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Discrete orthogonal polynomials and difference equations of several variables

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Abstract

The goal of this work is to characterize all second order difference operators of several variables that have discrete orthogonal polynomials as eigenfunctions. Under some mild assumptions, we give a complete solution of the problem.

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Contents

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1. Introduction

The goal of this study is to characterize all second order difference equations of several variables that have discrete orthogonal polynomials as eigenfunctions. More precisely, we consider difference operators of the form

$$D=\sum_{1\leqslant i,j\leqslant d}A_{i,j}\Delta_i\nabla_j+\sum_{i=1}^{d}B_i\Delta_i+C\mathcal{I},$$

where Δ_i and ∇_i are the forward and backward operator in the direction of the ith coordinate of $\mathbb{R}^d$, respectively, $\mathcal{I}$ is the identity operator, $A_{i,j}$, B_i and C are functions of $x\in\mathbb{R}^d$. The discrete orthogonal polynomials in our consideration are polynomials that are orthogonal with respect to an inner product of the form

$$\langle f,g\rangle=\sum_{x\in V}f(x)g(x)W(x),$$

where V is a lattice set in $\mathbb{R}^d$ and W is some positive weight function on V. There is a close correlation between D, W and V. Some restrictions need to be imposed on V due to the complexity of the geometry in higher dimensions. Under some mild and, we believe, reasonable assumptions on V, we give a complete solution of the problem.

For $d=1$, the one-dimensional case, the classification problem was studied by several authors early in the last century, we refer to [1,6] for references. It was found that the classical discrete orthogonal polynomials, namely, Hahn polynomials, Meixner polynomials, Krawtchouk polynomials, and Charlier polynomials are eigenfunctions of second order difference operators on the real line, and these are believed to be the only ones that have such property. To our great surprise, however, another family of solutions turns up when we analyze the problem carefully. The difference equation satisfied by this family of solutions is similar to that of Hahn polynomials but the parameters need to be chosen in a different way. In other words, the difference equation has two separate families of solutions when the parameters are chosen differently.

The same phenomenon also appears in the case of two variables. The second order difference equations that have orthogonal polynomials as eigenfunctions were identified in [12], but not all families of orthogonal polynomials were found. In fact, viewing it as an analogue of second order differential operators that have orthogonal polynomials as eigenfunctions, only one family of solutions that had been known in the literature was identified for each difference equation. The solutions of difference equations, however, turn out to be far richer than that of differential equations. For example, in the case of quadratic eigenvalues, only Hahn polynomials on the set

$V = \{(x, y)\colon\ x \geqslant 0,\ y \geqslant 0,\ x + y \leqslant N\}$ are identified in [12]; there are in fact several other families, including orthogonal polynomials on $\mathbb{N}_0^2$, on $V = \{(x, y)\colon\ 0 \leqslant x \leqslant N_1,\ 0 \leqslant y \leqslant N_2\}$, and several others. This will be shown in the present paper.

We start with identifying difference operators D that are self-adjoint with respect to the inner product $\langle\cdot,\cdot\rangle$. This leads to compatibility conditions between the coefficients of the operator D. The requirement that the difference equations have polynomial solutions and some mild restrictions on V reduce the coefficients of D to special simple forms, which allows us to use the compatibility conditions to determine the coefficients of the difference operator. A careful analysis of the result determines all possible solutions. It is well known that orthogonal bases in several variables are not unique [2]. For each solution, we give a family of mutually orthogonal polynomials explicitly and compute their norm; in other words, we give a family of orthonormal basis explicitly.

It should be mentioned that the analysis in several variables is by no means a straightforward extension of analysis in one or two variables. The complexity of the problem increases substantially as the dimension grows. For example, the number of solutions grows exponentially with the dimension, and the geometry of the admissible lattice sets becomes more involved as the dimension increases. Although our approach resembles the one used in [12], several new ideas are introduced to resolve various outstanding problems. The study in this paper is also much more systematic and thorough, as demonstrated by the new results obtained even in the case of $d = 2$ and by the new family of discrete orthogonal polynomials of one variable.

The paper is organized as follows. The self-adjoint operators are treated in Section 2, where the compatibility conditions are derived. In Section 3, we study difference equations that have polynomial solutions, called admissible equations, and show that the eigenvalues of such equations are either a quadratic polynomial or a linear polynomial of the index, and the coefficients of such equations need to have certain simple forms under some mild restrictions on V. The compatibility conditions can then be used to determine the coefficients, thus D, explicitly. The case of one variable is discussed in Section 4, in which the new family of discrete orthogonal polynomials is treated in details. For several variables, the case of quadratic eigenvalues is studied in Section 5 and the case of linear eigenvalues is developed in Section 6.

2. Self-adjoint difference operators

In this section we discuss the basic properties of self-adjoint second order difference operators with respect to an inner product defined on a discrete set. We start with some notations.

Let V be a subset of $\mathbb{R}^d$ and let $W : V \to \mathbb{R}$ be a positive function. Consider the inner product defined by the weight W as follows

$$\langle f, g\rangle = \sum_{x\in V} f(x)g(x)W(x), \tag{2.1}$$

for functions $f, g : \mathbb{R}^d \to \mathbb{R}$. We will be mainly interested in the case when $f(x)$ and $g(x)$ are polynomials of $x \in \mathbb{R}^d$, but the general discussion in this section does not depend essentially on the class of functions. Denote by $\{e_1, e_2, \dots, e_d\}$ the standard basis for $\mathbb{R}^d$. We denote by E_i, Δ_i and ∇_i, respectively, the customary shift, forward and backward difference operators acting on a function $f(x)$ as follows

$$E_i f(x) = f(x + e_i),$$

$$\Delta_i f(x) = f(x+e_i) - f(x) = (E_i - \mathcal{I}) f(x),$$
$$\nabla_i f(x) = f(x) - f(x-e_i) = \left(\mathcal{I} - E_i^{-1}\right) f(x),$$

where $\mathcal{I}$ is the identity operator.

We will consider second order difference operators of the form

$$D = \sum_{1\leqslant i,j\leqslant d} A_{i,j}\Delta_i\nabla_j + \sum_{i=1}^{d} B_i\Delta_i + C\mathcal{I}, \tag{2.2}$$

where $A_{i,j}$, B_i and C are some functions of x. Immediately from the definition one can see that D can also be rewritten as

$$D = \sum_{1\leqslant i\neq j\leqslant d} \alpha_{i,j} E_i E_j^{-1} + \sum_{i=1}^{d} \beta_i E_i + \sum_{i=1}^{d} \gamma_i E_i^{-1} + \delta\mathcal{I}, \tag{2.3}$$

where the new coefficients $\alpha_{i,j}$, β_i, γ_i and δ are related to the old ones via the formulas

$$\alpha_{i,j} = -A_{i,j} \quad \text{for } 1\leqslant i\neq j\leqslant d, \tag{2.4}$$

$$\beta_i = \sum_{k=1}^{d} A_{i,k} + B_i \quad \text{for } 1\leqslant i\leqslant d, \tag{2.5}$$

$$\gamma_i = \sum_{k=1}^{d} A_{k,i}, \tag{2.6}$$

$$\delta = C - \sum_{1\leqslant i\neq j\leqslant d} A_{i,j} - \sum_{i=1}^{d}(2A_{i,i} + B_i). \tag{2.7}$$

The representation (2.2) will be more convenient when we deal with polynomials, because the operators Δ_i and ∇_i decrease the total degree of a polynomial by one. However, necessary and sufficient conditions for an operator D to be self-adjoint with respect to the inner product (2.1) are much simpler and natural if we write it as in (2.3). Formulas (2.4)–(2.7) allow us to go easily from one representation to another.

Let us define "directional" boundaries of V as follows:

$$\partial_j^{\pm} V = \{x \in V\colon\ x \pm e_j \notin V\} \quad \text{for } j = 1,2,\dots,d,$$
$$\partial_{i,j} V = \{x \in V\colon\ x + e_i - e_j \notin V\} \quad \text{for } 1\leqslant i\neq j\leqslant d.$$

Example 2.1. Let $\mathbb{N}_0$ denote the set of all nonnegative integers. For the set $V = \mathbb{N}_0^d$ we have $\partial_j^- V = \partial_{i,j} V = V \cap \{x\colon\ x_j = 0\}$ and $\partial_j^+ V = \emptyset$ for $j = 1,2,\dots,d$ and $i \neq j$.

Other sets that play crucial role later are listed below

$$V_l^d = \left\{x \in \mathbb{N}_0^d\colon\ x_i \leqslant l_i \text{ for } i = 1,2,\dots,d\right\}, \tag{2.8a}$$

where l_i are positive integers;

$$V_N^d = \{x \in \mathbb{N}_0^d \colon x_1 + x_2 + \cdots + x_d \leqslant N\}, \tag{2.8b}$$

where N is a positive integer;

$$V_{N,S}^d = V_N^d \bigcap_{i\in S} \{x \colon x \leqslant l_i \text{ for } i \in S\}, \tag{2.8c}$$

where S is a nonempty subset of $\{1, 2, \dots, d\}$ and l_i are integers such that $1 \leqslant l_i \leqslant N$.

For example, for the parallelepiped V_l^d we have $\partial_j^- V_l^d = V_l^d \cap \{x \colon x_j = 0\}$, $\partial_j^+ V_l^d = V_l^d \cap \{x \colon x_j = l_j\}$ and $\partial_{i,j} V_l^d = \partial_j^- V_l^d \cup \partial_i^+ V_l^d$.

For V_N^d defined by (2.8b) we have $\partial_j^- V_N^d = \partial_{i,j} V_N^d = V_l^d \cap \{x \colon x_j = 0\}$, and $\partial_j^+ V_N^d = V_l^d \cap \{x \colon x_1 + x_2 + \cdots + x_d = N\}$.

The next proposition characterizes the self-adjoint operators with respect to the inner product (2.1) in terms of their coefficients.

