q^{(\alpha)}(j_1,\dots,j_n)$ is a weight vector of weight $\mathrm{wt}(\boldsymbol{j})$. For convenience, we put

$$\begin{aligned}\widetilde{\mathcal{L}}_n&:=\{\nu=\nu_1\varepsilon_1+\dots+\nu_n\varepsilon_n\in\mathcal{L}_n\mid \nu_j\geqslant 0,\ \nu_1+\dots+\nu_n=n\},\\ I_n(\lambda)&:=\bigl\{\boldsymbol{i}\in\{1,\dots,n\}^n\bigm|\mathrm{wt}(\boldsymbol{i})=\lambda\bigr\}\quad(\lambda\in\widetilde{\mathcal{L}}_n).\end{aligned}$$

By Lemma 3.1, we have the

Lemma 3.5. *For an integral weight* $\nu\in\widetilde{\mathcal{L}}_n$*, define the subspace*

$$\boldsymbol{V}_{n,q}^{(\alpha)}(\nu):=\sum_{\boldsymbol{i}\in I_n(\nu)}\mathbb{C}\cdot D_q^{(\alpha)}(\boldsymbol{i})$$

consisting of all weight vectors of weight ν*. Then the following decomposition*

$$\boldsymbol{V}_{n,q}^{(\alpha)}=\bigoplus_{\nu\in\widetilde{\mathcal{L}}_n}\boldsymbol{V}_{n,q}^{(\alpha)}(\nu)$$

holds.

3.2. Singular points for the decomposition

By Proposition 3.4, what remains important is to study the set $\mathrm{Sing}_{n,q}$ and determine the irreducible decomposition of $\boldsymbol{V}_{n,q}^{(\alpha)}$ for $\alpha\in\mathrm{Sing}_{n,q}$. We first give another description of $\mathrm{Sing}_{n,q}$ as a set of zeros of a certain polynomial defined below.

We notice that $D_q^{(0)}(j_1,\dots,j_n)$ is nothing but the monomial $x_{1j_1}\cdots x_{nj_n}$. It hence follows that the vectors $D_q^{(0)}(j_1,\dots,j_n)$ are linearly independent (i.e. $0\notin\mathrm{Sing}_{n,q}$) and each α-determinant $D_q^{(\alpha)}(j_1,\dots,j_n)$ is a linear combination of the monomials $D_q^{(0)}(j_1,\dots,j_n)$, say

$$D_q^{(\alpha)}(j_1,\dots,j_n)=\sum_{1\leqslant i_1,\dots,i_n\leqslant n}\widetilde{F}_{n,q}(\alpha;i_1,\dots,i_n;j_1,\dots,j_n)D_q^{(0)}(i_1,\dots,i_n)$$

for some $\widetilde{F}_{n,q}(\alpha;i_1,\dots,i_n;j_1,\dots,j_n)\in\mathbb{C}$. It is immediate to see that each $\widetilde{F}_{n,q}(\alpha;\boldsymbol{i};\boldsymbol{j})$ ($\boldsymbol{i},\boldsymbol{j}\in\{1,2,\dots,n\}^n$) is a polynomial in α and q with integral coefficients.

Consider the $n^n \times n^n$ matrix $\widetilde{F}_{n,q}(\alpha) := (\widetilde{F}_{n,q}(\alpha; \boldsymbol{i}; \boldsymbol{j}))_{\boldsymbol{i},\boldsymbol{j}\in\{1,\dots,n\}^n}$. The determinant $\widetilde{C}_{n,q}(\alpha) := \det \widetilde{F}_{n,q}(\alpha)$ is a polynomial in α and q with integral coefficients, and it is not identically zero because $\widetilde{F}_{n,q}(0)$ is the identity matrix. Thus we have the

Lemma 3.6. *The set* $\mathrm{Sing}_{n,q}$ *is given by*

$$\mathrm{Sing}_{n,q} = \{\alpha \in \mathbb{C} \mid \widetilde{C}_{n,q}(\alpha) = 0\}.$$

In particular, $\mathrm{Sing}_{n,q}$ *is a finite set.*

The cardinality $|\mathrm{Sing}_{n,q}|$ does depend on the parameter q. In what follows, for simplicity, we further impose an assumption on the parameter q that the value q maximize $|\mathrm{Sing}_{n,q}|$ as a function in q. (It is sufficient to assume that q is transcendental, for instance.)

Let us put

$$\widetilde{F}_q^{\lambda}(\alpha) := (\widetilde{F}_{n,q}(\alpha; \boldsymbol{i}; \boldsymbol{j}))_{\boldsymbol{i},\boldsymbol{j}\in I_n(\lambda)}$$

for an integral weight $\lambda \in \widetilde{\mathcal{L}}_n$. Then, by Lemma 3.5, the matrix $\widetilde{F}_{n,q}(\alpha)$ is a direct sum

$$\widetilde{F}_{n,q}(\alpha) \sim \bigoplus_{\lambda\in\widetilde{\mathcal{L}}_n} \widetilde{F}_q^{\lambda}(\alpha)$$

of the smaller matrices $\widetilde{F}_q^{\lambda}(\alpha)$ because $\widetilde{F}_{n,q}(\alpha; \boldsymbol{i}; \boldsymbol{j}) = 0$ if $\mathrm{wt}(\boldsymbol{i}) \neq \mathrm{wt}(\boldsymbol{j})$. Here, for given square matrices A and B, we write $A \sim B$ when $B = PAP^{-1}$ for some invertible matrix P. If we put $\widetilde{C}_q^{\lambda}(\alpha) = \det \widetilde{F}_q^{\lambda}(\alpha)$, it is clear that

$$\widetilde{C}_{n,q}(\alpha) = \prod_{\lambda\in\widetilde{\mathcal{L}}_n} \widetilde{C}_q^{\lambda}(\alpha). \tag{3.3}$$

The following lemma is immediately verified.

Lemma 3.7. *If* $\langle \lambda, \varepsilon_i \rangle = \langle \mu, \varepsilon_{\sigma(i)} \rangle$ $(1 \leqslant i \leqslant n)$ *for some* $\sigma \in \mathfrak{S}_n$*, then* $\widetilde{F}_q^{\lambda}(\alpha) \sim \widetilde{F}_q^{\mu}(\alpha)$ $(\lambda, \mu \in \widetilde{\mathcal{L}}_n)$*. In particular, for each* $\mu \in \widetilde{\mathcal{L}}_n$*, there exists a unique dominant integral weight* $\lambda \in \widetilde{\mathcal{L}}_n$ *such that* $\widetilde{F}_q^{\lambda}(\alpha) \sim \widetilde{F}_q^{\mu}(\alpha)$.

Consequently, together with (3.3), we have

$$\mathrm{Sing}_{n,q} = \bigcup_{\lambda\in\widetilde{\mathcal{L}}_n} \{\alpha \in \mathbb{C} \mid \widetilde{C}_q^{\lambda}(\alpha) = 0\} = \bigcup_{\lambda\in\widetilde{\mathcal{L}}_n^{\mathrm{dom}}} \{\alpha \in \mathbb{C} \mid \widetilde{C}_q^{\lambda}(\alpha) = 0\}.$$

Here $\widetilde{\mathcal{L}}_n^{\mathrm{dom}}$ is the set of dominant weights in $\widetilde{\mathcal{L}}_n$. We will regard a dominant integral weight $\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n \in \widetilde{\mathcal{L}}_n^{\mathrm{dom}}$ as a partition (or a Young diagram) $(\lambda_1, \dots, \lambda_n) \vdash n$. Also, we sometimes write $\lambda \vdash n$ to indicate $\lambda \in \widetilde{\mathcal{L}}_n^{\mathrm{dom}}$.

It seems quite difficult in general to determine the polynomials $\widetilde{C}_q^{\lambda}(\alpha)$ as well as the matrix $\widetilde{F}_q^{\lambda}(\alpha)$ explicitly. Therefore, any characterization of the matrix $\widetilde{F}_q^{\lambda}(\alpha)$, for instance, either by difference equations or in the framework of R-matrices would be interesting (if any). From this point of view, the following property of the matrices $\widetilde{F}_q^{\lambda}(\alpha)$ is considerably remarkable. (See Examples 4.14, and 4.17 in Section 4.6.)

Proposition 3.8. *The matrix $\widetilde{F}_q^{\lambda}(\alpha)$ is symmetric.*

Before proceeding to the proof, we prepare several convention. We associate a sequence

$$\boldsymbol{k}(\lambda)=\bigl(k_1(\lambda),\dots,k_n(\lambda)\bigr):=(\overbrace{1,\dots,1}^{\lambda_1},\overbrace{2,\dots,2}^{\lambda_2},\dots,\overbrace{n,\dots,n}^{\lambda_n})\in I_n(\lambda)$$

to each integral weight $\lambda=\lambda_1\varepsilon_1+\lambda_2\varepsilon_2+\cdots+\lambda_n\varepsilon_n\in\widetilde{\mathcal{L}}_n$. We define a right $\mathfrak{S}_n$-action on $I_n(\lambda)$ by $\boldsymbol{i}^{\sigma}=(i_{\sigma(1)},\dots,i_{\sigma(n)})$ for $\sigma\in\mathfrak{S}_n$ and $\boldsymbol{i}=(i_1,\dots,i_n)\in I_n(\lambda)$. Notice that the right action $I_n(\lambda)\curvearrowleft\mathfrak{S}_n$ is transitive and the stabilizer of $\boldsymbol{k}(\lambda)$ is the Young subgroup $\mathfrak{S}_\lambda$. Therefore we have

$$I_n(\lambda)\cong\mathfrak{S}_\lambda\backslash\mathfrak{S}_n\quad\text{and/or}\quad I_n(\lambda)=\boldsymbol{k}(\lambda)\cdot\mathfrak{S}_n.$$

Thus we have another expression

$$\widetilde{F}_q^{\lambda}(\alpha)\sim\bigl(\widetilde{F}_q^{\lambda}(\alpha;\tau,\sigma)\bigr)_{\tau,\sigma\in\mathfrak{S}_\lambda\backslash\mathfrak{S}_n},$$

where we put $\widetilde{F}_q^{\lambda}(\alpha;\tau,\sigma):=\widetilde{F}_{n,q}(\alpha;\boldsymbol{k}(\lambda)^{\tau};\boldsymbol{k}(\lambda)^{\sigma})$.

Proof of Proposition 3.8. For convenience, we put

$$X_{\boldsymbol{k}}(g,\sigma):=x_{g(1)k_{\sigma(1)}}\cdots x_{g(n)k_{\sigma(n)}}$$

for $g\in\mathfrak{S}_n$ and $\sigma\in\mathfrak{S}_\lambda\backslash\mathfrak{S}_n$. We also define $f^{g}_{\tau,\sigma}(\lambda)$ by

$$q^{\ell(g)}X_{\boldsymbol{k}}(g,\sigma)=\sum_{\tau\in\mathfrak{S}_\lambda\backslash\mathfrak{S}_n}f^{g}_{\tau,\sigma}(\lambda)X_{\boldsymbol{k}}(1,\tau)$$

for $g\in\mathfrak{S}_n$ and $\sigma,\tau\in\mathfrak{S}_\lambda\backslash\mathfrak{S}_n$. It follows that

$$\widetilde{F}_q^{\lambda}(\alpha;\tau,\sigma)=\sum_{g\in\mathfrak{S}_n}\alpha^{n-\nu_n(g)}f^{g}_{\tau,\sigma}(\lambda).$$

Suppose that a permutation $g\in\mathfrak{S}_n$ and a simple transposition s_i satisfies the condition $\ell(gs_i)>\ell(g)$, which is equivalent to the condition $g(i)<g(i+1)$. It follows that

$$X_{\boldsymbol{k}}(gs_i,\sigma)=\begin{cases}q^{-1}X_{\boldsymbol{k}}(g,\sigma s_i), & k_{\sigma(i)}=k_{\sigma(i+1)},\\ X_{\boldsymbol{k}}(g,\sigma s_i), & k_{\sigma(i)}<k_{\sigma(i+1)},\\ X_{\boldsymbol{k}}(g,\sigma s_i)-(q-q^{-1})X_{\boldsymbol{k}}(g,\sigma), & k_{\sigma(i)}>k_{\sigma(i+1)}.\end{cases}$$

This yields the relation

$$f^{gs_i}_{\tau,\sigma}(\lambda)=\theta_0(\sigma,i)f^{g}_{\tau,\sigma}(\lambda)+\theta_1(\sigma,i)f^{g}_{\tau,\sigma s_i}(\lambda)$$

where we put

$$\theta_0(w,i) := \begin{cases} 1-q^2, & k_{w(i)} > k_{w(i+1)}, \\ 0, & \text{otherwise}, \end{cases} \qquad \theta_1(w,i) := \begin{cases} 1, & k_{w(i)} = k_{w(i+1)}, \\ q, & \text{otherwise}. \end{cases}$$

Therefore, for a given permutation $g = s_{i_l} \cdots s_{i_1}$ of length l, we have

$$f_{\tau,\sigma}^{s_{i_l}\cdots s_{i_1}}(\lambda) = \sum_{(j_1,\dots,j_l)\in\{0,1\}^l} \Theta_{\boldsymbol{k}}\begin{pmatrix} i_1,\dots,i_l \\ j_1,\dots,j_l \end{pmatrix}; \sigma \Bigg) \delta^{\lambda}_{\tau,\sigma s_{i_1}^{j_1}\cdots s_{i_l}^{j_l}}$$

by induction. Here we put

$$\Theta_{\boldsymbol{k}}\left(\begin{matrix} i_1,\dots,i_l \\ j_1,\dots,j_l \end{matrix}; \sigma\right) := \theta_{j_1}(\sigma, i_1)\theta_{j_2}\big(\sigma s_{i_1}^{j_1}, i_2\big)\cdots\theta_{j_l}\big(\sigma s_{i_1}^{j_1}\cdots s_{i_{l-1}}^{j_{l-1}}, i_l\big),$$

$$\delta^{\lambda}_{\tau,\sigma} := \begin{cases} 1, & \tau\sigma^{-1} \in \mathfrak{S}_\lambda, \\ 0, & \text{otherwise}. \end{cases}$$

