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Mathematical Games

Finite games for a predicate logic without contractions*

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Abstract

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A predicate calculus **LS** is obtained from a multiset calculus for classical logic by discarding the rule for contraction of formulas. Provability in **LS** is characterized by the existence of winning strategies of finite games between two players. As an application, results concerning provability and non-provability of complex formulas are established.

1. Introduction

In recent years, much work has been done in substructural logics (i.e. logics without rules for exchange, weakening or contraction of formulas) and their links to theoretical computer science. The most prominent representative, Girard's linear logic [4], has two types of conjunction and disjunction: an *additive* type and a *multiplicative* type. Moreover, the modalities ! and ? permit recovery of the structural rules in a selective way. Whereas the syntax of substructural logics is given by appropriate restrictions of calculi for classical or intuitionistic logic, there are no obvious adaptations of the usual algebraic semantics based on Boolean or Heyting algebras. In the

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absence of structural rules, the connectives for conjunction and disjunction need not be idempotent. This has to be reflected by semantics for these systems; see, e.g. the constructions in [4] or [2]. Alternatively, one may try to interpret formulas by mathematical games between two players. Provability can then be characterized by the existence of winning strategies for one of the players (for notions about games, see, e.g. [5]). In this manner, the *dialogue* games initiated by Lorenzen [6] characterize provability of intuitionistic logic. These games may be infinite, i.e. plays may not terminate within a finite number of moves of the players. Recently, Blass [3] suggested infinite games which reflect the behaviour of the connectives in linear logic plus weakening. Closely related games have been shown to characterize provability of the multiplicative fragment of linear logic [1].

In this note, the predicate calculus **LS** (German: *logischer schwacher Kalkül*) will be characterized by *finite* games, following a suggestion of Ernst Specker. The rules of **LS** incorporate exchange and weakening of formulas, but neither contraction nor cut. Conjunction $\wedge$ is additive and disjunction $+$ is multiplicative. As will be seen later, this notation may be justified by the behaviour of the connectives in the calculus: whereas conjunction is idempotent, disjunction is not idempotent. In contrast to linear logic, negation ($\neg$) relates these connectives via the de Morgan laws. The definition of the suggested games (see Definition 3.1) is quite simple and intuitive: the players P and Q gradually reduce the subformulas of a given formula until a disjunction of literals $B^1 + \cdots + B^m$ is reached. P wins the play if this disjunction contains complementary literals. It is shown that these games characterize **LS**-provability, i.e. a formula is **LS**-provable if and only if P has a winning strategy for the associated game (Theorem 4.2). This characterization can be concretely applied to demonstrate provability and nonprovability of complex formulas in **LS**.

2. The calculus LS

In the following, the calculus **LC** for classical first-order predicate logic is defined. The *language* of **LC** is an arbitrary but fixed language of first-order predicate calculus, containing symbols for arbitrary constants, functions and predicates. It is defined in the following version. Free variables $u, v, w, \ldots$ and bound variables $x, y, z, \ldots$ are distinguished. Formulas are built from unnegated and negated prime formulas (*literals*, denoted by E, $\neg E$), using the connectives $\wedge$, $+$ and the quantifiers $\forall$, $\exists$. *Negation* $\neg$ of a formula is recursively defined by

$$\neg\neg E = E,$$

$$\neg(A \wedge B) = \neg A + \neg B,$$

$$\neg(A + B) = \neg A \wedge \neg B,$$

$$\neg(\forall x A(x)) = \exists x \neg A(x),$$

$$\neg(\exists x A(x)) = \forall x \neg A(x).$$

(No definition for prime formulas E since $\neg E$ is already in the language.)
Implication may be defined by $A \to B = \neg A + B$.
Multisets are finite unordered sequences of formulas, denoted by upper case greek letters. The multiset obtained by the concatenation of two multisets Γ and Π is denoted by Γ, Π.