Proposition 2.2. *The operator D is self-adjoint with respect to the inner product* (2.1)*, if and only if*

$$W(x)\gamma_i(x) = W(x - e_i)\beta_i(x - e_i), \quad x, x - e_i \in V, \tag{2.9}$$

$$W(x - e_i)\alpha_{i,j}(x - e_i) = W(x - e_j)\alpha_{j,i}(x - e_j), \quad x - e_i, x - e_j \in V, \ j \neq i, \tag{2.10}$$

and

$$\gamma_j(x) = 0, \quad x \in \partial_j^- V, \tag{2.11a}$$

$$\beta_j(x) = 0, \quad x \in \partial_j^+ V, \tag{2.11b}$$

$$\alpha_{i,j}(x) = 0, \quad x \in \partial_{i,j} V, \ i \neq j. \tag{2.11c}$$

Proof. Using $\{x + e_i - e_j \colon x \in V \setminus \partial_{i,j} V\} = V \setminus \partial_{j,i} V$ and changing the summation index we see that

$$\begin{aligned}\langle \alpha_{i,j} E_i E_j^{-1} u, v \rangle = &\sum_{x\in\partial_{i,j}V} \alpha_{i,j}(x) u(x + e_i - e_j) v(x) W(x) \\ &+ \sum_{x\in V\setminus\partial_{j,i}V} \alpha_{i,j}(x + e_j - e_i) u(x) v(x + e_j - e_i) W(x + e_j - e_i).\end{aligned}$$

Writing similar equalities for $\langle \beta_i E_i u, v\rangle$ and $\langle \gamma_i E_i^{-1} u, v\rangle$, using $\{x \pm e_i \colon x \in V \setminus \partial_i^{\pm} V\} = V \setminus \partial_i^{\mp} V$, and comparing $\langle Du, v\rangle$ with $\langle u, Dv\rangle$ gives the stated conditions. $\square$

As an immediate corollary from the relations established above we can deduce compatibility conditions that need to be satisfied by the coefficients of the operator D.

Corollary 2.3 *(Compatibility conditions). If the operator D is self-adjoint with respect to the inner product defined by* (2.1) *then for $1 \leqslant i \neq j \leqslant d$ we have*

$$\alpha_{i,j}(x-e_i)\beta_j(x-e_j)\gamma_i(x) = \alpha_{j,i}(x-e_j)\beta_i(x-e_i)\gamma_j(x), \tag{2.12a}$$

for $x, x-e_i, x-e_j \in V$ and

$$\begin{aligned}&\beta_i(x-e_i)\beta_j(x-e_i-e_j)\gamma_j(x)\gamma_i(x-e_j)\\ &\quad = \beta_j(x-e_j)\beta_i(x-e_i-e_j)\gamma_i(x)\gamma_j(x-e_i),\end{aligned} \tag{2.12b}$$

for $x, x-e_i, x-e_j, x-e_i-e_j \in V$.

Proof. Writing

$$\frac{W(x-e_j)}{W(x-e_i)} = \frac{W(x-e_j)}{W(x)}\frac{W(x)}{W(x-e_i)}$$

and using (2.10) on the left side and (2.9) for the right side we obtain (2.12a). Similarly if we use (2.9) for all ratios in the identity

$$\frac{W(x)}{W(x-e_i)}\frac{W(x-e_i)}{W(x-e_i-e_j)} = \frac{W(x)}{W(x-e_j)}\frac{W(x-e_j)}{W(x-e_i-e_j)},$$

we get (2.12b), which completes the proof. □

Remark 2.4. The compatibility conditions stated above can be easily extended to more general difference operators. For example, we can consider the operator $\widetilde{D}$ that adds two additional terms E_iE_j and $E_i^{-1}E_j^{-1}$ in the operator D in (2.3). However, the complexity of the computations in the sections below will increase significantly even with these two terms added. Furthermore, since $\Delta = \nabla + \nabla\Delta$, the operator $\widetilde{D}$ includes the 4th order term $\Delta_1\nabla_1\Delta_2\nabla_2$ (say, $d=2$) which is the discrete analog of $\partial_1^2\partial_2^2$, where ∂_i stands for the partial derivative with respect to x_i. Classifying orthogonal polynomials that are eigenfunctions of $\widetilde{D}$ appears to be an interesting but much harder problem.

3. Admissible equations and orthogonal polynomials

In this section we study difference operators D that have discrete orthogonal polynomials as eigenfunctions. For more details on the general theory of discrete orthogonal polynomials of several variables we refer the reader to [11].

Throughout the paper we use the standard multi-index notation. A multi-index will be denoted by $\mu = (\mu_1, \mu_2, \dots, \mu_d) \in \mathbb{N}_0^d$. For each μ we denote by x^μ the monomial

$$x^\mu = x_1^{\mu_1}x_2^{\mu_2}\cdots x_d^{\mu_d}$$

of total degree $|\mu| = \mu_1 + \mu_2 + \cdots + \mu_d$. The degree of a polynomial is defined as the highest degree of its monomials. We denote by $\mathbb{R}[x] = \mathbb{R}[x_1, x_2, \dots, x_d]$ the space of all polynomials in the variables $x_1, x_2, \dots, x_d$ and by Π_n^d the subspace of polynomials of degree at most n in the variables $\{x_1, x_2, \dots, x_d\}$. The latter has dimension $\dim(\Pi_n^d) = \binom{n+d}{n}$.

Let V be an at most countable set of isolated points in $\mathbb{R}^d$ and let $|V|$ denote the cardinality of V. If $\langle f, g\rangle = 0$ for the inner product defined by (2.1), we say that f and g are orthogonal with respect to W on the set V. Orthogonal polynomials on V depend on the structure of the ideal

$$\mathfrak{I}(V) = \bigl\{p \in \mathbb{R}[x_1, x_2, \dots, x_d]\colon\ p(x) = 0 \text{ for every } x \in V\bigr\}.$$

In fact, the orthogonal polynomials belong to the space

$$\mathbb{R}[V] = \mathbb{R}[x_1, x_2, \dots, x_d]/\mathfrak{I}(V).$$

There is a lattice set $\Lambda(V)$ such that every polynomial in $\mathbb{R}[V]$ can be written as

$$P(x) = \sum_{\mu\in\Lambda(V)} c_\mu x^\mu.$$

The lattice set is not uniquely determined by V. However, in all cases that arise here, there is a natural way to identify V and $\Lambda(V)$, see Example 3.1. Denote $\Lambda_k(V) = \{\mu \in \Lambda(V)\colon\ |\mu| = k\}$ and let r_k be the number of the elements in the set $\Lambda_k(V)$.

Example 3.1. It is clear that for $V = \mathbb{N}_0^d$ we have $\mathfrak{I}(\mathbb{N}_0^d) = (0)$ and therefore $\mathbb{R}[\mathbb{N}_0^d] = \mathbb{R}[x]$. For all other sets defined in Example 2.1 one can easily find (inductively on d and $|V|$) explicit sets of generators of the ideal $\mathfrak{I}(V)$. In particular, the generators listed below allow us to identify $\Lambda(V)$ and V.

For V_l^d, defined by (2.8a), the ideal $\mathfrak{I}(V_l^d)$ is generated by the set

$$G_l^d = \bigl\{(-x_1)_{l_1+1}, (-x_2)_{l_2+1}, \dots, (-x_d)_{l_d+1}\bigr\}.$$

For V_N^d, given by (2.8b), the ideal $\mathfrak{I}(V_N^d)$ is generated by the set

$$G_N^d = \bigl\{(-x_1)_{\mu_1}(-x_2)_{\mu_2}\cdots(-x_d)_{\mu_d}\colon\ |\mu| = N+1\bigr\}.$$

Finally, for $V_{N,S}^d$ in (2.8c), the corresponding ideal is generated by

$$G_{N,S}^d = G_N^d \cup \bigl\{(-x_i)_{l_i+1}\colon\ i \in S\bigr\}.$$

Next we want to consider difference operators D on the space $\mathbb{R}[V]$. In order to have a proper action of D on $\mathbb{R}[V]$ the ideal $\mathfrak{I}(V)$ must be D-invariant, i.e.

$$D\bigl(\mathfrak{I}(V)\bigr) \subset \mathfrak{I}(V).$$

One can easily show that if D is self-adjoint with respect to the inner product (2.1), then the above condition is satisfied. More precisely, we have the following proposition.

Proposition 3.2. *If the operator D, written in the form* (2.3)*, satisfies conditions* (2.11) *then the ideal $\mathfrak{I}(V)$ is D-invariant.*

Proof. Notice that if a polynomial $p(x)$ vanishes on V, then the polynomial $p(x-e_j)$ will vanish on $V \setminus \partial_j^- V$. Thus, if $p(x) \in \mathfrak{I}(V)$ and (2.11a) holds then $\gamma_j(x)p(x-e_j) \in \mathfrak{I}(V)$, i.e. the operator $\gamma_j E_j^{-1}$ preserves $\mathfrak{I}(V)$. Similar arguments show that (2.11b) (respectively (2.11c)) implies that $\beta_j E_j$ (respectively $\alpha_{i,j} E_i E_j^{-1}$) preserves $\mathfrak{I}(V)$, which completes the proof. □

Definition 3.3. Let $\mathfrak{I}(V)$ be D-invariant. The equation

$$Du = \lambda u \tag{3.1}$$

is called admissible on V if for any $k \in \mathbb{N}_0$ there is a number λ_k such that the equation $Du = \lambda_k u$ has r_k linearly independent polynomial solutions in $\mathbb{R}[V]$ and it has no nontrivial solutions in the set of polynomials of degree less than k.