We notice that

$$\theta_{j_p}\big(\sigma s_{i_1}^{j_1}\cdots s_{i_{p-1}}^{j_{p-1}}, i_p\big) = \theta_{j_p}\big(\sigma s_{i_1}^{j_1}\cdots s_{i_p}^{j_p}, i_p\big)$$

for each $p = 1, 2, \dots, l$. It therefore follows that if $\tau^{-1}\sigma s_{i_1}^{j_1}\cdots s_{i_l}^{j_l} \in \mathfrak{S}_\lambda$, then

$$\begin{aligned} \Theta_{\boldsymbol{k}}\left(\begin{matrix} i_1,\dots,i_l \\ j_1,\dots,j_l \end{matrix}; \sigma\right) &= \theta_{j_1}(\sigma, i_1)\theta_{j_2}\big(\sigma s_{i_1}^{j_1}, i_2\big)\cdots\theta_{j_l}\big(\sigma s_{i_1}^{j_1}\cdots s_{i_{l-1}}^{j_{l-1}}, i_l\big) \\ &= \theta_{j_1}\big(\sigma s_1^{j_1}, i_1\big)\theta_{j_2}\big(\sigma s_{i_1}^{j_1} s_{i_2}^{j_2}, i_2\big)\cdots\theta_{j_l}\big(\sigma s_{i_1}^{j_1}\cdots s_{i_l}^{j_l}, i_l\big) \\ &= \theta_{j_l}(\tau, i_l)\theta_{j_{l-1}}\big(\tau s_{i_l}^{j_l}, i_{l-1}\big)\cdots\theta_{j_1}\big(\tau s_{i_l}^{j_l}\cdots s_{i_2}^{j_2}, i_1\big) = \Theta_{\boldsymbol{k}}\left(\begin{matrix} i_l,\dots,i_1 \\ j_l,\dots,j_1 \end{matrix}; \tau\right). \end{aligned}$$

This immediately implies that $f_{\tau,\sigma}^{g}(\lambda) = f_{\sigma,\tau}^{g^{-1}}(\lambda)$, and hence the symmetry $\widetilde{F}^{\lambda}_{\tau,\sigma} = \widetilde{F}^{\lambda}_{\sigma,\tau}$ follows as we desired. □

3.3. Highest weight vectors in $\boldsymbol{V}_{n,q}^{(\alpha)}$

The aim of the present subsection is to construct a set of vectors $\{v^{(\alpha)}(T) \mid T \in \mathrm{STab}(\lambda), \lambda \vdash n\}$ in $\boldsymbol{V}_{n,q}^{(\alpha)}$ satisfying the following conditions: (a) if $T \in \mathrm{STab}(\lambda)$, then $v^{(\alpha)}(T) \in \boldsymbol{V}_{n,q}^{(\alpha)}(\lambda)$, (b) each $v^{(\alpha)}(T)$ is killed by $\rho(e_k)$ $(1 \leqslant k < n)$, (c) $\boldsymbol{V}_{n,q}^{(\alpha)} = \bigoplus_{\lambda\vdash n}\{\sum_{T\in\mathrm{STab}(\lambda)} \rho(\mathcal{U}_q(\mathfrak{gl}_n)) \cdot v^{(\alpha)}(T)\}$. To achieve this, we first construct such vectors $v^{(\alpha)}(T)$ for $\alpha \in \mathbb{C} \setminus \mathrm{Sing}_{n,q}$, and then extend the definition of them to any $\alpha \in \mathbb{C}$. So, we suppose that $\alpha \in \mathbb{C} \setminus \mathrm{Sing}_{n,q}$ for a while.

For a quantum α-determinant $D_q^{(\alpha)}(j_1,\dots,j_n)$ and $h_k \in \mathcal{H}_q(\mathfrak{S}_n)$, define

$$D_q^{(\alpha)}(j_1,\dots,j_n)\cdot\pi^{(\alpha)}(h_k) = \begin{cases} D_q^{(\alpha)}(j_1,\dots,j_{k+1},j_k,\dots,j_n), & j_k<j_{k+1},\\ q^{-1}D_q^{(\alpha)}(j_1,\dots,j_n), & j_k=j_{k+1},\\ D_q^{(\alpha)}(j_1,\dots,j_{k+1},j_k,\dots,j_n)-(q-q^{-1})D_q^{(\alpha)}(j_1,\dots,j_n), & j_k>j_{k+1}. \end{cases}$$

Each $\pi^{(\alpha)}(h_k)$ is extended as a linear operator on $\boldsymbol{V}_{n,q}^{(\alpha)}$ and defines a right $\mathcal{H}_q(\mathfrak{S}_n)$-module structure on $\boldsymbol{V}_{n,q}^{(\alpha)}$.

Remark 3.9. When $\alpha\in\mathrm{Sing}_{n,q}$, we cannot extend $\pi^{(\alpha)}(h_k)$ to a linear operator on $\boldsymbol{V}_{n,q}^{(\alpha)}$ as we see in the following example: When $\alpha=\frac{1}{q^3+q^2-q}\in\mathrm{Sing}_{3,q}$, we have a nontrivial linear relation

$$D_q^{(\alpha)}(1,1,2)+(1-q)D_q^{(\alpha)}(1,2,1)-qD_q^{(\alpha)}(2,1,1)=0.$$

However, since

$$D_q^{(\alpha)}(1,1,2)\cdot\pi^{(\alpha)}(h_1)=q^{-1}D_q^{(\alpha)}(1,1,2),\qquad D_q^{(\alpha)}(1,2,1)\cdot\pi^{(\alpha)}(h_1)=D_q^{(\alpha)}(2,1,1),$$
$$D_q^{(\alpha)}(2,1,1)\cdot\pi^{(\alpha)}(h_1)=D_q^{(\alpha)}(1,2,1)-\left(q-q^{-1}\right)D_q^{(\alpha)}(2,1,1),$$

it follows that

$$\begin{aligned} &D_q^{(\alpha)}(1,1,2)\cdot\pi^{(\alpha)}(h_1)+(1-q)D_q^{(\alpha)}(1,2,1)\cdot\pi^{(\alpha)}(h_1)-qD_q^{(\alpha)}(2,1,1)\cdot\pi^{(\alpha)}(h_1)\\ &\quad=\frac{(1-q^2)(1-q+q^2)}{q(1-q-q^2)}\left(D_q^{(0)}(1,1,2)+(1-q)D_q^{(0)}(1,2,1)-qD_q^{(0)}(2,1,1)\right)\neq 0, \end{aligned}$$

which means that $\pi^{(\alpha)}(h_1)$ cannot be extended to a linear operator on $\boldsymbol{V}_{3,q}^{(\alpha)}$ when $\alpha=\frac{1}{q^3+q^2-q}\in\mathrm{Sing}_{3,q}$.

It is directly checked that

$$\Phi_{n,q}^{(\alpha)}\left(\boldsymbol{e}_{j_1}\otimes\cdots\otimes\boldsymbol{e}_{j_n}\cdot\pi(h_k)\right)=D_q^{(\alpha)}(j_1,\dots,j_n)\cdot\pi^{(\alpha)}(h_k).$$

Hence, each operator $\pi^{(\alpha)}(h_k)$ commutes with the $\rho(\mathcal{U}_q(\mathfrak{gl}_n))$-action. In particular, $\pi^{(\alpha)}(\mathcal{H}_q(\mathfrak{S}_n))$ is the commutant of $\rho(\mathcal{U}_q(\mathfrak{gl}_n))$ in $\mathrm{End}\,\boldsymbol{V}_{n,q}^{(\alpha)}$ and vice versa.

For a standard tableau $T\in\mathrm{STab}(\lambda)$ of size n ($\lambda\vdash n$), we define $\boldsymbol{j}(T)=(j_1(T),\dots,j_n(T))\in I_n(\lambda)$ by

$$j_p(T)=i \quad\Leftrightarrow\quad \text{the number written in the } (i,j)\text{-box in } T \text{ is } p. \tag{3.4}$$

We set

$$v^{(\alpha)}(T):=D_q^{(\alpha)}\left(\boldsymbol{j}(T)\right)\cdot\pi^{(\alpha)}\left(\mathbb{E}_q(T)\right)\in\boldsymbol{V}_{n,q}^{(\alpha)}(\lambda)\quad\left(T\in\mathrm{STab}(\lambda)\right),$$

where $\mathbb{E}_q(T)$ is the q-Young symmetrizer for T (see Section 2.3). This is a highest weight vector of weight λ. By definition, each vector $v^{(\alpha)}(T)$ has an expression

$$\begin{aligned} v^{(\alpha)}(T) &= \sum_{\sigma\in\mathfrak{S}_\lambda\backslash\mathfrak{S}_n} Q_T^\sigma(q) D_q^{(\alpha)}\big(\boldsymbol{k}(\lambda)^\sigma\big) \\ &= \sum_{\tau\in\mathfrak{S}_\lambda\backslash\mathfrak{S}_n} \Big\{ \sum_{\sigma\in\mathfrak{S}_\lambda\backslash\mathfrak{S}_n} Q_T^\sigma(q) \widetilde{F}_q^\lambda(\alpha;\tau,\sigma) \Big\} D_q^{(0)}\big(\boldsymbol{k}(\lambda)^\tau\big) \end{aligned} \tag{3.5}$$

for certain *polynomials* $Q_T^\sigma(q)$ *in* q. For later use, we define the $f^\lambda \times |\mathfrak{S}_\lambda\backslash\mathfrak{S}_n|$ matrix $\widetilde{Q}_n^\lambda(q)$ by

$$\widetilde{Q}_n^\lambda(q) = \big(Q_T^\sigma(q)\big)_{T\in\mathrm{STab}(\lambda),\, \sigma\in\mathfrak{S}_\lambda\backslash\mathfrak{S}_n}. \tag{3.6}$$

Similar to the classical case, the vectors $v^{(\alpha)}(T)$ for $T \in \mathrm{STab}(\lambda)$ form a basis of the subspace

$$W_{n,q}^{(\alpha)}(\lambda) := \big\{ v \in \boldsymbol{V}_{n,q}^{(\alpha)}(\lambda) \;\big|\; \rho(e_k)\cdot v = 0 \ (1 \leqslant k < n) \big\}$$

consisting of the highest weight vectors of highest weight λ. Therefore, the cyclic module $\rho(\mathcal{U}_q(\mathfrak{gl}_n))\cdot v^{(\alpha)}(T)$ is equivalent to $\boldsymbol{E}_{n,q}^\lambda$ for each $T \in \mathrm{STab}(\lambda)$ and we have

$$\boldsymbol{V}_{n,q}^{(\alpha)} = \bigoplus_{\lambda\vdash n} \bigoplus_{T\in\mathrm{STab}(\lambda)} \rho\big(\mathcal{U}_q(\mathfrak{gl}_n)\big)\cdot v^{(\alpha)}(T).$$

In particular, every quantum α-determinant $D_q^{(\alpha)}(i_1,\ldots,i_n)$ is written in the form

$$D_q^{(\alpha)}(i_1,\ldots,i_n) = \sum_{\lambda\vdash n} \sum_{T\in\mathrm{STab}(\lambda)} \rho\big(a^{(\alpha)}(T)\big)\cdot v^{(\alpha)}(T) \quad \big(\exists a^{(\alpha)}(T) \in \mathcal{U}_q(\mathfrak{gl}_\alpha)\big). \tag{3.7}$$

Here we notice that the right-hand side of (3.5) makes sense even if $\alpha \in \mathrm{Sing}_{n,q}$, though the vector $v^{(\alpha)}(T)$ is defined only for $\alpha \in \mathbb{C}\setminus\mathrm{Sing}_{n,q}$. Actually, it is a linear combination of monomials $x_{1i_1}\cdots x_{ni_n}$ whose coefficient is a polynomial in α. So we *extend* the definition of $v^{(\alpha)}(T)$ for any $\alpha \in \mathbb{C}$ by the expression (3.5).

Lemma 3.10. *The vector $v^{(\alpha)}(T)$ is a highest weight vector in $\boldsymbol{V}_{n,q}^{(\alpha)}(\lambda)$ whenever $v^{(\alpha)}(T) \neq 0$.*

Proof. By definition, each vector $\rho(e_k)\cdot v^{(\alpha)}(T)$ is a polynomial function in α. Therefore, the property $\rho(e_k)\cdot v^{(\alpha)}(T) = 0$ is equivalent to some algebraic equation on α. Since any complex number in $\mathbb{C}\setminus\mathrm{Sing}_{n,q}$ is a root of the equation, we have $\rho(e_k)\cdot v^{(\alpha)}(T) = 0$ for any $\alpha \in \mathbb{C}$. This completes the proof. □

The formula (3.7) is valid for all $\alpha \in \mathbb{C}$ by a similar 'polynomial' discussion. It hence follows that

$$\boldsymbol{V}_{n,q}^{(\alpha)} = \bigoplus_{\lambda\vdash n} \Big\{ \sum_{T\in\mathrm{STab}(\lambda)} \rho\big(\mathcal{U}_q(\mathfrak{gl}_n)\big)\cdot v^{(\alpha)}(T) \Big\}$$

for *any* $\alpha \in \mathbb{C}$. We notice that $\{v^{(\alpha)}(T)\}_{T\in\mathrm{STab}(\lambda)}$ *generates* $W_{n,q}^{(\alpha)}(\lambda)$ and $m_q^\lambda(\alpha) = \dim W_{n,q}^{(\alpha)}(\lambda)$.