Definition 2.1. **LC** contains the following rules of inference:

axiom $\Gamma, E, \neg E$ (E prime formula)

$\wedge$-*rule* $\dfrac{\Gamma, T \quad \Gamma, Z}{\Gamma, T \wedge Z} \wedge$ $\qquad$ $+$-*rule* $\dfrac{\Gamma, T, Z}{\Gamma, T + Z} +$

$\forall$-*rule* $\dfrac{\Gamma, F(u)}{\Gamma, \forall x F(x)} \forall$ ... provided u is not in the lower multiset

$\exists$-*rule* $\dfrac{\Gamma, F(t)}{\Gamma, \exists x F(x)} \exists$

contraction-rule $\dfrac{\Gamma, T, T}{\Gamma, T} C$

Remark. It is easy to demonstrate that **LC** is indeed a sound and complete calculus for classical predicate logic, e.g. by showing that the system is equivalent to Gentzen's sequent calculus **LK**.

Definition 2.2. The calculus **LS** is obtained from **LC** by discarding the contraction rule.

Example 2.3. Let R be a unary predicate symbol, f a unary function symbol and $A = \exists x(R(x) + \neg R(f(x)))$. Below, an **LS**-proof of $A + A$ is given:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{R(f(u)), \neg R(f(f(u))), R(u), \neg R(f(u))}{R(f(u)) + \neg R(f(f(u))), R(u), \neg R(f(u))} +}{R(f(u)) + \neg R(f(f(u))), R(u) + \neg R(f(u))} +}{\exists x(R(x) + \neg R(f(x))), R(u) + \neg R(f(u))} \exists}{\exists x(R(x) + \neg R(f(x))), \exists x(R(x) + \neg R(f(x)))} \exists}{A + A} +$$

Replacing the lowest inference of this proof by a contraction yields an **LC**-proof of A. Thus, the formula is classically provable. However, it is not **LS**-provable; this can be seen by an exhaustive bottom-up search for possible proofs. Assume A is provable. Then it is at the bottom of an **LS**-proof. The lowest inference must be an $\exists$-inference with upper multiset $R(s) + \neg R(f(s))$ for some term s. Now, if $R(s) + \neg R(f(s))$ is provable, it must be the lower multiset of a $+$-inference with upper multiset

$R(s), \neg R(f(s))$, which only consists of prime formulas. Hence $R(s), \neg R(f(s))$ must be an axiom. This is a contradiction; therefore A is not **LS**-provable.

Two important results about the calculus **LS** are reviewed in the following. For proofs, see [7].

Theorem 2.4 (Decidability). **LS** *is decidable; it can be determined whether a given formula is provable or not.*

Essentially, this is done by an exhaustive bottom-up search for derivations of the formula, where its terms (which may contain function symbols) are computed using most general unifiers.

Theorem 2.5 (Semi-translation of classical logic to **LS**). *Given a formula A (without defined connectives) and a natural number n, $A^{(n)}$ is obtained from A by replacing every subformula of the form $\exists x F(x)$ by its n-fold disjunction. A is classically provable if and only if $A^{(n)}$ is* **LS**-*provable for some n.*

3. Definition of the games

Definition 3.1. The game $\mathfrak{G}(A)$ associated to a formula A is played between two players, a proponent P and an opponent Q. A move replaces one formula by another. A play of $\mathfrak{G}(A)$ starts with A and ends with a *final formula*. In order to define final formulas and legal moves, formulas are read as complete disjunctions $B^1 + \cdots + B^m$, i.e. disjunctions where every component B^j is a literal or of one of the forms $C \wedge D$, $\forall x F(x)$, $\exists x F(x)$.

(1) If every component in a complete disjunction $B^1 + \cdots + B^m$ of B is a literal, then B is final. P wins a play ending with such a formula B if and only if B contains complementary literals B^h, B^i (i.e. $B^h = \neg B^i$).

(2) If not all components in a complete disjunction $B^1 + \cdots + B^m$ of B are literals then a move replaces B by a formula $B^1_* + \cdots + B^m_*$, where $B^j_* = B^j$ for all j except one index k such that B^k is not a literal. This index k and B^k_* are determined as follows.