Remark 3.4. Notice that if (3.1) is admissible on V and if we have nontrivial solutions of the equations $Du = \lambda_k u$ and $Du = \lambda_l u$ for distinct integers $k, l \in \mathbb{N}_0$ then $\lambda_k \neq \lambda_l$. Thus if D is also a self-adjoint operator with respect to the inner product (2.1), the polynomial solutions for $Du = \lambda_k u$ and $Du = \lambda_l u$ will be mutually orthogonal.

From now on, we will assume that $\lambda_0 = 0$ (which is equivalent to $C = 0$ for operators D of the form (2.2)). This is not a real restriction on D since one can replace D by $D - \lambda_0 \mathcal{I}$ if necessary.

The next proposition shows that we can pick linearly independent polynomial solutions in such a way that the highest terms contain single monomials.

Proposition 3.5. *Equation* (3.1) *is admissible on V if and only if for each $k \in \mathbb{N}_0$ there exists a number λ_k such that the equation $Du = \lambda_k u$ has r_k linearly independent polynomial solutions of the form*

$$P_\mu(x) = x^\mu \mod \Pi_{k-1}^d. \tag{3.2}$$

The proof of Proposition 3.5 follows easily form Definition 3.3, see [12] in the case $d = 2$.

Remark 3.6. If D is an admissible operator on V we can trivially extend it to an admissible operator on the set $V' = \{(x,0)\colon\ x \in V\} \subset \mathbb{R}^{d+1}$. When we classify the possible admissible operators, we will exclude such degenerate situations.

The next proposition essentially characterizes the admissible difference operators of the form (2.2). In order to eliminate several singular cases we will assume that $\mu \in \Lambda(V)$ for $|\mu| = 1, 2, 3$. Instead of putting $\mu \in \Lambda(V)$ for $|\mu| = 3$ we can require that $3e_i \in \Lambda(V)$ for some $i \in \{1, 2, \ldots, d\}$ and for all $1 \leqslant i \neq j \leqslant d$ at least one of $\{2e_i + e_j, e_i + 2e_j\}$ belongs to $\Lambda(V)$, but this would make the statement of the theorem long and awkward.

Theorem 3.7. *Assume that $\mu \in \Lambda(V)$ for $|\mu| = 1, 2, 3$. Let D be a second order difference operator defined by* (2.2) *such that $\mathfrak{I}(V)$ is D-invariant. Then the following conditions are equivalent*:

(i) *The equation $Du = \lambda u$ is admissible*;
(ii) *The coefficients $B_i(x)$ are polynomials of degree at most* 1, *and $A_{i,j}(x)$ are polynomials of degree at most* 2 *satisfying*

$$B_i = bx_i \mod \Pi_0^d \quad \text{for } i = 1, 2, \ldots, d, \tag{3.3a}$$

$$A_{i,i} = ax_i^2 \mod \Pi_1^d \quad \text{for } i = 1, 2, \ldots, d, \tag{3.3b}$$

$$A_{i,j} + A_{j,i} = 2ax_i x_j \mod \Pi_1^d \quad \text{for } 1 \leqslant i \neq j \leqslant d \tag{3.3c}$$

for some constants a and b. The eigenvalues λ_k must be distinct and they are given by

$$\lambda_k = k(ka - a + b). \tag{3.4}$$

Proof. Assume first that the equation $Du = \lambda u$ is admissible. Applying Proposition 3.5 we see that there exist polynomials

$$P_{e_i}(x) = x_i \mod \Pi_0^d$$

satisfying

$$Du = \lambda_1 u. \tag{3.5}$$

Notice that $\Delta_j P_{e_i} = \delta_{i,j}$ and $\Delta_j \nabla_k P_{e_i} = 0$ for all $j, k = 1, 2, \ldots, d$. Thus $DP_{e_i} = B_i(x)$ and the last equation shows that formula (3.3a) holds, where we put $b = \lambda_1$. Similarly, we can find polynomials of the form

$$P_{2e_i}(x) = x_i^2 \mod \Pi_1^d \tag{3.6}$$

satisfying

$$Du = \lambda_2 u. \tag{3.7}$$

This time we have $\Delta_j P_{2e_i} = 2x_i \delta_{i,j} \mod \Pi_0^d$ and $\Delta_j \nabla_k P_{2e_i} = 2\delta_{i,j}\delta_{i,k}$. Using now (3.3a) we see that

$$DP_{2e_i} = 2A_{i,i} + 2bx_i^2 \mod \Pi_1^d,$$

which, combined with (3.6) and (3.7), shows that

$$A_{i,i} = \frac{\lambda_2 - 2b}{2} x_i^2 \mod \Pi_1^d,$$

thus proving (3.3b) if we denote $a = (\lambda_2 - 2b)/2$.

Next, for $i \neq j$ we take a polynomial of the form

$$P_{e_i + e_j}(x) = x_i x_j \mod \Pi_1^d$$

satisfying (3.7). A simple computation as above gives that

$$DP_{e_i+e_j} = A_{i,j} + A_{j,i} + 2bx_ix_j \mod \Pi_1^d,$$

which combined with (3.7) shows that (3.3c) holds. It remains to show that $A_{i,j}$ are polynomials of degree at most 2 and that (3.4) holds. First we use a polynomial of the form

$$P_{3e_i} = x_i^3 \mod \Pi_2^d$$

for some i satisfying

$$Du = \lambda_3 u, \tag{3.8}$$

to show that (3.4) holds for $k = 3$. Indeed, we already know that if P is a polynomial of degree at most 2, $D(P)$ will also be a polynomial of degree at most 2 (because we can write it as a linear combination of 1, P_{e_i}, $P_{e_i+e_j}$, for $i, j = 1, 2, \ldots, d$). Thus $D(P_{3e_i}) = D(x_i^3) \bmod \Pi_2^d$. On the other hand

$$D\big(x_i^3\big) = 6x_iA_{i,i} + \big(3x_i^2 + 3x_i + 1\big)B_i = (6a + 3b)x_i^3 \mod \Pi_2^d,$$

where in the last equality we used (3.3a) and (3.3b). Comparing now the coefficients of x_i^3 on both sides in (3.8), we see that $\lambda_3 = 6a + 3b$, which is exactly (3.4) for $k = 3$.

Next we consider the solution to Eq. (3.8) of the form

$$P_{2e_i+e_j}(x) = x_i^2x_j \mod \Pi_2^d$$

for $i \neq j$. Again $D(P_{2e_i+e_j}) = D(x_i^2x_j) \bmod \Pi_2^d$, but this time

$$\begin{aligned} D\big(x_i^2x_j\big) &= (2x_i + 1)A_{i,j} + (2x_i - 1)A_{j,i} + 2x_jA_{i,i} + (2x_i + 1)x_jB_i + x_i^2B_j \\ &= 2x_i(A_{i,j} + A_{j,i}) + A_{i,j} - A_{j,i} + 2x_jA_{i,i} + 2x_ix_jB_i + x_i^2B_j \mod \Pi_2^d \\ &= (6a + 3b)x_i^2x_j + A_{i,j} - A_{j,i} \mod \Pi_2^d \\ &= \lambda_3x_i^2x_j + A_{i,j} - A_{j,i} \mod \Pi_2^d, \end{aligned}$$

upon using (3.3) and $\lambda_3 = 6a + 3b$. Equation (3.8) shows that $A_{i,j} - A_{j,i} = 0 \bmod \Pi_2^d$, which combined with (3.3c) proves that $A_{i,j}$ and $A_{j,i}$ are polynomials of degree at most 2. Finally, let $\mu \in \Lambda(V)$ such that $|\mu| = k$ and let

$$P_\mu = x^\mu \mod \Pi_{k-1}^d$$

be a solution to

$$Du = \lambda_k u. \tag{3.9}$$

The admissibility implies that $D(u) = D(x^\mu) \bmod \Pi_{k-1}^d$. Notice that

$$\Delta_i x^\mu = \mu_i x^{\mu-e_i} \mod \Pi^d_{k-2} \quad \text{for all } i, \tag{3.10a}$$

$$\Delta_i \nabla_j x^\mu = \mu_i \mu_j x^{\mu-e_i-e_j} \mod \Pi^d_{k-3} \quad \text{for } i \neq j, \tag{3.10b}$$

$$\Delta_i \nabla_i x^\mu = \mu_i(\mu_i - 1) x^{\mu-2e_i} \mod \Pi^d_{k-3} \quad \text{for all } i \tag{3.10c}$$

and therefore

$$\begin{aligned} D\big(x^\mu\big) &= \sum_{1\leqslant i<j\leqslant d} \mu_i\mu_j x^{\mu-e_i-e_j}(A_{i,j}+A_{j,i}) + \sum_{i=1}^d \mu_i(\mu_i-1)x^{\mu-2e_i}A_{i,i} \\ &\quad + \sum_{i=1}^d \mu_i x^{\mu-e_i} B_i \mod \Pi^d_{k-1} \\ &= x^\mu\left(2a\sum_{1\leqslant i<j\leqslant d}\mu_i\mu_j + a\sum_{i=1}^d \mu_i(\mu_i-1) + b\sum_{i=1}^d \mu_i\right) \mod \Pi^d_{k-1} \\ &= |\mu|\big(a|\mu| - a + b\big)x^\mu \mod \Pi^d_{k-1}, \end{aligned} \tag{3.11}$$

which combined with (3.9) gives (3.4).