3.4. Explicit decomposition of $\boldsymbol{V}_{n,q}^{(\alpha)}$—Examples for $n = 2, 3$

Example 3.11. Let us see the simplest case, the $\mathcal{U}_q(\mathfrak{gl}_2)$-module $\boldsymbol{V}_{2,q}^{(\alpha)}$. Since

$$D_q^{(\alpha)}(1,1) = (1+\alpha)D_q^{(0)}(1,1), \qquad D_q^{(\alpha)}(1,2) = D_q^{(0)}(1,2) + \alpha q D_q^{(0)}(2,1),$$
$$D_q^{(\alpha)}(2,1) = \alpha q D_q^{(0)}(1,2) + \left(1+\alpha-\alpha q^2\right)D_q^{(0)}(2,1), \qquad D_q^{(\alpha)}(2,2) = (1+\alpha)D_q^{(0)}(2,2),$$

we have

$$\widetilde{F}_{2,q}(\alpha) = \begin{pmatrix} 1+\alpha & 0 & 0 & 0 \\ 0 & 1 & \alpha q & 0 \\ 0 & \alpha q & 1+\alpha-\alpha q^2 & 0 \\ 0 & 0 & 0 & 1+\alpha \end{pmatrix} = \widetilde{F}_q^{(2,0)}(\alpha) \oplus \widetilde{F}_q^{(1,1)}(\alpha) \oplus \widetilde{F}_q^{(0,2)}(\alpha),$$

$\widetilde{C}_{2,q}(\alpha) = \det \widetilde{F}_{2,q}(\alpha) = (1+\alpha)^3(1-\alpha q^2)$ and $\mathrm{Sing}_{2,q} = \{-1, q^{-2}\}$.

If $\alpha \in \mathbb{C} \setminus \mathrm{Sing}_{2,q}$, then we have $\boldsymbol{V}_{2,q}^{(\alpha)} \cong \boldsymbol{E}_{2,q}^{(2)} \oplus \boldsymbol{E}_{2,q}^{(1,1)}$ by Proposition 3.4. Each irreducible component is explicitly written as

$$\boldsymbol{E}_{2,q}^{(2)} : 0 \xleftarrow[f_1]{} \mathbb{C}D_q^{(\alpha)}(2,2) \underset{f_1}{\overset{e_1}{\rightleftarrows}} \mathbb{C}\left(qD_q^{(\alpha)}(1,2) + D_q^{(\alpha)}(2,1)\right) \underset{f_1}{\overset{e_1}{\rightleftarrows}} \mathbb{C}D_q^{(\alpha)}(1,1) \xrightarrow{e_1} 0,$$

$$\boldsymbol{E}_{2,q}^{(1,1)} : 0 \xleftarrow[f_1]{} \mathbb{C}\left(D_q^{(\alpha)}(1,2) - qD_q^{(\alpha)}(2,1)\right) \xrightarrow{e_1} 0.$$

The highest weight vectors of these modules are

$$D_q^{(\alpha)}(1,1) = (1+\alpha)x_{11}x_{21}, \qquad D_q^{(\alpha)}(1,2) - qD_q^{(\alpha)}(2,1) = \left(1-\alpha q^2\right)\mathrm{det}_q.$$

Hence the component $\boldsymbol{E}_{2,q}^{(2)}$ (respectively $\boldsymbol{E}_{2,q}^{(1,1)}$) disappears if $\alpha = -1$ (respectively $\alpha = q^{-2}$). We have thus

$$\boldsymbol{V}_{2,q}^{(\alpha)} \cong \begin{cases} \boldsymbol{E}_{2,q}^{(2)}, & \alpha = -1, \\ \boldsymbol{E}_{2,q}^{(1,1)}, & \alpha = q^{-2}, \\ \boldsymbol{E}_{2,q}^{(2)} \oplus \boldsymbol{E}_{2,q}^{(1,1)}, & \alpha \neq -1, q^{-2}. \end{cases}$$

Example 3.12. Look at the $\mathcal{U}_q(\mathfrak{gl}_3)$-module $\boldsymbol{V}_{3,q}^{(\alpha)}$. If we put

$$\begin{aligned}
v^{(3)} &= D_q^{(\alpha)}(1,1,1) = (1+\alpha)(1+2\alpha)x_{11}x_{21}x_{31}, \\
v_1^{(2,1)} &= D_q^{(\alpha)}(1,1,2) + (1-q)D_q^{(\alpha)}(1,2,1) - qD_q^{(\alpha)}(2,1,1) \\
&= (1+\alpha)\left(1+\left(q-q^2-q^3\right)\alpha\right)\left(x_{11}x_{21}x_{32} + (1-q)x_{11}x_{22}x_{31} - qx_{12}x_{21}x_{31}\right), \\
v_2^{(2,1)} &= D_q^{(\alpha)}(1,1,2) - (1+q)D_q^{(\alpha)}(1,2,1) + qD_q^{(\alpha)}(2,1,1) \\
&= (1+\alpha)\left(1+\left(-q-q^2+q^3\right)\alpha\right)\left(x_{11}x_{21}x_{32} - (1+q)x_{11}x_{22}x_{31} + qx_{12}x_{21}x_{31}\right),
\end{aligned}$$

$$\begin{aligned} v^{(1,1,1)} &= D_q^{(\alpha)}(1,2,3) - qD_q^{(\alpha)}(2,1,3) - qD_q^{(\alpha)}(1,3,2) - q^3 D_q^{(\alpha)}(3,2,1) + q^2 D_q^{(\alpha)}(2,3,1) \\ &\quad + q^2 D_q^{(\alpha)}(3,1,2) \\ &= \left(1 - 2\alpha q^2 + 2\alpha^2 q^4 - \alpha q^6\right) \det_q, \end{aligned}$$

then the $\mathcal{U}_q(\mathfrak{gl}_3)$-cyclic span of these vectors gives $\boldsymbol{V}_{3,q}^{(\alpha)}$ (see Example 3.2). Therefore we have

$$\boldsymbol{V}_{3,q}^{(\alpha)} \cong \begin{cases} \boldsymbol{E}_{3,q}^{(1,1,1)}, & \alpha = -1, \\ (\boldsymbol{E}_{3,q}^{(2,1)})^{\oplus 2} \oplus \boldsymbol{E}_{3,q}^{(1,1,1)}, & \alpha = -1/2, \\ \boldsymbol{E}_{3,q}^{(3)} \oplus \boldsymbol{E}_{3,q}^{(2,1)} \oplus \boldsymbol{E}_{3,q}^{(1,1,1)}, & \alpha = 1/(q^2 \pm (q - q^3)), \\ \boldsymbol{E}_{3,q}^{(3)} \oplus (\boldsymbol{E}_{3,q}^{(2,1)})^{\oplus 2}, & \alpha = (2q^{-2} + q^2 \pm \sqrt{q^4 + 4 - 4q^{-4}})/4, \\ \boldsymbol{E}_{3,q}^{(3)} \oplus (\boldsymbol{E}_{3,q}^{(2,1)})^{\oplus 2} \oplus \boldsymbol{E}_{3,q}^{(1,1,1)}, & \text{otherwise.} \end{cases}$$

In other words, we have

$$m_q^{(3)}(\alpha) = \begin{cases} 0, & \alpha = -1, -\frac{1}{2}, \\ 1, & \text{otherwise,} \end{cases}$$

$$m_q^{(2,1)}(\alpha) = \begin{cases} 0, & \alpha = -1, \\ 1, & \alpha = 1/(q^2 \pm (q - q^3)), \\ 2, & \text{otherwise,} \end{cases}$$

$$m_q^{(1,1,1)}(\alpha) = \begin{cases} 0, & \alpha = (2q^{-2} + q^2 \pm \sqrt{q^4 + 4 - 4q^{-4}})/4, \\ 1, & \text{otherwise.} \end{cases}$$

It also follows that

$$\mathrm{Sing}_{3,q} = \left\{ -1, -\frac{1}{2}, \frac{1}{q^2 \pm (q - q^3)}, \frac{2q^{-2} + q^2 \pm \sqrt{q^4 + 4 - 4q^{-4}}}{4} \right\}.$$

Notice that $0 < m_q^{(2,1)}(\alpha) < f^{(2,1)}$ when $\alpha = 1/(q^2 \pm (q - q^3))$. In particular, this shows that the classical result *cannot* be recovered from the quantum case by letting $q \to 1$. This is because the elementary divisors of the q-content transition matrices for generic q (defined in Section 4.2 below) are different from those of the 1-content transition matrices; see Example 4.1. See also Example 4.15 for the q-content discriminants.

4. Irreducible decomposition of $\boldsymbol{V}_{n,q}^{(\alpha)}$ for $\alpha \in \mathrm{Sing}_{n,q}$

In this section, we investigate the cases where some irreducible factors of the decomposition of the cyclic module $\boldsymbol{V}_{n,q}^{(\alpha)}$ may collapse.

4.1. *q*-Content discriminants

If $\alpha \in \mathrm{Sing}_{n,q}$, then there exists some $\lambda \vdash n$ such that $m_q^\lambda(\alpha) < f^\lambda$ by definition. To describe the sets

$$\mathrm{Sing}_q^\lambda := \bigl\{\alpha \in \mathrm{Sing}_{n,q} \bigm| m_q^\lambda(\alpha) < f^\lambda\bigr\}, \qquad \mathrm{Sing}_{n,q}(\alpha) := \bigl\{\lambda \in \widetilde{\mathcal{L}}_n^{\mathrm{dom}} \bigm| m_q^\lambda(\alpha) < f^\lambda\bigr\},$$

we introduce certain polynomials called the q-discriminants: Let $\lambda \in \widetilde{\mathcal{L}}_n^{\mathrm{dom}}$ and $\alpha, \beta \in \mathbb{C}$. When $\beta \in \mathbb{C} \setminus \mathrm{Sing}_{n,q}$, each vector $v^{(\alpha)}(T)$ is written as a linear combination of the vectors $\{v^{(\beta)}(T)\}_{T \in \mathrm{STab}(\lambda)}$

$$v^{(\alpha)}(T) = \sum_{S \in \mathrm{STab}(\lambda)} F_q^\lambda(\alpha, \beta; S, T) v^{(\beta)}(S) \quad \bigl(T \in \mathrm{STab}(\lambda)\bigr).$$

We introduce a $f^\lambda \times f^\lambda$ matrix $F_q^\lambda(\alpha, \beta)$ by

$$F_q^\lambda(\alpha, \beta) = \bigl(F_q^\lambda(\alpha, \beta; S, T)\bigr)_{S, T \in \mathrm{STab}(\lambda)}.$$

We call $F_q^\lambda(\alpha, \beta)$ the *q-content transition matrix* of λ. The function $C_q^\lambda(\alpha, \beta) := \det F_q^\lambda(\alpha, \beta)$ of α is called the *q-content discriminant* for λ with reference point β. If $\beta, \gamma \in \mathbb{C} \setminus \mathrm{Sing}_{n,q}$, then $F_q^\lambda(\alpha, \beta) F_q^\lambda(\beta, \gamma) = F_q^\lambda(\alpha, \gamma)$ and $C_q^\lambda(\alpha, \beta) C_q^\lambda(\beta, \gamma) = C_q^\lambda(\alpha, \gamma)$. In particular, if $\alpha, \beta \in \mathbb{C} \setminus \mathrm{Sing}_{n,q}$, then $C_q^\lambda(\alpha, \beta) = C_q^\lambda(\alpha, 0)/C_q^\lambda(\beta, 0)$. In what follows, we simply write $F_q^\lambda(\alpha)$ and $C_q^\lambda(\alpha)$ instead of $F_q^\lambda(\alpha, 0)$ and $C_q^\lambda(\alpha, 0)$. By definition, we have

$$m_q^\lambda(\alpha) = \dim_{\mathbb{C}} W_{n,q}^{(\alpha)}(\lambda) = \operatorname{rank} F_q^\lambda(\alpha),$$

and hence

$$\mathrm{Sing}_q^\lambda = \bigl\{\alpha \in \mathbb{C} \bigm| C_q^\lambda(\alpha) = 0\bigr\}.$$

Example 4.1. Since

$$\begin{aligned} v^{(\alpha)}\Bigl(\begin{smallmatrix}1&2\\3\end{smallmatrix}\Bigr) &= (1+\alpha)\bigl(1+\alpha-2\alpha q^2\bigr) v^{(0)}\Bigl(\begin{smallmatrix}1&2\\3\end{smallmatrix}\Bigr) + \alpha q(1+\alpha)\bigl(1-q^2\bigr) v^{(0)}\Bigl(\begin{smallmatrix}1&3\\2\end{smallmatrix}\Bigr), \\ v^{(\alpha)}\Bigl(\begin{smallmatrix}1&3\\2\end{smallmatrix}\Bigr) &= -\alpha q^{-1}(1+\alpha)\bigl(1-q^2\bigr)^2 v^{(0)}\Bigl(\begin{smallmatrix}1&2\\3\end{smallmatrix}\Bigr) + (1+\alpha)(1-\alpha) v^{(0)}\Bigl(\begin{smallmatrix}1&3\\2\end{smallmatrix}\Bigr), \end{aligned}$$

we have

$$F_q^{(2,1)}(\alpha) = \begin{pmatrix} (1+\alpha)(1+\alpha-2\alpha q^2) & \alpha q(1+\alpha)(1-q^2) \\ -\alpha q^{-1}(1+\alpha)(1-q^2)^2 & (1+\alpha)(1-\alpha) \end{pmatrix}$$

and

$$C_q^{(2,1)}(\alpha) = \det F_q^{(2,1)}(\alpha) = (1+\alpha)^2\bigl(1+\bigl(q-q^2-q^3\bigr)\alpha\bigr)\bigl(1+\bigl(-q-q^2+q^3\bigr)\alpha\bigr).$$

We note that the elementary divisors of the transition matrix $F^{(2,1)}(\alpha)$ are given by

$$\begin{cases} (1+\alpha)(1-\alpha), (1+\alpha)(1-\alpha), & q^2=1, \\ (1+\alpha), (1+\alpha)(1+(q-q^2-q^3)\alpha)(1+(-q-q^2+q^3)\alpha), & q^2 \neq 1. \end{cases}$$

The relation between the two collections $\{C_q^{\lambda}(\alpha)\}_{\lambda}$ and $\{\widetilde{C}_q^{\lambda}(\alpha)\}_{\lambda}$ is given by the

Lemma 4.2. *The equalities*

$$\operatorname{rank} \widetilde{F}_q^{\mu}(\alpha) = \sum_{\lambda \vdash n} K_{\lambda\mu} \operatorname{rank} F_q^{\lambda}(\alpha), \qquad \widetilde{C}_q^{\mu}(\alpha) = \prod_{\lambda \vdash n} C_q^{\lambda}(\alpha)^{K_{\lambda\mu}}$$

hold where $K_{\lambda\mu}$ is the Kostka number (*we refer* [5] *for the definition*).