(a) If some B^j $(j = 1, \ldots, m)$ is of the form $C \wedge D$ or $\forall x F(x)$ then Q chooses such a formula B^k. Furthermore,

(a1) if B^k is $C \wedge D$ then Q chooses B^k_* to be C or to be D,

(a2) if B^k is $\forall x F(x)$ then Q chooses a term t; B^k_* is $F(t)$.

(b) If no B^j $(j = 1, \ldots, m)$ is of the form $C \wedge D$ or $\forall x F(x)$ then P chooses a term t and a formula B^k of the form $\exists x F(x)$; B^k_* is $F(t)$.

Remarks. (1) A play of $\mathfrak{G}(A)$ is determined by a sequence of formulas $B_1, \ldots,$ where $B_1 = A$ and for $i > 0$, B_{i+1} is obtained from B_i by a move of either P or Q according to the above definition. Observe that B_{i+1} contains fewer logical symbols than B_i, so the length of the sequence $B_1, \ldots$ is limited by the number of logical symbols in A. In particular, $\mathfrak{G}(A)$ is finite.

(2) Note that P and Q need not alternate after each move: For example, if a complete disjunction $B^1 + \cdots + B^m$ contains several components of the form $C \wedge D$ then Q has to reduce one after the other according to rule 2(a), in an arbitrary sequence, *prior* to a possible next move of P. Whether a strategy is winning or not does not depend on the particular choice of this sequence, as will be seen in the proof of the equivalence theorem (Theorem 4.2).

(3) If a complete disjunction $B^1 + \cdots + B^m$ contains several components of the form $\exists x F(x)$ and rule 2(b) applies, the property of a strategy to be winning *does* depend on the particular choice of P for the component B^k. This will be illustrated in Example 3.3 below.

Example 3.2. Let $A = \exists x \forall y (\neg R(x) + R(y))$. (Note that A is classically valid.) Player Q has a winning strategy for $\mathfrak{G}(A)$: According to rule 2(b), P chooses a term s for x, yielding the formula $B_2 = \forall y (\neg R(s) + R(y))$. According to rule 2(a2), Q replaces y by a variable $u \neq s$, yielding $B_3 = \neg R(s) + R(u)$ and wins the game according to rule 1.

Player P has a winning strategy for $\mathfrak{G}(A + A)$: P chooses a term s for the first existential quantifier, then Q chooses a term t for the first universal quantifier, yielding $B_3 = \neg R(s) + R(t) + A$. Now, P chooses the *same* t for the second existential quantifier, yielding $B_4 = \neg R(s) + R(t) + \forall y (\neg R(t) + R(y))$. Finally, Q chooses an arbitrary t' for the second existential quantifier and loses the game.

Example 3.3. Let a, b be constant symbols and $A = \exists x (R(a) \wedge R(b)) + \exists x \neg R(x)$.

Player P has a winning strategy for $\mathfrak{G}(A)$: At the beginning, P chooses a term s for the first existential quantifier, yielding $B_2 = (R(a) \wedge R(b)) + \exists x \neg R(x)$. Then, Q has to choose one literal of the conjunction $(R(a) \wedge R(b))$, thus replacing B_2 by either $B_3 = R(a) + \exists x \neg R(x)$ or by $B_3 = R(b) + \exists x \neg R(x)$. According to the choice of Q, P then chooses a or b for the remaining existential quantifier and wins the game.

However, there does not exist a winning strategy of P for $\mathfrak{G}(A)$ where P chooses a term s for the *second* existential quantifier at the beginning, yielding $B_2 = \exists x (R(a) \wedge R(b)) + \neg R(s)$: Whatever term P chooses for the remaining existential quantifier, Q can win the game by choosing either the first part of the conjunction if $s \neq a$ or the second part of the conjunction if $s \neq b$.

4. Equivalence theorem

Equivalence of **LS**-provability with the existence of winning strategies in the defined games essentially relies on the following characteristic property of **LS**: all inference rules except the $\exists$-rule are *invertible*: if the lower multiset of such an inference is provable then its upper multiset(s) are also provable. This is made precise in the following lemma.