Conversely, assume that (ii) holds. One can easily show by induction on $|\mu|$ that for $\mu \in \mathbb{N}^d$ there exist polynomials of the form

$$P_\mu(x) = x^\mu + \sum_{|\nu|<|\mu|} c_{\mu,\nu} P_\nu(x) \tag{3.12}$$

satisfying (3.9) with eigenvalue $\lambda_{|\mu|}$ given by (3.4). Indeed, let us assume that this is true for $|\mu| \leqslant k-1$ and take μ such that $|\mu| = k$. Using computation (3.11) we see that

$$D\big(x^\mu\big) = \lambda_{|\mu|} x^\mu + \sum_{|\nu|<|\mu|} \gamma_{\mu,\nu} P_\nu(x),$$

for some constants $\gamma_{\mu,\nu}$. Then

$$D(P_\mu) - \lambda_{|\mu|} P_\mu = \sum_{|\nu|<|\mu|} \big(c_{\mu,\nu}(\lambda_{|\nu|} - \lambda_{|\mu|}) + \gamma_{\mu,\nu}\big) P_\nu,$$

i.e. if we pick $c_{\mu,\nu} = -\gamma_{\mu,\nu}/(\lambda_{|\nu|} - \lambda_{|\mu|})$ the polynomial P_μ defined by (3.12) will satisfy (3.9). Thus, in the quotient space $\mathbb{R}[V]$ we have r_k polynomials of total degree k satisfying (3.9). □

In the next sections we will use conditions (3.3) and the compatibility conditions (2.11) and (2.12) to determine the possible self-adjoint operators on appropriate sets V.

Since the difference equation $Du = \lambda u$ is invariant under translations $x \to x + h$, we can consider the difference equations modulo translations.

In the rest of the paper we will impose certain boundary conditions on the difference operator D. More precisely, we will assume that for every j the coefficients $A_{i,j}(x)$ vanish on the hyperplane $\{x\colon x_j = 0\}$. Roughly speaking, this means that V has sufficiently many boundary points belonging to $\{x\colon\ x_j = 0\}$. We show below, that if $a \neq 0$ and if V has "enough" points on its boundary then, up to a translation, this must be true.

Let us denote by $\partial_j V = \partial_j^- V \bigcap_{i \neq j} \partial_{i,j} V$ the "lower" jth boundary of V. We say that V has a nontrivial boundary if for every $j = 1, 2, \ldots, d$, $\partial_j V$ contains more than 2^d points and that these points do not belong to a variety of dimension less than $d - 1$. For example, a line in $d = 2$ is a nontrivial boundary, so is a plane in $d = 3$.

Proposition 3.8. *Let V be a discrete set with a nontrivial boundary and let D be an admissible, self-adjoint operator. If $a \neq 0$, then after an appropriate translation, $\partial_j V$ becomes a subset of $\{x\colon\ x_j = 0\}$ and the coefficients $A_{i,j}(x)$ vanish on $\{x\colon\ x_j = 0\}$.*

Proof. Using the fact that D is self-adjoint, Proposition 2.2, (2.4) and (2.6) we see that

$$A_{i,j}(x) = 0, \quad x \in \partial_j V \quad \text{for } 1 \leqslant j \leqslant d.$$

Applying Bezout's theorem for polynomials in d variables [7, Chapter IV, Section 2] for the polynomials $A_{1,j}(x), \ldots, A_{d,j}(x)$ which satisfy (3.3b)–(3.3c) and vanish on $\partial_j V$, one concludes that they must contain a common linear factor. Equation (3.3b) shows that this factor must be of the form $x_j - h_j$. Thus, applying the translation $x \to x + h$ with $h = (h_1, h_2, \ldots, h_d)$, $A_{i,j}$ takes the form

$$A_{i,j}(x) = x_j \times (\text{linear polynomial in } x_1, x_2, \ldots, x_d), \tag{3.13}$$

and completes the proof. $\square$

The compatibility conditions also imply that $A_{i,j} - A_{j,i} \in \Pi_1^d$ for $i \neq j$, which combined with (3.3c) shows that $A_{i,j} = a x_i x_j \mod \Pi_1^d$ (the argument works simultaneously for $a \neq 0$ and $a = 0$). The form of the coefficients determined below will be the staring point in the next sections.

Proposition 3.9. *Let $A_{i,j}$ and B_i satisfy the conditions in Theorem* 3.7(ii) *and assume that $A_{i,j}$ vanishes on $\{x\colon\ x_j = 0\}$. Then, the polynomial identities* (2.12a) *imply that*

$$A_{i,j} = x_j(a x_i + l_{i,j}), \tag{3.14}$$

$$B_i = b x_i + s_i, \tag{3.15}$$

or equivalently:

$$\alpha_{i,j} = -x_j(a x_i + l_{i,j}), \tag{3.16}$$

$$\beta_i = a x_i \sum_{k=1}^{d} x_k + \sum_{k=1}^{d} l_{i,k} x_k + b x_i + s_i, \tag{3.17}$$

$$\gamma_i = a x_i \sum_{k=1}^{d} x_k + x_i \sum_{k=1}^{d} l_{k,i}, \tag{3.18}$$

for some constants a, b, $l_{i,j}$ and s_i.

Proof. Equation (3.15) follows immediately from (3.3a). From Eqs. (3.3b), (3.3c) and the fact that $A_{i,j}$ vanishes on $\{x\colon\ x_j = 0\}$ we deduce that

$$A_{i,j} = x_j(q_{i,j}x_i + l_{i,j}), \tag{3.19}$$

where $q_{i,i} = a$ and $q_{i,j} + q_{j,i} = 2a$. It remains to show that $q_{i,j} = a$ for all $i \neq j$. From (2.4)–(2.6) we see that

$$\begin{aligned}
\alpha_{i,j} &= -q_{i,j}x_ix_j + \text{linear terms},\\
\beta_i &= x_i\sum_{k=1}^{n} q_{i,k}x_k + \text{linear terms},\\
\gamma_i &= x_i\sum_{k=1}^{n} q_{k,i}x_k + \text{linear terms}.
\end{aligned}$$

Comparing the coefficient of $x_i^3x_j^3$ on both sides of (2.12a) we obtain

$$q_{i,j}\big(q_{j,i}^2 + a^2\big) = q_{j,i}\big(q_{i,j}^2 + a^2\big),$$

which combined with $q_{i,j} + q_{j,i} = 2a$ gives $q_{i,j} = q_{j,i} = a$. $\square$

In the rest of the paper, we will find the general solution of the compatibility conditions (2.12), by equating the coefficients of the different powers of x, and we will determine the possible sets V, weights W as well as explicit bases of orthogonal polynomials.

Notice that (3.4) implies that λ_k must be at most quadratic in k. Thus, we have essentially two possible cases:

(i) λ_k is quadratic in k (i.e. $a \neq 0$);
(ii) λ_k is linear in k (i.e. $a = 0$).

Since we can always divide the equation $Du = \lambda_k u$ by a nonzero constant, we can take $a = -1$ in (i) and similarly we can normalize so that $b = -1$ in (ii). These two cases are discussed in Sections 5 and 6 respectively. In the next section we treat in details the case $d = 1$.

4. One-dimensional orthogonal polynomials

The one-dimensional case is the simplest case with no compatibility conditions. Our goal in this section is to provide a detailed study in this simplest situation and to show how the procedure works (i.e. how we determine the weight function $W(x)$ and the set V). The discrete orthogonal polynomials of one variable will also serve as building blocks in the higher-dimensional cases.

Historically, orthogonal polynomials satisfying difference equations of one variable were studied early in last century, we refer to [1,6] for references. It is well known that there are four families of such orthogonal polynomials of one variable, namely, Hahn polynomials, Meixner polynomials, Krawtchouk polynomials, and Charlier polynomials. However, to our great surprise, another family of orthogonal polynomials of discrete variable shows up in our study.

For simplicity of notation, we will drop all indices within this section, i.e. $x = (x_1) \in \mathbb{R}$, $E = E_1$, $\Delta = \Delta_1$, etc. The operator D will be of the form

$$D = A(x)\Delta\nabla + B(x)\Delta = \beta(x)E - \big(\beta(x) + A(x)\big)\mathcal{I} + A(x)E^{-1}, \tag{4.1}$$

where

$$A(x) = x(ax + l), \quad B(x) = bx + s \quad \text{and} \quad \beta(x) = A(x) + B(x). \tag{4.2}$$

As we explained at the end of the previous section, we have two possible cases:

(i) quadratic eigenvalue λ_k, we can fix $a = -1$;
(ii) linear eigenvalue λ_k, we can fix $a = 0$, $b = -1$.

4.1. *Quadratic eigenvalue: $a = -1$*

In order to make the formulas more symmetric, let us write $A(x)$, $\beta(x)$ and $B(x)$ as

$$A(x) = -x(x + \alpha_3), \tag{4.3a}$$

$$\beta(x) = -(x + \alpha_1 + 1)(x + \alpha_2 + 1), \tag{4.3b}$$

$$B(x) = \beta(x) - A(x) = (\alpha_3 - \alpha_1 - \alpha_2 - 2)x - (1 + \alpha_1)(1 + \alpha_2), \tag{4.3c}$$

i.e. we have replaced $l = -\alpha_3$, $b = \alpha_3 - \alpha_1 - \alpha_2 - 2$ and $s = -(1 + \alpha_1)(1 + \alpha_2)$. Thus (2.9) gives

$$\frac{W(x)}{W(x-1)} = \frac{\beta(x-1)}{A(x)} = \frac{(x + \alpha_1)(x + \alpha_2)}{x(x + \alpha_3)}. \tag{4.4}$$

There are essentially 2 possibilities, depending on if the numerator in (4.4) vanishes for some positive integer x or not.