Proof. We define

$$\boldsymbol{V}_{n,q}^{(\alpha)}(\lambda,\mu) := \mathcal{U}_q(\mathfrak{gl}_n) \cdot W_{n,q}^{(\alpha)}(\lambda) \cap H_{\mu}^{(\alpha)}, \qquad \boldsymbol{V}_{n,q}^{(\alpha)}(T,\mu) := \mathcal{U}_q(\mathfrak{gl}_n) \cdot v^{(\alpha)}(T) \cap H_{\mu}^{(\alpha)},$$

where

$$H_{\mu}^{(\alpha)} := \sum_{\sigma \in \mathfrak{S}_{\mu} \backslash \mathfrak{S}_n} \mathbb{C} \cdot D_q^{(\alpha)}\big(\boldsymbol{k}(\mu)^{\sigma}\big)$$

is the subspace of $\boldsymbol{V}_{n,q}^{(\alpha)}$ consisting of all weight vectors of weight μ. By definition, we have

$$H_{\mu}^{(\alpha)} = \bigoplus_{\lambda \vdash n} \boldsymbol{V}_{n,q}^{(\alpha)}(\lambda,\mu).$$

Since $W_{n,q}^{(\alpha)}(\lambda) = \sum_{T \in \mathrm{STab}(\lambda)} \mathbb{C} \cdot v^{(\alpha)}(T)$, we also have

$$\boldsymbol{V}_{n,q}^{(\alpha)}(\lambda,\mu) = \sum_{T \in \mathrm{STab}(\lambda)} \boldsymbol{V}_{n,q}^{(\alpha)}(T,\mu).$$

Notice that

$$\boldsymbol{V}_{n,q}^{(\alpha)}(T,\mu) \cong \begin{cases} \boldsymbol{E}_{n,q}^{\lambda}(\mu), & v^{(\alpha)}(T) \neq 0, \\ 0, & v^{(\alpha)}(T) = 0, \end{cases}$$

where $\boldsymbol{E}_{n,q}^{\lambda}(\mu)$ is the weight space in $\boldsymbol{E}_{n,q}^{\lambda}$ of weight μ. We denote by ι_T the intertwiner between $\boldsymbol{V}_{n,q}^{(\alpha)}(T,\mu)$ and $\boldsymbol{E}_{n,q}^{\lambda}(\mu)$ when $v^{(\alpha)}(T) \neq 0$. We also put $\iota_T(x) = 0 \in \boldsymbol{E}_{n,q}^{\lambda}(\mu)$ for any $x \in \boldsymbol{V}_{n,q}^{(\alpha)}(T,\mu)$ when $v^{(\alpha)}(T) = 0$. The map

$$\boldsymbol{V}_{n,q}^{(\alpha)}(\lambda,\mu) \supset \boldsymbol{V}_{n,q}^{(\alpha)}(T,\mu) \ni x \mapsto \iota_T(x) \otimes v^{(\alpha)}(T) \in \boldsymbol{E}_{n,q}^{\lambda}(\mu) \otimes W_{n,q}^{(\alpha)}(\lambda)$$

defines a linear isomorphism, and hence yields

$$\boldsymbol{V}_{n,q}^{(\alpha)}(\lambda,\mu)\cong \boldsymbol{E}_{n,q}^{\lambda}(\mu)\otimes W_{n,q}^{(\alpha)}(\lambda).$$

Consequently, we have the decomposition

$$H_{\mu}^{(\alpha)}\cong\bigoplus_{\lambda\vdash n}\boldsymbol{E}_{n,q}^{\lambda}(\mu)\otimes W_{n,q}^{(\alpha)}(\lambda)$$

as a vector space. Since $\dim \boldsymbol{E}_{n,q}^{\lambda}(\mu)=K_{\lambda\mu}$, we have the lemma. □

As a corollary, we also have the

Lemma 4.3. *The equalities*

$$\operatorname{rank} F_q^{\lambda}(\alpha)=\sum_{\mu\vdash n}K_{\lambda\mu}^{(-1)}\operatorname{rank}\widetilde{F}_q^{\mu}(\alpha),\qquad C_q^{\lambda}(\alpha)=\prod_{\mu\vdash n}\widetilde{C}_q^{\mu}(\alpha)^{K_{\lambda\mu}^{(-1)}}$$

hold where $K_{\lambda\mu}^{(-1)}$ is the reverse Kostka number (i.e. $\sum_{\nu\vdash n}K_{\lambda\nu}K_{\nu\mu}^{(-1)}=\delta_{\lambda\mu}$).

By Lemma 4.2, we notice that

$$\widetilde{C}_q^{\mu}(\alpha)=C_q^{\mu}(\alpha)\times\prod_{\substack{\lambda\vdash n\\ \lambda\neq\mu}}C_q^{\lambda}(\alpha)^{K_{\lambda\mu}},$$

which readily implies that

$$\mathrm{Sing}_{n,q}=\bigcup_{\lambda\vdash n}\{\alpha\in\mathbb{C}\mid \widetilde{C}_q^{\lambda}(\alpha)=0\}=\bigcup_{\lambda\vdash n}\{\alpha\in\mathbb{C}\mid C_q^{\lambda}(\alpha)=0\}.$$

Namely, the two collections $\{F_q^{\lambda}(\alpha)\}_{\lambda\vdash n}$ and $\{\widetilde{F}_q^{\lambda}(\alpha)\}_{\lambda\vdash n}$ of matrices have equivalent information on $\mathrm{Sing}_{n,q}$.

4.2. Classical result—A review

We recall the result of the classical case [6]. The set $\mathrm{Sing}_{n,1}$ is explicitly given by

$$\mathrm{Sing}_{n,1}=\left\{\pm1,\pm\frac{1}{2},\dots,\pm\frac{1}{n-1}\right\}.$$

The irreducible decomposition of $\boldsymbol{V}_{n,1}^{(\pm\frac{1}{k})}$ $(k=1,2,\dots,n-1)$ is

$$\boldsymbol{V}_{n,1}^{(-\frac{1}{k})}\cong\bigoplus_{\substack{\lambda\vdash n\\ \lambda_1\leqslant k}}\left(\boldsymbol{E}_1^{\lambda}\right)^{\oplus f^{\lambda}},\qquad \boldsymbol{V}_{n,1}^{(\frac{1}{k})}\cong\bigoplus_{\substack{\lambda\vdash n\\ \lambda_1'\leqslant k}}\left(\boldsymbol{E}_1^{\lambda}\right)^{\oplus f^{\lambda}}. \tag{4.1}$$

In other words, the multiplicity $m_1^\lambda(\alpha)$ of the Schur module $\boldsymbol{E}_1^\lambda$ in $\boldsymbol{V}_{n,1}^{(\pm 1/k)}$ is given by

$$m_1^\lambda\left(-\frac{1}{k}\right)=\begin{cases} f^\lambda, & \lambda_1 \leqslant k,\\ 0, & \text{otherwise,}\end{cases}\qquad m_1^\lambda\left(\frac{1}{k}\right)=\begin{cases} f^\lambda, & \lambda'_1 \leqslant k,\\ 0, & \text{otherwise.}\end{cases} \tag{4.2}$$

In particular, since $f^{\lambda'}=f^\lambda$, we see that $m_1^\lambda(-\frac{1}{k})=m_1^{\lambda'}(\frac{1}{k})$, where λ' denotes the transposition of λ as a Young diagram. Thus, in the classical case, each isotypic component either exists with full multiplicity or disappears completely. Using the (modified) content polynomial

$$c^\lambda(\alpha):=\prod_{(i,j)\in\lambda}\bigl(1+(j-i)\alpha\bigr)$$

of a Young diagram λ, one finds that (4.2) is equivalent to

$$m_1^\lambda(\alpha)=\begin{cases} f^\lambda, & c^\lambda(\alpha)\neq 0,\\ 0, & c^\lambda(\alpha)=0.\end{cases}$$

In the quantum case, if $\alpha\in\mathrm{Sing}_{n,q}$, then we have $m_q^\lambda(\alpha)<f^\lambda$ for some λ. In contrast to the classical case, however, it could be that $m_q^\lambda(\alpha)\neq 0$ as we see in Example 3.12.

4.3. Symmetric and skew-symmetric cases

The cases where $\lambda=(n)$ and $(1,\dots,1)$ are much easier because the respective multiplicities $m_q^\lambda(\alpha)$ of $\boldsymbol{E}_{n,q}^\lambda$ in the irreducible decomposition of $\boldsymbol{V}_{n,q}^{(\alpha)}$ are always either 0 or 1. Actually, we have the following.

Proposition 4.4. *The highest weight vectors for* (n) *and* $(1,\dots,1)$ *in* $\boldsymbol{V}_{n,q}^{(\alpha)}$ *are*

$$\begin{aligned} v^{(\alpha)}\bigl(\boxed{1}\boxed{2}\boxed{\cdot}\boxed{n}\bigr)&=\Bigl(\sum_{\sigma\in\mathfrak{S}_n}\alpha^{n-\nu_n(\sigma)}\Bigr)v^{(0)}\bigl(\boxed{1}\boxed{2}\boxed{\cdot}\boxed{n}\bigr),\\ v^{(\alpha)}\left(\begin{smallmatrix}\boxed{1}\\\boxed{2}\\\boxed{\cdot}\\\boxed{n}\end{smallmatrix}\right)&=\Bigl(\sum_{\sigma\in\mathfrak{S}_n}\alpha^{n-\nu_n(\sigma)}\bigl(-q^2\bigr)^{\ell(\sigma)}\Bigr)v^{(0)}\left(\begin{smallmatrix}\boxed{1}\\\boxed{2}\\\boxed{\cdot}\\\boxed{n}\end{smallmatrix}\right). \end{aligned}\tag{4.3}$$

In other words, the corresponding q-content discriminants (and/or q-transition matrices) are

$$C_q^{(n)}(\alpha)=F_q^{(n)}(\alpha)=\sum_{\sigma\in\mathfrak{S}_n}\alpha^{n-\nu_n(\sigma)},\qquad C_q^{(1,\dots,1)}(\alpha)=F_q^{(1,\dots,1)}(\alpha)=\sum_{\sigma\in\mathfrak{S}_n}\alpha^{n-\nu_n(\sigma)}\bigl(-q^2\bigr)^{\ell(\sigma)}.$$

In particular,

$$\begin{aligned} m_q^{(n)}(\alpha)&=\begin{cases}0, & \alpha=-1,-\frac{1}{2},\dots,-\frac{1}{n-1},\\ 1, & \textit{otherwise,}\end{cases}\\ m_q^{(1,\dots,1)}(\alpha)&=\begin{cases}0, & \sum_{\sigma\in\mathfrak{S}_n}\alpha^{n-\nu_n(\sigma)}(-q^2)^{\ell(\sigma)}=0,\\ 1, & \textit{otherwise.}\end{cases} \end{aligned}\tag{4.4}$$

Proof. The first equation in (4.3) is straightforward. To prove the second one, notice that $D_q^{(\alpha)}(1,2,\dots,n)\cdot\pi^{(\alpha)}(\mathbb{E}_q(\boxed{\begin{smallmatrix}1\\2\\ \cdot\\ n\end{smallmatrix}}))$ is contained in the one dimensional invariant subspace $\mathbb{C}\cdot\det_q$, and hence it must be a scalar multiple of $\det_q$. The scalar is given by the coefficient of $x_{11}\cdots x_{nn}$ in

$$v^{(\alpha)}\left(\boxed{\begin{smallmatrix}1\\2\\ \cdot\\ n\end{smallmatrix}}\right)=D_q^{(\alpha)}(1,2,\dots,n)\cdot\pi^{(\alpha)}\left(\mathbb{E}_q\left(\boxed{\begin{smallmatrix}1\\2\\ \cdot\\ n\end{smallmatrix}}\right)\right)=\sum_{w\in\mathfrak{S}_n}(-q)^{\ell(w)}D_q^{(\alpha)}\big(w(1),\dots,w(n)\big)$$
$$=\sum_{w\in\mathfrak{S}_n}(-q)^{\ell(w)}\sum_{\sigma\in\mathfrak{S}_n}\alpha^{n-\nu_n(\sigma)}q^{\ell(\sigma)}x_{\sigma(1)w(1)}\cdots x_{\sigma(n)w(n)}.$$

The coefficient is equal to

$$\sum_{w=\sigma\in\mathfrak{S}_n}(-q)^{\ell(w)}\alpha^{n-\nu_n(\sigma)}q^{\ell(\sigma)}=\sum_{\sigma\in\mathfrak{S}_n}\alpha^{n-\nu_n(\sigma)}\big(-q^2\big)^{\ell(\sigma)}$$

as desired. The last statement (4.4) about the multiplicity $m_q^{(n)}(\alpha)$ follows immediately from the fact

$$\sum_{\sigma\in\mathfrak{S}_n}\alpha^{n-\nu_n(\sigma)}=\prod_{k=1}^{n-1}(1+k\alpha)$$

(see, e.g. [6] or [9]). This shows the proposition. □

As a corollary, we have the following.