Lemma 4.1 (Inversion lemma). (1) *If* **LS** $\vdash \Gamma, T \wedge Z$, *then* **LS** $\vdash \Gamma, T$, *and* **LS** $\vdash \Gamma, Z$.

(2) *If* **LS** $\vdash \Gamma, T+Z$, *then* **LS** $\vdash \Gamma, T, Z$.

(3) *If* **LS** $\vdash \Gamma, \forall x F(x)$, *then* **LS** $\vdash \Gamma, F(t)$ *for any term* t.

Proof. Every implication is proved by induction on the depth of a proof P of its left-hand side.

(1) Inspecting the last inference of P, the following cases are distinguished.

- If the last inference is an axiom $\Gamma', E, \neg E$, then Γ, T and Γ, Z are axioms, too, since it is impossible that $T \wedge Z$ is either E or $\neg E$ (recall that E is a prime formula).
- If the principal formula of the last inference (i.e. the formula introduced by it) is different from $T \wedge Z$, then apply the induction hypothesis to the upper multisets of this inference and obtain proofs of Γ, T and of Γ, Z by modifying the last inference. For example, let the last inference of P be

$$\frac{\Gamma', T \wedge Z, T' \quad \Gamma', T \wedge Z, Z'}{\Gamma', T \wedge Z, T' \wedge Z'} \wedge .$$

  By the induction hypothesis, obtain proofs of Γ', T, T', of Γ', T, Z', of Γ, Z, T' and of Γ, Z, Z'. Combine them by $\wedge$-inferences to proofs of $\Gamma', T, T' \wedge Z'$ and $\Gamma', Z, T' \wedge Z'$.
- If the principal formula of the last inference is $T \wedge Z$, then the required proofs are obtained by discarding the inference.

(2) The implication is proved in a similar way.

(3) Special attention is required for the case where the last inference of P is of the form

$$\frac{\Gamma', \forall x F(x), F'(u)}{\Gamma', \forall x F(x), \forall x F'(x)} \forall.$$

By the substitution lemma for **LS** (which is proved as the substitution lemma for classical logic), first obtain a proof with the same depth of $\Gamma', \forall x F(x), F'(v)$ with v not in $\Gamma', F(t), \forall x F'(x)$. For $\Gamma', \forall x F(x), F'(v)$, the induction hypothesis then yields a proof of $\Gamma', F(t), F'(v)$, a $\forall$-inference finally gives a proof of $\Gamma', F(t), \forall x F'(x)$.

The remaining cases are treated in a similar way. □

Theorem 4.2 (Equivalence theorem). *Let A be a formula.* **LS** $\vdash A$ *if and only if P has a winning strategy for* $\mathfrak{G}(A)$.

Proof. The proof is by induction on the number d of logical symbols $\wedge$, $\forall$ and $\exists$ in A. Let $B^1 + \cdots + B^m$ be a complete disjunction of A and observe that the inversion lemma (Lemma 4.1) for $+$ implies

$$\mathbf{LS} \vdash B^1 + \cdots + B^m \Leftrightarrow \mathbf{LS} \vdash B^1, \ldots, B^m.$$

- $d=0$. A is a disjunction of literals $B^1 + \cdots + B^m$. P wins $\mathfrak{G}(A)$ if and only if A contains complementary literals B^h, B^i. This is the case if and only if **LS** $\vdash A$.

- $d>0$. Let A be provable. If Q chooses a B^k of the form $C \wedge D$, A is replaced by either.