Hahn polynomials. Let (4.4) vanish for $x = N + 1$ but be positive for $x \leqslant N$. Then one of α_1 and α_2 must be equal to $-N - 1$, say $\alpha_2 = -N - 1$ and we can rewrite (4.4) as

$$\frac{W(x)}{W(x-1)} = \frac{(x + \alpha_1)(N + 1 - x)}{x(\beta_1 + 1 + N - x)}, \tag{4.5}$$

where we denoted $\beta_1 = -\alpha_3 - N - 1$. Thus, up to a constant factor, the weight becomes

$$W(x) = \binom{\alpha_1 + x}{x}\binom{\beta_1 + N - x}{N - x}.$$

The corresponding orthogonal polynomials on $V_N^1 = \{0, 1, \ldots, N\}$ are the Hahn polynomials $Q_n(x; \alpha_1, \beta_1, N)$ given by

$$Q_n(x; \alpha_1, \beta_1, N) = {}_3F_2\left(\begin{matrix} -n, n + \alpha_1 + \beta_1 + 1, -x \\ \alpha_1 + 1, -N \end{matrix}; 1\right). \tag{4.6}$$

In order to have a positive weight at all $x \in V_N^1$ the right-hand side of (4.5) must be positive for $x = 1, 2, \dots, N$. Putting $x = 1$ and $x = N$ we see that $(\alpha_1 + 1)(\beta_1 + N) > 0$ and $(\alpha_1 + N) \times (\beta_1 + 1) > 0$. Starting with $\beta_1 > -1$ or $\beta_1 < -1$ shows that there are two solutions,

(i) $\alpha_1 > -1$, $\beta_1 > -1$,
(ii) $\alpha_1 < -N$, $\beta_1 < -N$.

Conversely, it is clear that if α_1 and β_1 satisfy (i) or (ii) the weight function $W(x)$ will have a constant sign on V_N^1. Hahn polynomials satisfy the following orthogonal relation

$$\sum_{x=0}^{N} \frac{(\alpha_1+1)_x(\beta_1+1)_{N-x}}{x!(N-x)!} Q_n(x;\alpha_1,\beta_1,N) Q_m(x;\alpha_1,\beta_1,N) = \frac{(-1)^n n!(\beta_1+1)_n(n+\alpha_1+\beta_1+1)_{N+1}}{N!(2n+\alpha_1+\beta_1+1)(-N)_n(\alpha_1+1)_n} \delta_{n,m}. \tag{4.7}$$

The difference operator D takes the form

$$D = x(\beta_1+N+1-x)\Delta\nabla + \big((\alpha_1+1)N - x(\alpha_1+\beta_1+2)\big)\Delta. \tag{4.8}$$

Hahn-type polynomials on $\mathbb{N}_0$. Assume now that the right-hand side of (4.4) does not vanish for any $x \in \mathbb{N}$. Then it must be positive for all $x \in \mathbb{N}$. In order to have a positive function $W(x)$ and to have convergent series $\sum_{x\in\mathbb{N}_0} x^n W(x)$ for some $n \in \mathbb{N}_0$, the parameter α_3 must be positive and α_1, α_2 must satisfy one of the following

(i) α_1 and α_2 are both positive, or there exists a negative integer κ such that $\alpha_1, \alpha_2 \in (\kappa, \kappa+1)$;
(ii) $\alpha_2 = \bar{\alpha}_1 \in \mathbb{C} \setminus \mathbb{R}$.

For such $\alpha_1, \alpha_2, \alpha_3$ we can write the weight $W(x)$ as

$$W(x) = \frac{\binom{x+\alpha_1}{x}\binom{x+\alpha_2}{x}}{\binom{x+\alpha_3}{x}} = \frac{(\alpha_1+1)_x(\alpha_2+1)_x}{x!(\alpha_3+1)_x}. \tag{4.9}$$

The series $\sum_{x\geqslant 0} x^n W(x)$ converges absolutely for $n < \alpha_3 - \alpha_1 - \alpha_2 - 1$, i.e. the corresponding polynomials will be orthogonal on $\mathbb{N}_0$ up to a given degree. It seems that these polynomials have not appeared in the literature before. Therefore, we derive below their basic properties: explicit formula in terms of hypergeometric functions, the three term recurrence formula, etc.

Proposition 4.1. *The polynomials*

$$R_n(x) = {}_3F_2\left(\begin{matrix} -n, n-\alpha_3+\alpha_1+\alpha_2+1, -x \\ \alpha_1+1, \alpha_2+1 \end{matrix}; 1\right), \tag{4.10}$$

satisfy

$$D\big(R_n(x)\big) = n(\alpha_3-\alpha_1-\alpha_2-1-n)R_n(x) \tag{4.11}$$

where

$$D = -x(x+\alpha_3)\nabla\Delta + \big((\alpha_3-\alpha_1-\alpha_2-2)x - (1+\alpha_1)(1+\alpha_2)\big)\Delta, \tag{4.12}$$

and the orthogonal relation

$$\sum_{x=0}^{\infty} R_n(x)R_m(x)\frac{(\alpha_1+1)_x(\alpha_2+1)_x}{x!\,(\alpha_3+1)_x} = 0,$$

for $n \neq m$ such that $n+m < \alpha_3-\alpha_1-\alpha_2-1$.

Proof. We look for a solution of (4.11) of the form

$$R_n(x) = \sum_{k=0}^{n} a_k m_k(x), \quad \text{where } m_k(x) = \frac{(-x)_k}{k!}. \tag{4.13}$$

Notice that

$$\Delta m_k = -m_{k-1}, \quad x\nabla m_k = k m_k \quad \text{and} \quad x m_{k-1} = (k-1)m_{k-1} - k m_k. \tag{4.14}$$

Using these formulas we can easily calculate $D(R_n)$:

$$\begin{aligned}
D\big(R_n(x)\big) &= \sum_{k=1}^{n}\big[(x+\alpha_3)x\nabla\big(a_k m_{k-1}(x)\big) - \big((\alpha_3-\alpha_1-\alpha_2-2)x \\
&\quad - (1+\alpha_1)(1+\alpha_2)\big)a_k m_{k-1}(x)\big] \\
&= \sum_{k=1}^{n}\big[(k-\alpha_3+\alpha_1+\alpha_2+1)x + \alpha_3(k-1) + (1+\alpha_1)(1+\alpha_2)\big]a_k m_{k-1}(x) \\
&= \sum_{k=0}^{n}\big[k(\alpha_3-\alpha_1-\alpha_2-1-k)a_k + (\alpha_1+k+1)(\alpha_2+k+1)a_{k+1}\big],
\end{aligned}$$

where in the last formula we assumed that $a_{n+1}=0$. Thus the difference equation (4.11) leads to the recursive relation

$$a_{k+1} = -\frac{(n-k)(n+k-\alpha_3+\alpha_1+\alpha_2+1)}{(\alpha_1+k+1)(\alpha_2+k+1)}a_k,$$

whose solution with $a_0 = 1$ is

$$a_k = \frac{(-n)_k(n-\alpha_3+\alpha_1+\alpha_2+1)_k}{(\alpha_1+1)_k(\alpha_2+1)_k}. \tag{4.15}$$

This shows that the polynomials $R_n(x)$ defined by (4.10) satisfy (4.11). The orthogonality follows from Remark 3.4. □

From the explicit formula we can derive other properties of these orthogonal polynomials.

Proposition 4.2. *The polynomials R_n satisfy the three-term relation*

$$xR_n(x) = \mathfrak{a}_n R_{n+1}(x) - (\mathfrak{a}_n + \mathfrak{c}_n)R_n(x) + \mathfrak{c}_n R_{n-1}(x), \tag{4.16}$$

where the coefficients are given by

$$\mathfrak{a}_n = \frac{(n+\alpha_1+1)(n+\alpha_2+1)(n-\alpha_3+\alpha_1+\alpha_2+1)}{(2n-\alpha_3+\alpha_1+\alpha_2+1)(2n-\alpha_3+\alpha_1+\alpha_2+2)} \tag{4.17a}$$

and

$$\mathfrak{c}_n = -\frac{n(n-\alpha_3+\alpha_1)(n-\alpha_3+\alpha_2)}{(2n-\alpha_3+\alpha_1+\alpha_2)(2n-\alpha_3+\alpha_1+\alpha_2+1)}. \tag{4.17b}$$

Proof. Using again (4.14) we see that

$$xR_n = \sum_{k=1}^{n} k(a_k - a_{k-1})m_k(x) - (n+1)a_n m_{n+1}(x),$$

where a_k are given by (4.15). It is now a simple matter to compare it with $\mathfrak{a}_n R_{n+1}(x) - (\mathfrak{a}_n + \mathfrak{c}_n)R_n(x) + \mathfrak{c}_n R_{n-1}(x)$ and verify that the coefficients of $m_k(x)$ agree for $k = 0, 1, \ldots, n+1$. $\Box$

Proposition 4.3. *The norm of the polynomial R_n can be computed from the formula*

$$\begin{aligned}\langle R_n, R_n\rangle = {} & \frac{n!}{(\alpha_3-\alpha_1-\alpha_2-2n-1)(1+\alpha_1)_n(1+\alpha_2)_n} \\ & \times \frac{\Gamma(\alpha_3-\alpha_1-\alpha_2-n)\Gamma(\alpha_3+1)}{\Gamma(\alpha_3-\alpha_1-n)\Gamma(\alpha_3-\alpha_2-n)}.\end{aligned} \tag{4.18}$$

Proof. Using the three-term relation, a standard computation shows that

$$\langle R_n, R_n\rangle = \frac{c_n c_{n-1}\cdots c_1}{a_{n-1}a_{n-2}\cdots a_0}\langle R_0, R_0\rangle.$$

The quantity $\langle R_0, R_0\rangle = \langle 1, 1\rangle$ is a ${}_2F_1$ evaluated at 1, which can be computed in closed form using Gauss' formula

$$\langle 1, 1\rangle = \frac{\Gamma(\alpha_3+1)\Gamma(\alpha_3-\alpha_1-\alpha_2-1)}{\Gamma(\alpha_3-\alpha_1)\Gamma(\alpha_3-\alpha_2)}.$$

The statement follows from the above relations, (4.17) and the formula

$$\Gamma(x) = (-1)^n(-x+1)_n\Gamma(x-n). \qquad \Box$$

Remark 4.4. Notice that although $\langle R_n, R_m\rangle$ converges only for finitely many (n, m), formula (4.10) defines polynomials for every $n \in \mathbb{N}_0$. The operator D given in (4.12) is admissible according to Definition 3.3. Moreover, Eqs. (4.11) and (4.16) hold for every $n \in \mathbb{N}_0$. In particular, this implies that the polynomials $R_n(x)$ provide a solution to the discrete–discrete version of the bispectral problem discussed in [3].