Corollary 4.5. *Define the set* $\mathrm{prSing}_{n,q}$ *by* $\mathrm{prSing}_{n,q}=\mathrm{Sing}_q^{(n)}\cup\mathrm{Sing}_q^{(1,\dots,1)}$. *Then*

$$\mathrm{prSing}_{n,q}=\left\{-1,-\frac{1}{2},\dots,-\frac{1}{n-1}\right\}\cup\left\{\alpha\in\mathbb{C}\;\Big|\;\sum_{w\in\mathfrak{S}_n}\big(-q^2\big)^{\ell(w)}\alpha^{n-\nu_n(w)}=0\right\}$$

and $\mathrm{prSing}_{n,q}$ *is a subset of* $\mathrm{Sing}_{n,q}$.

We call the singular points in $\mathrm{Sing}_q^{(n)}$ *classical* and the one in $\mathrm{Sing}_q^{(1,\dots,1)}$ *semi-classical*. We notice that $\mathrm{prSing}_{n,1}=\{\pm1,\pm\frac{1}{2},\dots,\pm\frac{1}{n-1}\}=\mathrm{Sing}_{n,1}$ in the classical case. However, it could be true that $\mathrm{Sing}_{n,q}\supsetneq\mathrm{prSing}_{n,q}$ in the quantum case (see, e.g. Example 3.12). When $\alpha\in\mathrm{Sing}_{n,q}\setminus\mathrm{prSing}_{n,q}$, we call it a *quantum* singular point.

4.4. Several explicit points in $\mathrm{Sing}_{n,q}$

We use the following lemma.

Lemma 4.6. *(See [4, Lemma 2.1].) For any* $g\in\mathfrak{S}_n$, *the equality*

$$\sum_{w\in\mathfrak{S}_k}\alpha^{n-\nu_n(wg)}=\alpha^{n-\nu_n(w_0g)}(1+\alpha)\cdots\big(1+(k-1)\alpha\big)$$

holds. Here $\mathfrak{S}_k$ is regarded as a subgroup $\mathfrak{S}_k = \{w \in \mathfrak{S}_n;\ w(x) = x,\ x > k\}$ of $\mathfrak{S}_n$, and w_0 is the element in $\mathfrak{S}_k$ (depending on g) such that $\nu_n(w_0 g) \geqslant \nu_n(wg)$ for any $w \in \mathfrak{S}_k$.

The following lemma plays a key role for understanding the multiplicity $m_q^\lambda(-\frac{1}{k})$.

Lemma 4.7. *Let k be a positive integer less that n. If $\alpha = -\frac{1}{k}$, then*

$$D_q^{(\alpha)}\big(a_1, \dots, a_m, 1^{k+1}, b_1, \dots, b_l\big) = 0 \tag{4.5}$$

for any $a_1, \dots, a_m, b_1, \dots, b_l$. Here 1^{k+1} denotes the $k+1$ consecutive sequence $\overbrace{1, \dots, 1}^{k+1}$.

Proof. Let $I_{n,k}$ be the set consisting of finite sequences $(i_1, \dots, i_k) \in \{1, 2, \dots, n\}^k$ such that the entries are distinct. We write $\boldsymbol{i} \cap \boldsymbol{j} = \emptyset$ if $\boldsymbol{i} \in I_{n,m}$ and $\boldsymbol{j} \in I_{n,l}$ have no common entry. For any pair $\boldsymbol{i} = (i_1, \dots, i_m) \in I_{n,m}$ and $\boldsymbol{j} = (j_1, \dots, j_l) \in I_{n,l}$ such that $\boldsymbol{i} \cap \boldsymbol{j} = \emptyset$, we put

$$W_n(\boldsymbol{i}, \boldsymbol{j}) = \big\{w \in \mathfrak{S}_n \bigm| w(x) = i_x\ (1 \leqslant x \leqslant m),\ w(n-l+y) = j_y\ (1 \leqslant y \leqslant l)\big\}.$$

Then we have

$$\begin{aligned} &D_q^{(\alpha)}\big(a_1, \dots, a_m, 1^{k+1}, b_1, \dots, b_l\big) \\ &\quad = \sum_{w \in \mathfrak{S}_n} \alpha^{n-\nu_n(w)} q^{\ell(w)} x_{w(1)a_1} \cdots x_{w(m)a_m} x_{w(m+1)1} \cdots x_{w(m+k+1)1} x_{w(n-l+1)b_1} \cdots x_{w(n)b_l} \\ &\quad = \sum_{\substack{\boldsymbol{i}=(i_1,\dots,i_m)\in I_{n,m} \\ \boldsymbol{j}=(j_1,\dots,j_l)\in I_{n,l} \\ \boldsymbol{i}\cap\boldsymbol{j}=\emptyset}} \sum_{w \in W_n(\boldsymbol{i},\boldsymbol{j})} \alpha^{n-\nu_n(w)} q^{\ell(w)} x_{i_1a_1} \cdots x_{i_ma_m} x_{w(m+1)1} \cdots x_{w(m+k+1)1} x_{j_1b_1} \cdots x_{j_lb_l}, \end{aligned} \tag{4.6}$$

where we suppose that $m + (k+1) + l = n$. To prove the lemma, we show that each sum

$$\sum_{w \in W_n(\boldsymbol{i},\boldsymbol{j})} \alpha^{n-\nu_n(w)} q^{\ell(w)} x_{i_1a_1} \cdots x_{i_ma_m} x_{w(m+1)1} \cdots x_{w(m+k+1)1} x_{j_1b_1} \cdots x_{j_lb_l} \tag{4.7}$$

in (4.6) has a factor $(1+\alpha)\cdots(1+k\alpha)$. Consider the group

$$\mathfrak{S}_n(m,l) = \big\{w \in \mathfrak{S}_n \bigm| w(x) = x\ (x \leqslant m,\ n-l+1 \leqslant x)\big\}.$$

The group $\mathfrak{S}_n(m,l)$ acts on $W_n(\boldsymbol{i}, \boldsymbol{j})$ from right transitively and faithfully. We take the unique element $w_0 \in W_n(\boldsymbol{i}, \boldsymbol{j})$ such that $w_0(x) < w_0(y)$ for $m+1 \leqslant \forall x < \forall y \leqslant m+k+1$. Then we have $W_n(\boldsymbol{i}, \boldsymbol{j}) = w_0 \cdot \mathfrak{S}_n(m,l)$ and $\ell(w_0 w) = \ell(w_0) + \ell(w)$ for $w \in \mathfrak{S}_n(m,l)$. Therefore, the sum (4.7) is rewritten as

$$\begin{aligned} &\sum_{w \in \mathfrak{S}_n(m,l)} \alpha^{n-\nu_n(w_0w)} q^{\ell(w_0w)} x_{i_1a_1} \cdots x_{i_ma_m} x_{w_0w(m+1)1} \cdots x_{w_0w(m+k+1)1} x_{j_1b_1} \cdots x_{j_lb_l} \\ &\quad = \Bigg(\sum_{w \in \mathfrak{S}_n(m,l)} \alpha^{n-\nu_n(w_0w)}\Bigg) q^{\ell(w_0)} x_{i_1a_1} \cdots x_{i_ma_m} x_{w_0(m+1)1} \cdots x_{w_0(m+k+1)1} x_{j_1b_1} \cdots x_{j_lb_l}. \end{aligned}$$

Since $\nu_n(\cdot)$ is a class function and $\mathfrak{S}_n(m,l) = g\mathfrak{S}_{k+1}g^{-1}$ for some $g \in \mathfrak{S}_n$, we have

$$\sum_{w\in\mathfrak{S}_n(m,l)} \alpha^{n-\nu_n(w_0w)} = \sum_{w\in\mathfrak{S}_{k+1}} \alpha^{n-\nu_n(w_0gwg^{-1})} = \sum_{w\in\mathfrak{S}_{k+1}} \alpha^{n-\nu_n(w\cdot g^{-1}w_0g)},$$

which indeed has a factor $(1+\alpha)\cdots(1+k\alpha)$ by Lemma 4.6. Thus we have the lemma. □

Remark 4.8. Even though there are $k+1$ identical columns, if they are not consecutive like (4.5), then Lemma 4.7 does not hold. For the classical case, such a consecutiveness condition is unnecessary to hold the vanishment. See [4].

As a corollary of the lemma above, we have the

Corollary 4.9. *If* $\frac{\lambda_1}{1+|\lambda|} > 1 - \frac{1}{1+k}$, *then* $m_q^{\lambda}(-\frac{1}{k}) = 0$.

Proof. Recall that the vectors $v^{(\alpha)}(T) = D_q^{(\alpha)}(\boldsymbol{j}(T)) \cdot \pi^{(\alpha)}(\mathbb{E}_q(T)) \in \boldsymbol{V}_{n,q}^{(\alpha)}(\lambda)$ for $T \in \mathrm{STab}(\lambda)$ of size n ($\lambda \vdash n$) form a basis of the space of highest weight vectors $W_{n,q}^{(\alpha)}(\lambda)$ in $\boldsymbol{V}_{n,q}^{(\alpha)}(\lambda)$. Here $\boldsymbol{j}(T) = (j_1(T), \ldots, j_n(T)) \in I_n(\lambda)$ is defined in (3.4). Now it is easy to see that the inequality $\frac{\lambda_1}{1+|\lambda|} > 1 - \frac{1}{1+k}$ can be written as $k(|\lambda| - \lambda_1 + 1) < \lambda_1$. In other words, we have

$$k\big(\big|\{p;\ j_p(T) \neq 1\}\big| + 1\big) < \big|\{p;\ j_p(T) = 1\}\big|.$$

By the pigeonhole principle, this implies that there exists at least one $k+1$ consecutive sequence 1^{k+1} in $\boldsymbol{j}(T)$. Hence, by Lemma 4.7 the result is immediate. □

Practically, using Lemma 4.7, we can estimate the multiplicity $m_q^{\lambda}(-\frac{1}{k})$ for each positive integer k more accurately if the partition λ is given explicitly.

Example 4.10. The vectors

$$D_q^{(\alpha)}(1, \ldots, 1, \overset{j\text{th}}{2}, 1, \ldots, 1) \cdot \pi^{(\alpha)}\Big(\mathbb{E}_q\Big(\begin{smallmatrix} \boxed{1}\boxed{*}\boxed{\cdot}\boxed{*} \\ \boxed{j} \end{smallmatrix}\Big)\Big) \quad (j = 2, \ldots, n)$$

generate the space of highest weight vectors $W_{n,q}^{(\alpha)}(\lambda)$ when $\lambda = (n-1, 1)$. Since $D_q^{(\alpha)}(1, \ldots, 1, \overset{j\text{th}}{2}, 1, \ldots, 1) \neq 0$ holds only if $n-k \leqslant j \leqslant k+1$, we have $m_q^{(n-1,1)}(-\frac{1}{k}) \leqslant \max\{2k+2-n, 0\}$. In particular, we have $m_q^{(n-1,1)}(-\frac{1}{k}) = 0$ whenever $n > 2k+1$. Moreover, suppose $k = n-1$. Then $2k+2-n = n > f^{\lambda} = n-1$. Therefore, in this case, there is at least one nonzero vector of the form $D_q^{(\alpha)}(1, \ldots, 1, \overset{j\text{th}}{2}, 1, \ldots, 1)$ which is killed by the q-Young symmetrizer $\pi^{(\alpha)}(\mathbb{E}_q(\begin{smallmatrix} \boxed{1}\boxed{*}\boxed{\cdot}\boxed{*} \\ \boxed{j} \end{smallmatrix}))$.

For classical singular points, we have the following general result.

Theorem 4.11. *It holds that*

$$\mathrm{Sing}_{n,q}\left(-\frac{1}{k}\right) = \big\{\lambda \in \widetilde{\mathcal{L}}_n^{\mathrm{dom}} \bigm| \langle\lambda, \varepsilon_1\rangle > k\big\}$$

for each positive integer k.