$$H_1 = B^1 + \cdots + B^{k-1} + C + B^{k+1} + \cdots + B^m, \text{ or by}$$

$$H_2 = B^1 + \cdots + B^{k-1} + D + B^{k+1} + \cdots + B^m.$$

By the inversion lemma for $\wedge$, these formulas are provable. By the induction hypothesis, P has a winning strategy for the corresponding games and thus wins the game. The case where Q chooses a B^k of the form $\forall x F(x)$ is similar, using the inversion lemma for $\forall$. Finally, if no B^j is of the form $C \wedge D$ or $\forall x F(x)$, every B^j is either a literal or of the form $\exists x F(x)$. Since there is a proof of the multiset $B^1, \ldots, B^m$, its last inference must be an $\exists$-inference with principal formula $B^k = \exists x F(x)$ and upper multiset $B^1, \ldots, B^{k-1}, B^{k+1}, \ldots, B^m, F(t)$. Now, P chooses the term t and the formula B^k, replacing A by the provable formula $B^1 + B^{k-1} + B^{k+1} + \cdots + B^m + F(t)$. By the induction hypothesis, P has a winning strategy for the corresponding game and thus wins the game.

Conversely, let P have a winning strategy for $\mathfrak{G}(A)$. If there is a B^k of the form $C \wedge D$, Q may choose this formula and replace A by either H_1 or H_2 as above. Since P has a winning strategy for $\mathfrak{G}(A)$, it has winning strategies for $\mathfrak{G}(H_1)$ and for $\mathfrak{G}(H_2)$ as well. Applying the induction hypothesis to these formulas yields proofs of H_1 and of H_2, from which a proof of A can be constructed. If there is a B^k of the form $\forall x F(x)$, Q may choose this formula and replace A by $B^1 + \cdots + B^{k-1} + F(u) + B^{k+1} + \cdots + B^m$ for a variable u not in A. Since P has a winning strategy for the associated game, the induction hypothesis yields a proof of this formula, from which a proof of A can be constructed. Finally, if no B^i is of the form $C \wedge D$ or $\forall x F(x)$, P is able to choose a term t and a formula B^k of the form $\exists x F(x)$ and to replace A by $B^1 + \cdots + B^{k-1} + F(t) + B^{k+1} + \cdots + B^m$, such that P still has a winning strategy for the associated game. Applying the induction hypothesis yields a proof of this formula, from which a proof of A can be obtained. □

Remark. The above equivalence theorem states that provability in the calculus **LS** is equivalent to the existence of winning strategies for the defined games. The same characterization holds for *classical* provability provided the definition of the games (Definition 3.1) is slightly modified by reformulating rule 2(b) as follows:

(b)^c If no B^j ($j = 1, \ldots, m$) is of the form $C \wedge D$ or $\forall x F(x)$ then P has two alternatives:

(b1)^c either P chooses a term t and a formula B^k of the form $\exists x F(x)$ and B^k_* is $F(t)$;
(b2)^c or P chooses a formula B^k of the form $\exists x F(x)$ and B^k_* is $B^k + B^k$.

An equivalence theorem for these modified games can be proved using the approximation theorem of classical logic (Theorem 2.5). Note that the plays of such a modified game may not be finite: whenever rule 2(b)^c applies to a formula, P may continue to duplicate existential subformulas forever (rule (b2)^c).

5. Applications

In Example 3.2, a winning strategy of Q for $\mathfrak{G}(A)$ and a winning strategy of P for $\mathfrak{G}(A+A)$ was described, where $A=\exists x\forall y(\neg R(x)+R(y))$. By the equivalence theorem, **LS** $\not\vdash A$ and **LS** $\vdash A+A$. Looking at example 3.3, the theorem implies **LS** $\vdash \exists x(R(a)\wedge R(b))+\exists x\neg R(x)$. Describing winning strategies for the games, $\mathfrak{G}(A)$ can be used to establish results concerning provability or nonprovability of much more complex formulas A. This is illustrated by the following example and the following remark.

Example 5.1. Let $R_1,\dots,R_k$ be unary predicate symbols and for $j>0$,

$$S_j(x,y)=R_1(x)+\cdots+R_j(x)+\neg R_j(y).$$

For $n>0$, define

$$A_n=\exists x\forall y(S_1(x,y)\wedge\cdots\wedge S_n(x,y)).$$

Claim. (i) **LS** $\not\vdash \underbrace{A_n+\cdots+A_n}_{n\text{-times}}$.

(ii) **LS** $\vdash \underbrace{A_n+\cdots+A_n}_{(n+1)\text{-times}}$.