4.2. *Linear eigenvalue: $a = 0$ and $b = -1$*

If $a = 0$ and $b = -1$, Eqs. (4.2) give $A(x) = lx$, $B(x) = -x + s$, and $\beta(x) = A(x) + B(x) = (l-1)x + s$. Thus

$$\frac{W(x)}{W(x-1)} = \frac{(l-1)(x-1)+s}{lx}. \tag{4.19}$$

We have now 2 possibilities: $l = 1$ or $l \neq 1$.

Charlier polynomials. If $l = 1$, Eq. (4.19) essentially reduces to

$$\frac{W(x)}{W(x-1)} = \frac{s}{x},$$

i.e. s must be a positive number and (up to a constant factor) $W(x) = s^x/x!$. The corresponding polynomials are the Charlier polynomials

$$C_n(x; s) = {}_2F_0\left(\begin{matrix} -n, -x \\ - \end{matrix}; -\frac{1}{s}\right)$$

orthogonal on $\mathbb{N}_0$. The difference operator takes the form

$$D = x\Delta\nabla + (s-x)\Delta = sE - (x+s)\mathcal{I} + xE^{-1}. \tag{4.20}$$

Assume now that $l \neq 1$. Then, we can rewrite (4.19) as

$$\frac{W(x)}{W(x-1)} = c\frac{x+d}{x}, \tag{4.21}$$

where we have put $c = (l-1)/l \neq 0$ and $d = (1+s-l)/(l-1)$.

We now have two possibilities depending on the sign of c.

Krawtchouk polynomials. If $c < 0$ then (4.21) can be rewritten as

$$\frac{W(x)}{W(x-1)} = (-c)\frac{-d-x}{x}, \tag{4.22}$$

from which it follows that $(-d)$ must be a positive integer (otherwise, $W(x)$ will become negative at some point). If we put $d = -N-1$ and $p = c/(c-1) \in (0,1)$ we obtain the Krawtchouk polynomials

$$K_n(x; p, N) = {}_2F_1\left(\begin{matrix} -n, -x \\ -N \end{matrix}; \frac{1}{p}\right)$$

orthogonal on $V_N^1 = \{0, 1, \dots, N\}$ with respect to the weight $W(x) = p^x(1-p)^{N-x}\binom{N}{x}$. The difference operator takes the form

$$D = x(1-p)\Delta\nabla + (Np-x)\Delta. \tag{4.23}$$

Meixner polynomials. If $c > 0$ then $W(1) > 0$ implies that $\beta := d + 1 > 0$. In order to have convergent series, we need $c < 1$. This leads to the Meixner polynomials

$$M_n(x; \beta, c) = {}_2F_1\left(\begin{matrix} -n, -x \\ \beta \end{matrix}; 1 - \frac{1}{c}\right), \tag{4.24}$$

depending on two parameters $0 < c < 1$ and $\beta > 0$, orthogonal on $\mathbb{N}_0$ with respect to the weight $W(x) = c^x(\beta)_x/x!$. The difference operator is

$$D = \frac{x}{1-c}\Delta\nabla + \left(-x - \frac{c\beta}{1-c}\right)\Delta. \tag{4.25}$$

The Meixner polynomials satisfy the orthogonal relation

$$\sum_{x=0}^{\infty} \frac{(\beta)_x}{x!} c^x M_n(x, \beta, c) M_m(x, \beta, c) = \frac{c^{-n} n!}{(\beta)_n (1-c)^\beta} \delta_{m,n}. \tag{4.26}$$

5. Multivariable case with quadratic eigenvalue: $a = -1$

We first solve the compatibility conditions (2.12) in arbitrary dimension $d > 1$, and then we write explicitly all possible orthogonal polynomials with the corresponding weights. For $a = -1$ formulas (3.16)–(3.18) give:

$$\alpha_{i,j} = x_j(x_i - l_{i,j}), \tag{5.1a}$$

$$\beta_i = -x_i \sum_{k=1}^{d} x_k + \sum_{k=1}^{d} l_{i,k} x_k + b x_i + s_i, \tag{5.1b}$$

$$\gamma_i = x_i\left(-\sum_{k=1}^{d} x_k + \sum_{k=1}^{d} l_{k,i}\right). \tag{5.1c}$$

The next proposition gives the general solution to the compatibility conditions (2.12).

Proposition 5.1. *For $a = -1$ there are 2 solutions to the compatibility conditions.*

Solution 1:

$$\alpha_{i,j} = x_j(x_i - l_i), \tag{5.2a}$$

$$\beta_i = -(x_i - l_i)\left(\sum_{k=1}^{d} x_k - b - r\right), \tag{5.2b}$$

$$\gamma_i = -x_i\left(\sum_{k=1}^{d} x_k - \sum_{k=1}^{d} l_k - r\right), \tag{5.2c}$$

where b, r, and $\{l_i\}_{i=1}^{d}$ are free parameters.

Solution 2:

$$\alpha_{i,j} = x_j(x_i - l_i), \tag{5.3a}$$

$$\beta_i = -(x_i - l_i)\left(\sum_{k=1}^{d} x_k - \sum_{k=1}^{d} l_k + 1 - r_i\right), \tag{5.3b}$$

$$\gamma_i = -x_i\left(\sum_{k=1}^{d} x_k - \sum_{k=1}^{d} l_k - r_i\right), \tag{5.3c}$$

where $\{l_i, r_i\}_{i=1}^{d}$ *are free parameters and* $b = \sum_{k=1}^{d} l_k - 1$.

Proof. Let us take $i \neq j$ and let $k \neq i, j$. Comparing the coefficients of $x_i^3 x_j x_k$ and $x_i^3 x_j$ on both sides of (2.12a) we get:

$$-(l_{j,k} + l_{j,i} + 2) = -2(l_{j,i} + 1), \tag{5.4}$$

and

$$(l_{j,i} + 1)\left(\sum_{k=1}^{d} l_{k,i}\right) - (s_j - 1 - b - l_{j,j}) + (1 + l_{i,j})(1 + l_{j,i})$$
$$= -(l_{j,i} + 1)\left[-\left(\sum_{k=1}^{d} l_{k,j}\right) - (b + l_{i,i} + 2)\right]. \tag{5.5}$$

Equation (5.4) simply means that for $i \neq j$, $l_{j,i}$ is independent of i, i.e. we can put $l_j := l_{j,i}$ for $j \neq i$. Using this in (5.5) we obtain that

$$s_j = l_j(l_j - l_{j,j} - b). \tag{5.6}$$

Our formulas simplify as follows

$$\alpha_{i,j} = x_j(x_i - l_i), \tag{5.7a}$$

$$\beta_i = -x_i \sum_{k=1}^{d} x_k + l_i \sum_{k \neq i} x_k + (l_{i,i} + b)x_i - l_i(l_{i,i} + b - l_i)$$
$$= -(x_i - l_i)\left(\sum_{k=1}^{d} x_k - l_{i,i} - b + l_i\right), \tag{5.7b}$$

$$\gamma_i = x_i\left(-\sum_{k=1}^{d} x_k + \sum_{k \neq i} l_k + l_{i,i}\right). \tag{5.7c}$$

With the above formulas it is very easy to solve completely (2.12a). Indeed, after canceling the common factor $x_i x_j (x_i - l_i - 1)(x_j - l_j - 1)$ Eq. (2.12a) gives

$$(l_{i,i} - l_i - l_{j,j} + l_j)\left(\sum_{k=1}^{d} l_k - b - 1\right) = 0. \tag{5.8}$$

Let us denote $r_i = l_{i,i} - l_i$. Then the first factor in the last formula is simply $r_i - r_j$.

From (5.8) it follows that we have 2 possibilities: $r_i = r_j$ for all $i \neq j$, i.e. $r := r_i$ is independent of i, or $b = \sum_{k=1}^{d} l_k - 1$. The first leads to Solution 1, and the second leads to Solution 2. It is immediate to see that the functions defined by (5.2) or (5.3) satisfy (2.12b). □

Next, we want to determine the possible weights and orthogonal polynomials. It is easy to see that there is no weight corresponding to Solution 2. Otherwise, from (5.3) and (2.9) we will have

$$\frac{W(x)}{W(x - e_i)} = \frac{x_i - l_i - 1}{x_i} \quad \text{for } i = 1, 2, \ldots, d. \tag{5.9}$$

Then, for $x = e_i$ it follows that $l_i < 0$, i.e. we can put $-l_i = r_i > 0$ and therefore, up to a constant factor, $W(x)$ is given by

$$W(x) = \prod_{i=1}^{d} \frac{(r_i)_{x_i}}{x_i!},$$

but then even $\langle 1, 1\rangle$ will diverge.

Let us now concentrate on Solution 1, given by (5.2). Equations (5.2) and (2.9) give

$$\frac{W(x)}{W(x - e_i)} = \frac{x_i - l_i - 1}{x_i} \frac{(\sum_{k=1}^{d} x_k - b - r - 1)}{(\sum_{k=1}^{d} x_k - \sum_{k=1}^{d} l_k - r)}, \tag{5.10}$$

for $i = 1, 2, \ldots, d$. For a vector $y \in \mathbb{R}^k$ we will denote

$$|y| = \sum_{i=1}^{k} y_i. \tag{5.11}$$

If we put

$$\sigma_i = -(l_i + 1), \tag{5.12}$$

and

$$\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_d) \tag{5.13}$$

we can write (5.10) as

$$\frac{W(x)}{W(x - e_i)} = \frac{x_i + \sigma_i}{x_i} \frac{(|x| - b - r - 1)}{(|x| + |\sigma| + d - r)}. \tag{5.14}$$

We have several possible sub-cases, depending on whether the right-hand side in Eq. (5.14) can vanish or not. We divide them first in two big subclasses:

- $b+r \notin \mathbb{N}_0$;
- $b+r \in \mathbb{N}_0$.