Proof. If $\lambda_1 > k$, then there exists a standard tableau $T \in \mathrm{STab}(\lambda)$ such that the highest weight vector

$$v^{(\alpha)}(T) = D_q^{(\alpha)}\bigl(1^{k+1}, *, \dots, *\bigr) \cdot \pi^{(\alpha)}\bigl(\mathbb{E}_q(T)\bigr)$$

vanishes when $\alpha = -\frac{1}{k}$ by Lemma 4.7. Conversely, suppose $\lambda \in \mathrm{Sing}_{n,q}(-\frac{1}{k})$. Then, since $m_q^\lambda(-\frac{1}{k}) < f^\lambda$, there exists a standard tableau $T \in \mathrm{STab}(\lambda)$ such that $v^{(-\frac{1}{k})}(T) = 0$. Since we have the expression

$$v^{(-\frac{1}{k})}(T) = \sum_{\sigma \in \mathfrak{S}_\lambda \backslash \mathfrak{S}_n} Q_T^\sigma(q) D_q^{(-\frac{1}{k})}\bigl(\boldsymbol{k}(\lambda)^\sigma\bigr)$$

by (3.5) where $Q_T^\sigma(q)$ is a polynomial in q, the identity $v^{(-\frac{1}{k})}(T) = 0$ remains true at $q = 1$. It hence follows that $\lambda \in \mathrm{Sing}_{n,1}(-\frac{1}{k})$, whence $\lambda_1 > k$ by (4.1). This proves the theorem. □

Remark 4.12. Let $\alpha = \alpha(q) \in \mathrm{Sing}_{n,q}$. Then there is an integer k ($1 \leqslant |k| \leqslant n-1$) such that $\alpha(q) \to \frac{1}{k}$ when $q \to 1$. This is because the algebraic function $\alpha(q)$ is a root of some q-content discriminant and this discriminant reduces to a content polynomial when $q \to 1$. Therefore, there is a canonical map $\mathrm{Sing}_{n,q}(\alpha(q)) \to \mathrm{Sing}_{n,1}(\frac{1}{k})$ when $\alpha(q) \to \frac{1}{k}$, that is, $\lambda \in \mathrm{Sing}_{n,q}(\alpha(q))$ implies $\lambda \in \mathrm{Sing}_{n,1}(\frac{1}{k})$ ($q \to 1$). However, the limit formula $\lim_{q\to 1} m_q^\lambda(\alpha(q)) = m_1^\lambda(\frac{1}{k})$ does not hold in general.

By (3.5), we see that $m_q^\lambda(\alpha) = \dim W_{n,q}^{(\alpha)}(\lambda) = \mathrm{rank}\, \widetilde{Q}_n^\lambda(q) \widetilde{F}_q^\lambda(\alpha)$, where the matrix $\widetilde{Q}_n^\lambda(q)$ is defined in (3.6). Since $\widetilde{F}_q^\lambda(\alpha)$ is the identity matrix when $\alpha = 0$, we have in particular

$$\mathrm{rank}\, \widetilde{Q}_n^\lambda(q) = \dim W_{n,q}^{(0)}(\lambda) = f^\lambda.$$

Therefore, we obtain the following rough estimation of the multiplicity

$$f^\lambda + \mathrm{rank}\, \widetilde{F}_q^\lambda(\alpha) - |\mathfrak{S}_\lambda \backslash \mathfrak{S}_n| \leqslant m_q^\lambda(\alpha) \leqslant \min\bigl\{f^\lambda, \mathrm{rank}\, \widetilde{F}_q^\lambda(\alpha)\bigr\}.$$

Since $\det_q^{(-1)}$ equals the quantum determinant $\det_q$, we see that $\boldsymbol{V}_{n,q}^{(-1)}$ defines the one dimensional representation. It follows that

$$\mathrm{rank}\, \widetilde{F}_q^\lambda(-1) = \begin{cases} 1, & \lambda = \varepsilon_1 + \dots + \varepsilon_n, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, in particular

$$\mathrm{Sing}_{n,q}(-1) = \widetilde{\mathcal{L}}_n^{\mathrm{dom}} \setminus \{\varepsilon_1 + \dots + \varepsilon_n\} = \bigl\{\lambda \in \widetilde{\mathcal{L}}_n^{\mathrm{dom}} \bigm| \langle \lambda, \varepsilon_1 \rangle > 1\bigr\}$$

as we stated in Theorem 4.11 and $m_q^\lambda(-1) = 0$ for $\lambda \in \mathrm{Sing}_{n,q}(-1)$.

From these observation with the facts from Examples 3.11 and 3.12, we naturally reach the

Conjecture A. *A singular value $\alpha \in \mathrm{Sing}_{n,q}$ is a quantum (i.e. $\alpha \notin \mathrm{prSing}_{n,q}$) if and only if $m_q^\lambda(\alpha) \neq 0$ for any $\lambda \in \widetilde{\mathcal{L}}_n^{\mathrm{dom}}$.*

Remark 4.13. The following question comes up naturally but is nontrivial: What can one say about the relation between the multiplicities of the roots of q-content discriminants and the multiplicities of the irreducible subrepresentations in $\boldsymbol{V}_{n,q}^{(\alpha)}$?

4.5. $(\mathcal{U}_q(\mathfrak{gl}_n), \mathcal{H}_q(\mathfrak{S}_n))$-bimodule $\mathcal{V}_{n,q}^{(\alpha)}$

We define the representation π of the algebra $\mathcal{H}_q(\mathfrak{S}_n)$ on the space $\boldsymbol{V}_{n,q}^{(0)} = \bigoplus_{1\leqslant j_1,\dots,j_n\leqslant n} \mathbb{C}\cdot x_{1j_1}\cdots x_{nj_n}$ by

$$x_{1j_1}\cdots x_{kj_k}x_{k+1j_{k+1}}\cdots x_{nj_n}\cdot\pi(h_k) = x_{1j_1}\cdots x_{k+1j_k}x_{kj_{k+1}}\cdots x_{nj_n}.$$

It is immediate to see that the $\mathcal{U}_q(\mathfrak{gl}_n)$-intertwiner $\Phi_{n,q}^{(0)} : (\mathbb{C}^n)^{\otimes n} \to \boldsymbol{V}_{n,q}^{(0)}$ is also $\mathcal{H}_q(\mathfrak{S}_n)$-equivariant. Hence $\pi(\mathcal{H}_q(\mathfrak{S}_n))$ is the commutant of $\rho(\mathcal{U}_q(\mathfrak{gl}_n))$ in $\mathrm{End}(\boldsymbol{V}_{n,q}^{(0)})$ by Schur–Weyl duality and vice versa [2] (cf. [10]). Therefore, as a $(\mathcal{U}_q(\mathfrak{gl}_n), \mathcal{H}_q(\mathfrak{S}_n))$-bimodule, we have the irreducible decomposition

$$\boldsymbol{V}_{n,q}^{(0)} \cong \bigoplus_{\lambda\vdash n} \boldsymbol{E}_{n,q}^{\lambda} \boxtimes \boldsymbol{M}_{n,q}^{\lambda}, \tag{4.8}$$

where $\boldsymbol{M}_{n,q}^{\lambda}$ is the irreducible $\mathcal{H}_q(\mathfrak{S}_n)$-module corresponding to the partition λ.

Consider the $(\mathcal{U}_q(\mathfrak{gl}_n), \mathcal{H}_q(\mathfrak{S}_n))$-cyclic module

$$\mathcal{V}_{n,q}^{(\alpha)} := \rho\big(\mathcal{U}_q(\mathfrak{gl}_n)\big)\cdot \det_q^{(\alpha)}\cdot\pi\big(\mathcal{H}_q(\mathfrak{S}_n)\big) \subset \boldsymbol{V}_{n,q}^{(0)}.$$

This is the smallest $\pi(\mathcal{H}_q(\mathfrak{S}_n))$-invariant subspace in $\boldsymbol{V}_{n,q}^{(0)}$ containing $\boldsymbol{V}_{n,q}^{(\alpha)}$. By (4.8), every irreducible component in $\mathcal{V}_{n,q}^{(\alpha)}$ is of the form $\boldsymbol{E}_{n,q}^{\lambda} \boxtimes \boldsymbol{M}_{n,q}^{\lambda}$ with multiplicity at most one. Hence the irreducible decomposition of the $(\mathcal{U}_q(\mathfrak{gl}_n), \mathcal{H}_q(\mathfrak{S}_n))$-bimodule $\mathcal{V}_{n,q}^{(\alpha)}$ is given as

$$\mathcal{V}_{n,q}^{(\alpha)} \cong \bigoplus_{\lambda\in Y_n(\alpha)} \boldsymbol{E}_{n,q}^{\lambda} \boxtimes \boldsymbol{M}_{n,q}^{\lambda},$$

where $Y_n(\alpha)$ is a certain subset of partitions of n. Therefore, if the multiplicity $m_q^{\lambda}(\alpha)$ of the irreducible representation $\boldsymbol{E}_{n,q}^{\lambda}$ in $\boldsymbol{V}_{n,q}^{(\alpha)}$ is not zero, then the irreducible component $\boldsymbol{E}_{n,q}^{\lambda} \boxtimes \boldsymbol{M}_{n,q}^{\lambda}$ does appear in the irreducible decomposition of $\mathcal{V}_{n,q}^{(\alpha)}$. Thus Conjecture A is restated as the following form.

Conjecture A′. *We have $\mathcal{V}_{n,q}^{(\alpha)} \cong \bigoplus_{\lambda\vdash n} \boldsymbol{E}_{n,q}^{\lambda} \boxtimes \boldsymbol{M}_{n,q}^{\lambda}$ if and only if $\alpha \in \mathbb{C} \setminus \mathrm{prSing}_{n,q}$.*

We note that $\mathrm{prSing}_{n,1} = \mathrm{Sing}_{n,1}$. Hence, the conjecture can be regarded as a quantum counterpart of the fact that $\mathcal{V}_{n,1}^{(\alpha)} \cong \bigoplus_{\lambda\vdash n} \boldsymbol{E}_{n,1}^{\lambda} \boxtimes \boldsymbol{M}_{n,1}^{\lambda}$ if and only if $\alpha \in \mathbb{C} \setminus \mathrm{Sing}_{n,1}$.

4.6. Examples of q-content discriminants

We here collect several examples of q-content discriminants with some of their corresponding q-transition matrices.

Example 4.14. The matrix $\widetilde{F}_q^{(2,1)}(\alpha)$ corresponding to $\widetilde{C}_q^{(2,1)}(\alpha)$ in the example above is given by

$$\widetilde{F}_q^{(2,1)}(\alpha) = (1+\alpha)\begin{pmatrix} 1 & \alpha q & \alpha q^2 \\ \alpha q & 1 & \alpha q(2-q^2) \\ \alpha q^2 & \alpha q(2-q^2) & 1+2\alpha-2\alpha q^2 \end{pmatrix}.$$

The matrix $\widetilde{F}_q^{(1,1,1)}(\alpha)$ corresponding to $\widetilde{C}_q^{(1,1,1)}(\alpha)$ is also given by

$$\widetilde{F}_q^{(1,1,1)}(\alpha) = \begin{pmatrix} 1 & \alpha q & \alpha q & \alpha q^3 & \alpha^2 q^2 & \alpha^2 q^2 \\ \alpha q & \gamma_1 & \alpha^2 q^2 & \alpha q^2\gamma_3 & \alpha q^3 & \alpha q\gamma_1 \\ \alpha q & \alpha^2 q^2 & \gamma_1 & \alpha q^2\gamma_3 & \alpha q\gamma_1 & \alpha q^3 \\ \alpha q^3 & \alpha q^2\gamma_3 & \alpha q^2\gamma_3 & \gamma_5 & \gamma_4 & \gamma_4 \\ \alpha^2 q^2 & \alpha q^3 & \alpha q\gamma_1 & \gamma_4 & \gamma_2 & \alpha q^2\gamma_3 \\ \alpha^2 q^2 & \alpha q\gamma_1 & \alpha q^3 & \gamma_4 & \alpha q^2\gamma_3 & \gamma_2 \end{pmatrix},$$

where $\gamma_1, \dots, \gamma_5$ are given by

$$\gamma_1 = 1+\alpha-\alpha q^2, \qquad \gamma_2 = 1+\alpha-\alpha q^4, \qquad \gamma_3 = 1+\alpha-q^2,$$
$$\gamma_4 = \alpha q\bigl(2+2\alpha-2q^2-2\alpha q^2+q^4\bigr),$$
$$\gamma_5 = 1+3\alpha+2\alpha^2-4\alpha q^2-4\alpha^2 q^2+2\alpha q^4+2\alpha^2 q^4-\alpha q^6.$$

If we put

$$P = \begin{pmatrix} q^3 & q & q & -q & -q & 1 \\ q^2 & 1-q^2 & q & -1+q^2 & q & -q \\ q^2 & q & 1-q^2 & q & -1+q^2 & -q \\ 1 & -q & -q & -q & -q & -q^3 \\ q & 0 & 1-q-q^2 & 0 & 1+q-q^2 & q^2 \\ q & 1-q-q^2 & 0 & 1+q-q^2 & 0 & q^2 \end{pmatrix},$$

then we have

$$P^{-1}\widetilde{F}_q^{(1,1,1)}(\alpha)P = \operatorname{diag}\bigl(\alpha_{(3)}, \alpha^+_{(2,1)}, \alpha^+_{(2,1)}, \alpha^-_{(2,1)}, \alpha^-_{(2,1)}, \alpha_{(1,1,1)}\bigr),$$

where $\alpha_{(3)} = (1+\alpha)(1+2\alpha)$, $\alpha^{\pm}_{(2,1)} = (1+\alpha)(1+\alpha q^2(1\pm(q-q^{-1})))$, $\alpha_{(1,1,1)} = 1-2\alpha q^2+2\alpha^2 q^4-\alpha q^6$.