Proof. The claim is proved by describing winning strategies of Q and of P, respectively. To simplify notation, let $A_n^{(p)}=\underbrace{A_n+\cdots+A_n}_{p\text{-times}}$ for $p>0$.

(i) A winning strategy of player Q for $\mathfrak{G}(A_n^{(n)})$ is described:

$B_1=A_n^{(n)}$

(P chooses s_1 for existential quantifier of an A_n:)

$B_2=\forall y(S_1(s_1,y)\wedge\cdots\wedge S_n(s_1,y))+A_n^{(n-1)}$

(Q chooses variable u_1 different from s_1:)

$B_3=(S_1(s_1,u_1)\wedge\cdots\wedge S_n(s_1,u_1))+A_n^{(n-1)}$

(Q chooses last part of $S_1(s_1,u_1)\wedge\cdots\wedge S_n(s_1,u_1)$:)

$B_4=R_1(s_1)+\cdots+R_n(s_1)+\neg R_n(u_1)+A_n^{(n-1)}$

(P chooses s_2 for existential quantifier of an A_n:)

$B_5=R_1(s_1)+\cdots+R_n(s_1)+\neg R_n(u_1)+\forall y(S_1(s_2,y)\wedge\cdots\wedge S_n(s_2,y))+A_n^{(n-2)}$

(Q chooses variable u_2 different from s_1,s_2:)

$B_6=R_1(s_1)+\cdots+R_n(s_1)+\neg R_n(u_1)+(S_1(s_2,u_2)\wedge\cdots\wedge S_n(s_2,u_2))+A_n^{(n-2)}$

(Q chooses last but second part of $S_1(s_2,u_2)\wedge\cdots\wedge S_n(s_2,u_2)$:)

$B_7=R_1(s_1)+\cdots+R_n(s_1)+\neg R_n(u_1)+$
$R_1(s_2)+\cdots+R_{n-1}(s_2)+\neg R_{n-1}(u_2)+A_n^{(n-2)}$.

(Note that Q may not have chosen the last part of $S_1(s_2, u_2) \wedge \cdots \wedge S_n(s_2, u_2)$: if P chose $s_2 = u_1$, B_7 would already contain the complementary literals $\neg R_n(u_1)$ and $R_n(u_1)$ and P would win the game.)

$$\vdots$$

$$B_{3n+1} = R_1(s_1) + \cdots + R_n(s_1) + \neg R_n(u_1) + \cdots + R_1(s_n) + \neg R_1(u_n).$$

Observe that B_{3n+1} does not contain complementary literals, so Q wins the game.

(ii) A winning strategy of player P for $B_n^{(n+1)}$ is described:
At the beginning, P chooses an arbitrary s_1 for the existential quantifier of the first A_n. Q then chooses t_1 for the universal quantifier of this A_n and a certain part of $S_1(s_1, t_1) \wedge \cdots \wedge S_n(s_1, t_1)$. Now, P chooses $s_2 = t_1$ for the existential quantifier of the second A_n; Q chooses t_2 for the universal quantifier of this A_n and a certain part of the conjunction. P then chooses $s_3 = t_2$ etc.

By writing down the sequence of formulas $B_1, B_2, \ldots$ as above, it is easy to see that if P applies this strategy, the last formula B_{3n+4} must contain complementary literals. Hence P wins the game. □

Remark. The following more involved nonprovability result can be established in a manner similar to the above example. Recall that the approximation theorem of classical logic (Theorem 2.5) states that a formula A is classically provable if and only if $A^{(n)}$ is provable for some n. Now, given a recursive function $\mathfrak{f}$, one can explicitly construct a sequence $A_1, A_2, \ldots$ of classically valid formulas such that for every $n \in \mathbb{N}$, A_n contains $\mathrm{O}(n)$ symbols and

$$\mathbf{LS} \not\vdash A_n^{(\mathfrak{f}(n))}.$$

The construction is based on a finite axiomatization of arithmetic where recursive functions can be formally represented.

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