5.1. *Case 1: $b+r \notin \mathbb{N}_0$*

Then $|x|-b-r-1$ does not vanish. We have two possibilities:

5.1.1. For some $i \in \{1,2,\dots,d\}$, $(-\sigma_i) \notin \mathbb{N}$. We show below that this implies that $\sigma_j > -1$ for all $j \in \{1,2,\dots,d\}$. Take $j \neq i$, arbitrary $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$. From (5.14) we see that

$$\frac{W(ke_i+ne_j)}{W((k-1)e_i+ne_j)} = \frac{k+\sigma_i}{k}\frac{k+n-b-r-1}{k+n+|\sigma|+d-r} > 0. \tag{5.15}$$

The second ratio is positive for large n, and therefore $k+\sigma_i > 0$ for every $k \in \mathbb{N}$, i.e. $\sigma_i > -1$. Equation (5.15) for $n=0$ implies that

$$\frac{k-b-r-1}{k+|\sigma|+d-r} > 0,$$

for every $k \in \mathbb{N}$. But then for every $j \in \{1,2,\dots,d\}$ we must have

$$\frac{W(e_j)}{W(0)} = \frac{1+\sigma_j}{1}\frac{1-b-r-1}{1+|\sigma|+d-r} > 0,$$

showing that $\sigma_j > -1$.

Since $\sigma_j > -1$ for all j and $b+r \notin \mathbb{N}_0$ we must have $V = \mathbb{N}_0^d$. Up to a constant, the weight $W(x)$ is given by

$$W(x) = \prod_{i=1}^{d}\frac{(\sigma_i+1)_{x_i}}{x_i!}\frac{(-b-r)_{|x|}}{(|\sigma|+d-r+1)_{|x|}}.$$

Changing the parameters

$$\gamma = |\sigma|+d-r, \qquad \beta = -(b+r)-1,$$

the weight takes the form

$$W(x) = \prod_{i=1}^{d}\frac{(\sigma_i+1)_{x_i}}{x_i!}\frac{(\beta+1)_{|x|}}{(\gamma+1)_{|x|}},$$

and the free parameters are $\{\sigma_i\}$, β and γ. The difference operator becomes

$$D = -\sum_{1 \leqslant i,j \leqslant d} x_j \big[\sigma_i + 1 + x_i + \delta_{i,j}\big(\gamma - |\sigma| - d\big)\big] \Delta_i \nabla_j$$
$$+ \sum_{i=1}^{d} \big[x_i\big(\gamma - \beta - |\sigma| - d - 1\big) - (1+\beta)(1+\sigma_i)\big]\Delta_i, \tag{5.16}$$

and $\lambda_n = n(-n + \gamma - \beta - |\sigma| - d)$.

For a vector $y = (y_1, y_2, \ldots, y_d) \in \mathbb{R}^d$ we will denote

$$y^j = (y_j, y_{j+1}, \ldots, y_d) \quad \text{and} \quad Y_j = (y_1, y_2, \ldots, y_j), \tag{5.17}$$

with the convention that $y^{d+1} = 0$ and $Y_0 = 0$. Let us denote by $R_n(x; \alpha_1, \alpha_2, \alpha_3)$ the orthogonal polynomials given by (4.10).

Theorem 5.2. *For $\nu \in \mathbb{N}_0^d$, such that $2|\nu| < \gamma - |\sigma| - \beta - d - 1$, the polynomials*

$$R_\nu(x; \sigma, \beta, \gamma) = \prod_{j=1}^{d} (\alpha_{2,j} + 1)_{\nu_j} R_{\nu_j}(x_j; \sigma_j, \alpha_{2,j}, \alpha_{3,j}),$$

where $\alpha_{2,j}$ and $\alpha_{3,j}$ are given by

$$\alpha_{2,j} = \beta + \big|\nu^{j+1}\big| + |X_{j-1}|,$$
$$\alpha_{3,j} = \gamma - \big|\nu^{j+1}\big| - \big|\sigma^{j+1}\big| - (d - j) + |X_{j-1}|$$

satisfy the difference equation

$$D\phi_\nu = \lambda_{|\nu|}\phi_\nu$$

and the orthogonal relation

$$\sum_{x \in \mathbb{N}_0^d} R_\nu(x; \sigma, \beta, \gamma) R_\mu(x; \sigma, \beta, \gamma) \prod_{i=1}^{d} \frac{(\sigma_i + 1)_{x_i}}{x_i!} \frac{(\beta + 1)_{|x|}}{(\gamma + 1)_{|x|}} = A_\nu \delta_{\nu,\mu}, \tag{5.18}$$

where

$$A_\nu = \frac{(1+\beta)_{|\nu|}\Gamma(1+\gamma)\Gamma(\gamma - \beta - |\sigma| - 2|\nu| - d)}{\Gamma(\gamma - |\sigma| - |\nu| + 1 - d)\Gamma(\gamma - \beta)}$$
$$\times \prod_{j=1}^{d} \frac{\nu_j!(\alpha_{3,j} - \alpha_{2,j} - \nu_j)_{\nu_j}(\alpha_{3,j-1} - \alpha_{2,j-1} + 1)_{\nu_j}}{(1+\sigma_j)_{\nu_j}}. \tag{5.19}$$

Proof. Notice that $\alpha_{3,j} - \alpha_{2,j}$ is independent of x, and therefore from (4.10) one can immediately see that ϕ_ν is indeed a polynomial of x of total degree $|\nu|$. From Theorem 3.7 we know that for every $k \in \mathbb{N}_0$, the equation $Du = \lambda_k u$ has $r_k = \binom{k+d-1}{k} = \dim(\Pi_k^d / \Pi_{k-1}^d)$ linearly independent solutions. Therefore, it is enough to prove that (5.18) holds.

In (5.18) we will first sum with respect to x_d, then with respect to x_{d-1}, and so on. Writing

$$(\beta+1)_{|x|} = \bigl(\beta+1+|X_{d-1}|\bigr)_{x_d}(\beta+1)_{|X_{d-1}|}$$

and extracting only the terms depending on x_d we get the sum

$$\begin{aligned} I_{\nu_d,\mu_d} &:= \sum_{x_d\geqslant 0} R_{\nu_d}(x_d)R_{\mu_d}(x_d)\frac{(\sigma_d+1)_{x_d}}{x_d!}\frac{(\beta+1+|X_{d-1}|)_{x_d}}{(\gamma+1+|X_{d-1}|)_{x_d}} \\ &= \sum_{x_d\geqslant 0} R_{\nu_d}(x_d)R_{\mu_d}(x_d)\frac{(\sigma_d+1)_{x_d}}{x_d!}\frac{(\alpha_{2,d}+1)_{x_d}}{(\alpha_{3,d}+1)_{x_d}}, \end{aligned}$$

where $R_m(x_d) = R_m(x_d;\sigma_d,\alpha_{2,d},\alpha_{3,d})$. Using now (4.18) and the fact that $\alpha_{3,d}-\alpha_{2,d} = \gamma-\beta$ is independent of x, we see that

$$I_{\nu_d,\mu_d} = \delta_{\nu_d,\mu_d} B_{\nu_d}\frac{\Gamma(\gamma+|X_{d-1}|+1)}{(1+\beta+|X_{d-1}|)_{\nu_d}\Gamma(\gamma-\sigma_d-\nu_d+|X_{d-1}|)},$$

where B_{ν_d} is a constant (independent of x), whose value can be extracted from (4.18)

$$B_{\nu_d} = \frac{\nu_d!\,\Gamma(\gamma-\beta-\sigma_d-\nu_d)}{(1+\sigma_d)_{\nu_d}(\gamma-\beta-\sigma_d-2\nu_d-1)\Gamma(\gamma-\beta-\nu_d)}.$$

Using the fact that

$$\Gamma(x) = (x-n)_n\Gamma(x-n), \tag{5.20}$$

we see that

$$I_{\nu_d,\mu_d} = \delta_{\nu_d,\mu_d} C_{\nu_d}\frac{(\gamma+1)_{|X_{d-1}|}}{(1+\beta+|X_{d-1}|)_{\nu_d}(\gamma-\sigma_d-\nu_d)_{|X_{d-1}|}},$$

with $C_{\nu_d} = B_{\nu_d}\Gamma(\gamma+1)/\Gamma(\gamma-\sigma_d-\nu_d)$. This shows, in particular, that

$$\begin{aligned} &\bigl[\bigl(\beta+1+|X_{d-1}|\bigr)_{\nu_d}\bigr]^2\frac{(\beta+1)_{|X_{d-1}|}}{(\gamma+1)_{|X_{d-1}|}}I_{\nu_d,\mu_d} \\ &\quad = \delta_{\nu_d,\mu_d}C_{\nu_d}(\beta+1)_{\nu_d}\frac{(\beta+1+\nu_d)_{|X_{d-1}|}}{(\gamma-\sigma_d-\nu_d)_{|X_{d-1}|}}, \end{aligned} \tag{5.21}$$

where we have used the identity

$$(\beta+1)_{|X_{d-1}|}\bigl(\beta+1+|X_{d-1}|\bigr)_{\nu_d} = (\beta+1)_{|\nu_d|}(\beta+1+\nu_d)_{|X_{d-1}|}. \tag{5.22}$$