Example 4.15. The q-content discriminants for 3-box diagrams are given by

$$\begin{aligned}
C_q^{(3)}(\alpha) &= (1+\alpha)(1+2\alpha),\\
C_q^{(2,1)}(\alpha) &= (1+\alpha)^2\big(1+\big(q-q^2-q^3\big)\alpha\big)\big(1+\big(-q-q^2+q^3\big)\alpha\big),\\
C_q^{(1,1,1)}(\alpha) &= 1-2\alpha q^2+2\alpha^2q^4-\alpha q^6.
\end{aligned}$$

These are obtained by the equations (Lemma 4.2)

$$\begin{aligned}
&\widetilde{C}_q^{(3)}(\alpha) = C_q^{(3)}(\alpha), \qquad \widetilde{C}_q^{(2,1)}(\alpha) = C_q^{(3)}(\alpha)C_q^{(2,1)}(\alpha),\\
&\widetilde{C}_q^{(1,1,1)}(\alpha) = C_q^{(3)}(\alpha)C_q^{(2,1)}(\alpha)^2C_q^{(1,1,1)}(\alpha),
\end{aligned}$$

and $\widetilde{C}_q^{\lambda}(\alpha)=\det \widetilde{F}_q^{\lambda}(\alpha)$ are calculated from the results in Example 4.14.

Example 4.16. Using the fact that

$$\begin{aligned}
\mathbb{E}_q\big(\begin{smallmatrix}1&2\\3&4\end{smallmatrix}\big) &= 1+q^{-1}h_1+\big(q^{-1}-q\big)h_2+q^{-1}h_3-q^{-1}h_1h_2h_1-q^{-1}h_2h_3h_2-h_1h_2\\
&\quad+\big(q^{-2}-1\big)h_2h_1+\big(q^{-2}-1\big)h_2h_3-h_3h_2-q^{-2}h_2h_3h_2h_1-q^{-2}h_1h_2h_1h_3\\
&\quad-q^{-1}h_1h_2h_3-q^{-1}h_3h_2h_1+\big(q^{-3}-q^{-1}\big)h_2h_1h_3+qh_1h_2h_1h_3h_2\\
&\quad+qh_2h_3h_2h_1h_2+q^{-2}h_1h_3+q^2h_2h_1h_3h_2+h_1h_2h_1h_3h_2h_1,\\
\mathbb{E}_q\big(\begin{smallmatrix}1&3\\2&4\end{smallmatrix}\big) &= 1-qh_1-qh_3+q^{-1}h_1h_2h_1+q^{-1}h_2h_3h_2-\big(q^{-1}-q\big)h_1h_2h_3h_2h_1\\
&\quad-q^{-2}h_2h_1-q^{-2}h_2h_3-h_1h_2h_3h_2+\big(q^{-2}-1\big)h_2h_3h_2h_1+\big(q^{-2}-1\big)h_1h_2h_1h_3\\
&\quad-h_1h_3h_2h_1+q^{-1}h_1h_2h_3+q^{-1}h_3h_2h_1+\big(q^{-1}-q^{-3}\big)h_2h_1h_3-q^{-1}h_1h_2h_1h_3h_2\\
&\quad-q^{-1}h_2h_3h_2h_1h_2+q^2h_1h_3+q^{-2}h_2h_1h_3h_2+h_1h_2h_1h_3h_2h_1,
\end{aligned}$$

it follows that

$$\begin{aligned}
v^{(\alpha)}\big(\begin{smallmatrix}1&2\\3&4\end{smallmatrix}\big) &= (1+\alpha)\big(1+2\alpha+\alpha^2-3\alpha q^2-3\alpha^2q^2+2\alpha^2q^6\big)v^{(0)}\big(\begin{smallmatrix}1&2\\3&4\end{smallmatrix}\big)\\
&\quad+\alpha q(1+\alpha)^2\big(1-q^2\big)^2v^{(0)}\big(\begin{smallmatrix}1&3\\2&4\end{smallmatrix}\big),\\
v^{(\alpha)}\big(\begin{smallmatrix}1&3\\2&4\end{smallmatrix}\big) &= -\alpha q^{-1}\big(1-q^2\big)(1+\alpha)\big(2+\alpha-q^2-4\alpha q^2+q^4+\alpha q^4\big)v^{(0)}\big(\begin{smallmatrix}1&2\\3&4\end{smallmatrix}\big)\\
&\quad+(1+\alpha)\big(1-\alpha-\alpha^2+3\alpha^2q^2-\alpha q^4-2\alpha^2q^4+\alpha q^6\big)v^{(0)}\big(\begin{smallmatrix}1&3\\2&4\end{smallmatrix}\big).
\end{aligned}$$

Hence the q-transition matrix $F_q^{(2,2)}(\alpha)$ is given by

$$\begin{aligned}
&(1+\alpha)\\
&\quad\times\begin{pmatrix}1+2\alpha+\alpha^2-3\alpha q^2-3\alpha^2q^2+2\alpha^2q^6 & \alpha q(1+\alpha)(1-q^2)^2\\
-\alpha q^{-1}(1-q^2)(2+\alpha-q^2-4\alpha q^2+q^4+\alpha q^4) & (1-\alpha-\alpha^2+3\alpha^2q^2-\alpha q^4-2\alpha^2q^4+\alpha q^6)\end{pmatrix}
\end{aligned}$$

and hence the corresponding q-content discriminant $C_q^{(2,2)}(\alpha)=\det F_q^{(2,2)}(\alpha)$ is

$$(1+\alpha)^2\big(1+\big(1-3q^2-q^4+q^6\big)\alpha-q^2\big(4-6q^2+q^4-q^6+q^8\big)\alpha^2$$
$$-2q^2\big(1-q^2\big)\big(1-5q^2+3q^4-q^6+q^8\big)\alpha^3-q^2\big(1-q^2\big)^2\big(1-3q^2+5q^4\big)\alpha^4\big).$$

Notice that the transition matrix $F_q^{(2,2)}(\alpha)$ does become a scalar matrix $(1-\alpha^2)I_2$ if we let $q=1$.

Example 4.17. The matrix $\widetilde{F}_q^{(3,1)}(\alpha)$ corresponding to $\widetilde{C}_q^{(3,1)}(\alpha)$ is given by

$$(1+\alpha)\begin{pmatrix} (1+2\alpha) & \alpha q(1+2\alpha) & \alpha q^2(1+2\alpha) & \alpha q^3(1+2\alpha) \\ \alpha q(1+2\alpha) & 1+\alpha+\alpha q^2 & \alpha q(2+2\alpha-q^2) & \alpha q^2(1+2\alpha)(2-q^2) \\ \alpha q^2(1+2\alpha) & \alpha q(2+2\alpha-q^2) & 1+2\alpha+2\alpha^2q^2(1-q^2) & \alpha q(1+2\alpha)(3-2q^2) \\ \alpha q^3(1+2\alpha) & \alpha q^2(1+2\alpha)(2-q^2) & \alpha q(1+2\alpha)(3-2q^2) & (1+2\alpha)(1+3\alpha(1-q^2)) \end{pmatrix}.$$

Further, the matrix $\widetilde{F}_q^{(2,2)}(\alpha)$ corresponding to $\widetilde{C}_q^{(2,2)}(\alpha)$ is given by

$$(1+\alpha)\begin{pmatrix} (1+\alpha) & \alpha q(1+\alpha) & \alpha q^2(1+\alpha) & \alpha q^2(1+\alpha) & \alpha^2q^3(1+\alpha) & 2\alpha^2q^4 \\ \alpha q(1+\alpha) & 1+\alpha q^2 & \alpha q\gamma_1 & \alpha q\gamma_1 & \alpha^2q^2(3-q^2) & \alpha q^3\gamma_2 \\ \alpha q^2(1+\alpha) & \alpha q\gamma_1 & \gamma_4 & 2\alpha^2q^2(2-q^2) & \alpha q\gamma_3 & \alpha q^2\gamma_5 \\ \alpha q^2(1+\alpha) & \alpha q\gamma_1 & 2\alpha^2q^2(2-q^2) & \gamma_4 & \alpha q\gamma_3 & \alpha q^2\gamma_5 \\ \alpha q^3(1+\alpha) & \alpha^2q^2(3-q^2) & \alpha q\gamma_3 & \alpha q\gamma_3 & \gamma_7 & \alpha q\gamma_6 \\ 2\alpha^2q^4 & \alpha q^3\gamma_2 & \alpha q^2\gamma_5 & \alpha q^2\gamma_5 & \alpha q\gamma_6 & \gamma_8 \end{pmatrix},$$

where $\gamma_1,\dots,\gamma_8$ are given by

$$\gamma_1=2-q^2+\alpha q^2,\qquad \gamma_2=1+3\alpha-2\alpha q^2,\qquad \gamma_3=1+2\alpha+q^2-\alpha q^2-q^4,$$
$$\gamma_4=1+2\alpha-2\alpha q^2+\alpha q^4,\qquad \gamma_5=2+4\alpha-q^2-3\alpha q^2,$$
$$\gamma_6=\alpha q\big(4+6\alpha-4q^2-6\alpha q^2+q^4+\alpha q^4\big),$$
$$\gamma_7=1+\alpha+2\alpha q^2+2\alpha^2q^2-3\alpha q^4-2\alpha^2q^4+\alpha q^6,$$
$$\gamma_8=1+5\alpha+6\alpha^2-4\alpha q^2-6\alpha^2q^2-2\alpha^2q^4+2\alpha^2q^6.$$

Example 4.18. We have

$$C_q^{(4)}(\alpha)=(1+\alpha)(1+2\alpha)(1+3\alpha),$$
$$C_q^{(3,1)}(\alpha)=(1+\alpha)^3(1+2\alpha)^2\big(1-q^2\big(2-q^2\big)\alpha\big)$$
$$\times\big(1+\big(1-q^4\big)\alpha-q^2\big(4-5q^2+4q^4\big)\alpha^2-2q^2\big(2-4q^2+2q^4-q^6\big)\alpha^3\big),$$
$$C_q^{(2,2)}(\alpha)=(1+\alpha)^2\big(1+\big(1-3q^2-q^4+q^6\big)\alpha-q^2\big(4-6q^2+q^4-q^6+q^8\big)\alpha^2$$
$$-2q^2\big(1-q^2\big)\big(1-5q^2+3q^4-q^6+q^8\big)\alpha^3-q^2\big(1-q^2\big)^2\big(1-3q^2+5q^4\big)\alpha^4\big).$$

These are obtained by the equations (Lemma 4.2)

$$\widetilde{C}_q^{(4)}(\alpha) = C_q^{(4)}(\alpha), \qquad \widetilde{C}_q^{(3,1)}(\alpha) = C_q^{(4)}(\alpha) C_q^{(3,1)}(\alpha),$$
$$\widetilde{C}_q^{(2,2)}(\alpha) = C_q^{(4)}(\alpha) C_q^{(3,1)}(\alpha) C_q^{(2,2)}(\alpha),$$

and $\widetilde{C}_q^{\lambda}(\alpha) = \det \widetilde{F}_q^{\lambda}(\alpha)$ are calculated from the results in Examples 4.16 and 4.17.

5. Quantum α-permanent

A theory similar to the one developed in Section 3 for the quantum α-determinants can be also established for the quantum α-permanent defined below. We hence close the paper by observing two examples concerning the cyclic $\mathcal{U}_q(\mathfrak{gl}_n)$-module $\boldsymbol{U}_{n,q}^{(\alpha)}$ generated by a quantum α-permanent, and give a conjecture on a 'reciprocity' between the multiplicities of the irreducible summands of two modules $\boldsymbol{V}_{n,q}^{(\alpha)}$ and $\boldsymbol{U}_{n,q}^{(\alpha)}$. Introducing partition functions for the multiplicities of respective irreducible decompositions, we close the paper by restating a certain weaker version of the conjecture (and also Conjecture A) in terms of the partition functions.

5.1. Quantum α-permanent cyclic modules $\boldsymbol{U}_{n,q}^{(\alpha)}$

We define a *quantum* (*column*) *α-permanent* by $\mathrm{per}_q^{(\alpha)} = \det_{-q^{-1}}^{(\alpha)} = \det_{q^{-1}}^{(-\alpha)}$. Namely, we have

$$\mathrm{per}_q^{(\alpha)} := \sum_{w \in \mathfrak{S}_n} \alpha^{n-\nu_n(w)} (-q)^{-\ell(w)} x_{w(1)1} \cdots x_{w(n)n} \in \mathcal{A}_q(\mathrm{Mat}_n).$$

We notice that $\mathrm{per}_q^{(1)}$ is the quantum permanent per_q. We also remark that $\mathrm{per}_q^{(\alpha)}(X) = \mathrm{per}_q^{(\alpha)}({}^t X)$ as in the case of $\det_q^{(\alpha)}(X)$. For convenience, we put

$$P_q^{(\alpha)}(j_1, \dots, j_n) := \sum_{w \in \mathfrak{S}_n} \alpha^{n-\nu_n(w)} (-q)^{-\ell(w)} x_{w(1),j_1} \cdots x_{w(n),j_n}.$$

We notice that $\mathrm{per}_q^{(\alpha)} = P_q^{(\alpha)}(1, 2, \dots, n)$. Let us consider the cyclic module

$$\boldsymbol{U}_{n,q}^{(\alpha)} := \rho\big(\mathcal{U}_q(\mathfrak{gl}_n)\big) \cdot \mathrm{per}_q^{(\alpha)},$$

and we also define $\mathrm{Sing}_{n,q}^{\mathrm{per}}$ and $m_q^n(\alpha)_{\mathrm{per}}$ similarly to $\mathrm{Sing}_{n,q}$ and $m_q^n(\alpha)$. By the same discussion as in the proofs of Proposition 3.3 and Lemma 3.1, we have the

Lemma 5.1. *We have*

$$\rho\big(q^{\lambda}\big) \cdot P_q^{(\alpha)}(j_1, \dots, j_n) = q^{\langle \lambda, \varepsilon_{j_1} + \cdots + \varepsilon_{j_n} \rangle} P_q^{(\alpha)}(j_1, \dots, j_n),$$
$$\rho(e_k) \cdot P_q^{(\alpha)}(j_1, \dots, j_n) = \sum_{l=1}^{n} \delta_{j_l, k+1} q_k^l(j_1, \dots, j_n) P_q^{(\alpha)}(j_1, \dots, j_{l-1}, k, j_{l+1}, \dots, j_n),$$
$$\rho(f_k) \cdot P_q^{(\alpha)}(j_1, \dots, j_n) = \sum_{l=1}^{n} \delta_{j_l, k} q_k^l(j_1, \dots, j_n) P_q^{(\alpha)}(j_1, \dots, j_{l-1}, k+1, j_{l+1}, \dots, j_n),$$

and

$$\boldsymbol{U}_{n,q}^{(\alpha)} = \sum_{1 \leqslant i_1, \dots, i_n \leqslant n} \mathbb{C} \cdot P_q^{(\alpha)}(i_1, \dots, i_n).$$

We also define

$$\begin{aligned} &P_q^{(\alpha)}(j_1, \dots, j_n) \cdot \pi_{\mathrm{per}}^{(\alpha)}(h_k) \\ &= \begin{cases} P_q^{(\alpha)}(j_1, \dots, j_{k+1}, j_k, \dots, j_n), & j_k < j_{k+1}, \\ q^{-1} P_q^{(\alpha)}(j_1, \dots, j_n), & j_k = j_{k+1}, \\ P_q^{(\alpha)}(j_1, \dots, j_{k+1}, j_k, \dots, j_n) - (q - q^{-1}) P_q^{(\alpha)}(j_1, \dots, j_n), & j_k > j_{k+1}. \end{cases} \end{aligned}$$

for a quantum α-permanent $P_q^{(\alpha)}(j_1, \dots, j_n)$ and $h_k \in \mathcal{H}_q(\mathfrak{S}_n)$ as in Section 3.3. We notice that, also in the present permanent case, the result corresponding to Proposition 3.4 holds when $\alpha \in \mathbb{C} \setminus \mathrm{Sing}_{n,q}^{\mathrm{per}}$.

Example 5.2. The highest weight vectors in $\boldsymbol{U}_{2,q}^{(\alpha)}$ are

$$P_q^{(\alpha)}(1,1) = \bigl(1 - \alpha q^{-2}\bigr) x_{11} x_{21}, \qquad P_q^{(\alpha)}(1,2) - q P_q^{(\alpha)}(2,1) = (1+\alpha) \det\nolimits_q.$$

Hence the component $\boldsymbol{E}_{2,q}^{(2)}$ (respectively $\boldsymbol{E}_{2,q}^{(1,1)}$) does not appear if $\alpha = -q^2$ (respectively $\alpha = 1$). Thus we have

$$m_q^{(2)}(\alpha)_{\mathrm{per}} = \begin{cases} 0, & \alpha = q^2, \\ 1, & \text{otherwise}, \end{cases} \qquad m_q^{(1,1)}(\alpha)_{\mathrm{per}} = \begin{cases} 0, & \alpha = -1, \\ 1, & \text{otherwise}, \end{cases}$$

and $\mathrm{Sing}_{2,q}^{\mathrm{per}} = \{-1, q^2\}$.

Example 5.3. Look at the $\mathcal{U}_q(\mathfrak{gl}_3)$-module $\boldsymbol{U}_{3,q}^{(\alpha)}$. We can take the highest weight vectors as follows:

$$\begin{aligned} u^{(3)} &= \bigl(1 - 2\alpha q^{-2} + 2\alpha^2 q^{-4} - \alpha q^{-6}\bigr) x_{11} x_{21} x_{31}, \\ u_1^{(2,1)} &= (1+\alpha)\bigl(1 - \bigl(q^{-1} + q^{-2} - q^{-3}\bigr)\alpha\bigr)\bigl(x_{11}x_{21}x_{32} + (1-q)x_{11}x_{22}x_{31} - q x_{12}x_{21}x_{31}\bigr), \\ u_2^{(2,1)} &= (1+\alpha)\bigl(1 - \bigl(-q^{-1} + q^{-2} + q^{-3}\bigr)\alpha\bigr)\bigl(x_{11}x_{21}x_{32} - (1+q)x_{11}x_{22}x_{31} + q x_{12}x_{21}x_{31}\bigr), \\ u^{(1,1,1)} &= (1+\alpha)(1+2\alpha)\det\nolimits_q. \end{aligned}$$

Therefore, we conclude that

$$\boldsymbol{U}_{3,q}^{(\alpha)} \cong \begin{cases} \boldsymbol{E}_{3,q}^{(3)}, & \alpha=-1, \\ \boldsymbol{E}_{3,q}^{(3)} \oplus (\boldsymbol{E}_{3,q}^{(2,1)})^{\oplus 2}, & \alpha=-\frac{1}{2}, \\ \boldsymbol{E}_{3,q}^{(3)} \oplus \boldsymbol{E}_{3,q}^{(2,1)} \oplus \boldsymbol{E}_{3,q}^{(1,1,1)}, & \alpha=1/(q^{-2} \pm (q^{-1}-q^{-3})), \\ (\boldsymbol{E}_{3,q}^{(2,1)})^{\oplus 2} \oplus \boldsymbol{E}_{3,q}^{(1,1,1)}, & \alpha=(2q^2+q^{-2} \pm \sqrt{q^{-4}+4-4q^4})/4, \\ \boldsymbol{E}_{3,q}^{(3)} \oplus (\boldsymbol{E}_{3,q}^{(2,1)})^{\oplus 2} \oplus \boldsymbol{E}_{3,q}^{(1,1,1)}, & \text{otherwise}. \end{cases}$$

In other words, each multiplicity can be described as

$$m_q^{(3)}(\alpha)_{\text{per}} = \begin{cases} 0, & \alpha=(2q^2+q^{-2} \pm \sqrt{q^{-4}+4-4q^4})/4, \\ 1, & \text{otherwise}, \end{cases}$$

$$m_q^{(2,1)}(\alpha)_{\text{per}} = \begin{cases} 0, & \alpha=-1, \\ 1, & \alpha=1/(q^{-2} \pm (q^{-1}-q^{-3})), \\ 2, & \text{otherwise}, \end{cases}$$

$$m_q^{(1,1,1)}(\alpha)_{\text{per}} = \begin{cases} 0, & \alpha=-1,-\frac{1}{2}, \\ 1, & \text{otherwise}. \end{cases}$$

Hence, as a counterpart of the $\text{Sing}_{3,q}$ for the quantum α determinant case (Example 3.12),

$$\text{Sing}_{3,q}^{\text{per}} = \left\{-1, -\frac{1}{2}, \frac{1}{q^{-2} \pm (q^{-1}-q^{-3})}, \frac{2q^2+q^{-2} \pm \sqrt{q^{-4}+4-4q^4}}{4}\right\}$$

is the corresponding singular set $\text{Sing}_{3,q}^{\text{per}}$ for this permanent.

5.2. *Reciprocity for multiplicities between* $\text{Sing}_{n,q}$ *and* $\text{Sing}_{n,q}^{\text{per}}$

By the same calculation as we did in the proof of Proposition 4.4, we get

$$m_q^{(n)}(\alpha)_{\text{per}} = \begin{cases} 0, & \sum_{\sigma \in \mathfrak{S}_n} \alpha^{n-\nu_n(\sigma)}(-q^{-2})^{\ell(\sigma)}=0, \\ 1, & \text{otherwise}, \end{cases}$$
$$m_q^{(1,\dots,1)}(\alpha)_{\text{per}} = \begin{cases} 0, & \alpha=-1,-\frac{1}{2},\dots,-\frac{1}{n-1}, \\ 1, & \text{otherwise}. \end{cases} \tag{5.1}$$

This shows in particular that, for the permanent case, the singular points $-\frac{1}{k}$ $(1 \leqslant k < n)$ should be also called *classical*. Moreover, comparing (4.4) and (5.1), we have the following remarkable relations

$$m_q^{(n)}\big(\alpha(q)\big)_{\text{per}} = m_q^{(1,\dots,1)}\big(\alpha\big(q^{-1}\big)\big), \qquad m_q^{(1,\dots,1)}\big(\alpha(q)\big)_{\text{per}} = m_q^{(n)}\big(\alpha\big(q^{-1}\big)\big).$$

Furthermore, we find the same relations in the case where $n=3$ by comparing Example 3.12 with Example 5.3 with respect to the transposition of the diagram λ. Thus, we naturally come to expect the following ‘reciprocity’ (or ‘mirror symmetry’ with respect to the reflection in the main diagram of λ).

Conjecture B.

(1) *If* $\alpha(q) \in \mathrm{Sing}^{\mathrm{per}}_{n,q}$, *then* $\alpha(q^{-1}) \in \mathrm{Sing}_{n,q}$.
(2) *The map*

$$\mathrm{Sing}^{\mathrm{per}}_{n,q} \ni \alpha(q) \mapsto \alpha\bigl(q^{-1}\bigr) \in \mathrm{Sing}_{n,q}$$

is bijective.
(3) *Let* $\alpha(q) \in \mathrm{Sing}^{\mathrm{per}}_{n,q}$. *Then the equality*

$$m^{\lambda}_q\bigl(\alpha(q)\bigr)_{\mathrm{per}} = m^{\lambda'}_q\bigl(\alpha\bigl(q^{-1}\bigr)\bigr)$$

holds for each $\lambda \in \widetilde{\mathcal{L}}^{\mathrm{dom}}_n$.

We note that the conjecture is true when $q = 1$ (see Section 4.2).

Remark 5.4. If the conjecture is true, then it follows from Corollary 4.9 that $m^{\lambda}_q(-\frac{1}{k})_{\mathrm{per}} = 0$ if $\frac{\lambda'_1}{1+|\lambda|} > 1 - \frac{1}{1+k}$. However, we notice that there is no permanent counterpart of Lemma 4.7.

Let $\mathcal{P}$ be the set of all partitions. Define generating functions of the multiplicities $m^{\lambda}_q(\alpha)$ and $m^{\lambda}_q(\alpha)_{\mathrm{per}}$ by

$$\vartheta^{\det}_q(t,\alpha) := \sum_{\lambda\in\mathcal{P}} \frac{m^{\lambda}_q(\alpha)}{f^{\lambda}} t^{|\lambda|} = \sum_{n=0}^{\infty}\sum_{\lambda\vdash n} \frac{m^{\lambda}_q(\alpha)}{f^{\lambda}} t^{n},$$

$$\vartheta^{\mathrm{per}}_q(t,\alpha) := \sum_{\lambda\in\mathcal{P}} \frac{m^{\lambda}_q(\alpha)_{\mathrm{per}}}{f^{\lambda}} t^{|\lambda|} = \sum_{n=0}^{\infty}\sum_{\lambda\vdash n} \frac{m^{\lambda}_q(\alpha)_{\mathrm{per}}}{f^{\lambda}} t^{n}.$$

We call $\vartheta^{\det}_q(t,\alpha)$ (respectively $\vartheta^{\mathrm{per}}_q(t,\alpha)$) the *partition function* of the cyclic module $\boldsymbol{V}^{(\alpha)}_{n,q}$ (respectively $\boldsymbol{U}^{(\alpha)}_{n,q}$). Obviously, one has $\vartheta^{\det}_1(t,\alpha) = \vartheta^{\mathrm{per}}_1(t,\alpha)$. If $\alpha \notin \bigcup_{n=1}^{\infty} \mathrm{Sing}_{n,q}$ (or $\alpha \notin \bigcup_{n=1}^{\infty} \mathrm{Sing}^{\mathrm{per}}_{n,q}$), then it is readily seen that

$$\vartheta^{\det}_q(t,\alpha) = \vartheta^{\mathrm{per}}_q(t,\alpha) = \prod_{i=1}^{\infty} \frac{1}{1-t^i}.$$

When $q = 1$, it is also easily calculated as

$$\vartheta^{\det}_1\biggl(t, \pm\frac{1}{k}\biggr) = \prod_{i=1}^{k} \frac{1}{1-t^i}$$

for $k = 1, 2, \ldots, \infty$.

In terms of these partition functions $\vartheta_q^{\det}(t,\alpha)$ and $\vartheta_q^{\mathrm{per}}(t,\alpha)$, (a slightly weaker version of) Conjectures A and B are respectively restated as follows.

Conjecture AB.

(1) *If α is a quantum singular point and $k = 1, 2, \dots, \infty$, then*

$$\vartheta_q^{\det}(t,\alpha) \neq \prod_{i=1}^{k} \frac{1}{1-t^i}.$$

(2) *The equality*

$$\vartheta_q^{\det}\bigl(t,\alpha\bigl(q^{-1}\bigr)\bigr) = \vartheta_q^{\mathrm{per}}\bigl(t,\alpha(q)\bigr)$$

holds for $\alpha(q) \in \bigcup_{n=1}^{\infty} \mathrm{Sing}_{n,q}^{\mathrm{per}}$.

As a step for approaching the conjecture, one perhaps needs to examine it first when the singularity is classical: Calculate explicitly $\vartheta_q^{\det}(t, -\frac{1}{k})$ and $\vartheta_q^{\mathrm{per}}(t, -\frac{1}{k})$ for $k \in \mathbb{Z}_{>0}$.

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