Equation (5.21) shows that the remaining $d-1$ fold sums of $\langle R_\nu, R_\mu\rangle$ have exactly the same structure as that of the original d fold sums with β and γ replaced by $\beta+\nu_d$ and $\gamma-\sigma_d-\nu_d-1$, respectively. In other words, it shows that we can use induction to complete the proof. For A_ν we obtain the following formula:

$$A_\nu = \frac{(1+\beta)_{|\nu|}\Gamma(1+\gamma)}{\Gamma(\gamma-|\sigma|-|\nu|+1-d)} \prod_{j=1}^{d} \frac{\nu_j!}{(1+\sigma_j)_{\nu_j}(\gamma-\beta-|\sigma^j|-2|\nu^j|+j-d-1)}$$
$$\times \prod_{j=1}^{d} \frac{\Gamma(\gamma-\beta-|\sigma^j|-2|\nu^{j+1}|-\nu_j+j-d)}{\Gamma(\gamma-\beta-|\sigma^{j+1}|-2|\nu^{j+1}|-\nu_j+j-d)}. \tag{5.23}$$

Using several times (5.20) one can rewrite the right-hand side as in (5.19). □

5.1.2. For every $i \in \{1, 2, \dots, d\}$ we have $(-\sigma_i) = l_i + 1 \in \mathbb{N}$, i.e. $l_i \in \mathbb{N}_0$. The corresponding V is the parallelepiped

$$V_l^d = \{x \in \mathbb{N}_0^d\colon\ x_i \leqslant l_i\}.$$

By (5.10) the weight function in this case is given by

$$W(x) = \prod_{i=1}^{d} \frac{(-l_i)_{x_i}}{x_i!} \frac{(\beta+1)_{|x|}}{(-|l|-r+1)_{|x|}}, \tag{5.24}$$

where we again set $\beta = -(b+r)-1$. Putting $x = e_i$ and $x = (l_1, l_2, \dots, l_d)$ in (5.10) we get that

$$\frac{\beta+1}{|l|+r-1} > 0 \quad \text{and} \quad \frac{|l|+\beta}{r} > 0.$$

From this we see that the parameters β and r must satisfy one of the following conditions:

(i) $\beta > -1$ and $r > 0$;
(ii) $\beta < -|l|$ and $r < -|l| + 1$.

Recall that $b+r \notin \mathbb{N}_0$, but everything will hold even if $b+r = -\beta-1 \in \mathbb{N}_0$ as long as $\beta+|l| < 0$ (i.e. if (ii) holds), because $(\beta+1)_{|x|}$ will not vanish for $x \in V_l^d$.

In the following we will use the notations y^j and Y_j defined in (5.17). For example, $|L_j| = l_1+\dots+l_j$. If $\nu, l \in \mathbb{N}_0^d$, then $\nu \leqslant l$ means $\nu_i \leqslant l_i$ for $1 \leqslant i \leqslant d$. Recall that the Hahn polynomials are denoted by $Q_n(x; \alpha_1, \beta_1, N)$.

Theorem 5.3. *Let $l_i \in \mathbb{N}_0$, $1 \leqslant i \leqslant d$. For $\nu \in \mathbb{N}_0^d$, $\nu_i \leqslant l_i$, the polynomials*

$$\phi_\nu(x; \beta, r, l) = \prod_{i=1}^{d} (\alpha_{1,j}+1)_{\nu_j} Q_{\nu_j}(x_j; \alpha_{1,j}, \alpha_{2,j}, l_j), \tag{5.25}$$

where $\alpha_{1,j}$ and $\alpha_{2,j}$ are given by

$$\alpha_{1,j} = \beta + |\nu^{j+1}| + |X_{j-1}| \quad \text{and} \quad \alpha_{2,j} = |L_{j-1}| - |X_{j-1}| + |\nu^{j+1}| + r - 1,$$

satisfy the difference equation

$$D\phi_\nu = \lambda_{|\nu|}\phi_\nu$$

and the orthogonal relation

$$\sum_{\nu\leqslant l}\phi_\nu(x;\beta,r,l)\phi_\mu(x;\beta,r,l)\prod_{i=1}^{d}\frac{(-l_i)_{x_i}}{x_i!}\frac{(\beta+1)_{|x|}}{(-|l|-r+1)_{|x|}}=B_\nu\delta_{\nu,\mu}, \tag{5.26}$$

where the normalization constant is given by

$$B_\nu=\frac{(-1)^{|\nu|}(1+\beta)_{|\nu|}}{(r+|\nu|)_{|l|-|\nu|}}\prod_{j=1}^{d}\frac{\nu_j!(\beta+r+2|\nu^{j+1}|+\nu_j+|L_{j-1}|)_{l_j+1}}{(-l_j)_{\nu_j}(\beta+r+2|\nu^j|+|L_{j-1}|)}. \tag{5.27}$$

Proof. Since $\alpha_{1,j}+\alpha_{2,j}$ is independent of x, it is easy to see from (4.6) that ϕ_ν is indeed a polynomial of x of total degree ν. We proceed as in the proof of Theorem 5.2. In (5.26) we will first sum with respect to x_d. Using the fact that

$$\begin{gathered}\left(-|l|-r+1\right)_{|x|}=\left(-|l|-r+1+|X_{d-1}|\right)_{x_d}\left(-|l|-r+1\right)_{|X_{d-1}|},\\ (\beta+1)_{|x|}=(\beta+1)_{|X_{d-1}|}\left(\beta+1+|X_{d-1}|\right)_{x_d},\end{gathered}$$

we can split the weight function as a product

$$W(x)=\frac{(-l_d)_{x_d}(\beta+1+|X_{d-1}|)_{x_d}}{x_d!(-|l|-r+1+|X_{d-1}|)_{x_d}}W'(X_{d-1})$$

where

$$W'(X_{d-1})=\prod_{i=1}^{d-1}\frac{(-l_i)_{x_i}}{x_i!}\frac{(\beta+1)_{|X_{d-1}|}}{(-|l|-r+1)_{|X_{d-1}|}}.$$

It is easy to verify that

$$\frac{(-l_d)_{x_d}(\beta+1+|X_{d-1}|)_{x_d}}{x_d!(-|l|-r+1+|X_{d-1}|)_{x_d}}=\frac{l_d!}{(\alpha_{2,d}+1)_{l_d}}\binom{x_d+\alpha_{1,d}}{x_d}\binom{l_d-x_d+\alpha_{2,d}}{l_d-x_d}.$$

Hence, using the fact that $\alpha_{1,d}$ and $\alpha_{2,d}$ are independent of x_d, the sum over x_d in (5.26) becomes

$$\begin{aligned}I_{\nu_d,\mu_d}:={}&\frac{l_d!}{(\alpha_{2,d}+1)_{l_d}}(\alpha_{1,d}+1)_{\nu_d}(\alpha_{1,d}+1)_{\mu_d}\\ &\times\sum_{x_d=0}^{l_d}Q_{\nu_d}(x_d)Q_{\mu_d}(x_d)\binom{x_d+\alpha_{1,d}}{x_d}\binom{l_d-x_d+\alpha_{2,d}}{l_d-x_d},\end{aligned}$$

where $Q_m(x_d)=Q_m(x_d,\alpha_{1,d},\alpha_{2,d},l_d)$. Using (4.7) and simplifying, we get

$$I_{\nu_d,\mu_d}=\delta_{\nu_d,\mu_d}\frac{(-1)^{\nu_d}\nu_d!(\nu_d+\beta+|L_{d-1}|+r)_{l_d+1}}{(-l_d)_{\nu_d}(2\nu_d+\beta+|L_{d-1}|+r)}\frac{(\alpha_{2,d}+1)_{\nu_d}}{(\alpha_{2,d}+1)_{l_d}}(\alpha_{1,d}+1)_{\nu_d}.$$

In the above expression, the term $(\alpha_{1,d}+1)_{\nu_d}$ contains variables $x_1,\dots,x_{d-1}$. Combining this term with $W'(X_{d-1})$ and using (5.22), we see that the weight function for $x_1,\dots,x_{d-1}$ becomes

$$W_{d-1}(X_{d-1}) = (\beta+1)_{\nu_d}\prod_{i=1}^{d-1}\frac{(-l_i)_{x_i}}{x_i!}\frac{(\beta+1+\nu_d)_{|X_{d-1}|}}{(-|l|-r+1)_{|X_{d-1}|}}\frac{(\alpha_{2,d}+1)_{\nu_d}}{(\alpha_{2,d}+1)_{l_d}}.$$

By the definition of $\alpha_{2,d}$ and expanding the Pochhammer symbols gives

$$\begin{aligned}&\bigl(-|l|-r+1\bigr)_{|X_{d-1}|}\frac{(\alpha_{2,d}+1)_{l_d}}{(\alpha_{2,d}+1)_{\nu_d}}\\ &\quad=(-1)^{l_d-\nu_d}\bigl(-|l|-r+1\bigr)\bigl(-|l|-r+2\bigr)\cdots\bigl(-|L_{d-1}|-r-\nu_d+|X_{d-1}|\bigr)\\ &\quad=\bigl(-|L_{d-1}|-r+1-\nu_d\bigr)_{|X_{d-1}|}\bigl(|L_{d-1}|+r+\nu_d\bigr)_{l_d-\nu_d}.\end{aligned}$$

Consequently, the weight function W_{d-1} becomes

$$W_{d-1}(X_{d-1}) = \frac{(\beta+1)_{\nu_d}}{(|L_{d-1}|+r+\nu_d)_{l_d-\nu_d}}\prod_{i=1}^{d-1}\frac{(-l_i)_{x_i}}{x_i!}\frac{(\beta+1+\nu_d)_{|X_{d-1}|}}{(-|L_{d-1}|-r+1-\nu_d)_{|X_{d-1}|}}.$$

Apart from a constant multiple, the weight function W_{d-1} has exactly the same structure of $W(x)$ with β replaced by $\beta+\nu_d$, r replaced by $r+\nu_d$, l replaced by L_{d-1}, respectively, and one variable less. Hence, we can use induction to complete the proof. For B_ν we obtain the following formula

$$B_\nu = (-1)^{|\nu|}(1+\beta)_{|\nu|}\prod_{j=1}^{d}\frac{\nu_j!(\beta+r+2|\nu^{j+1}|+\nu_j+|L_{j-1}|)_{l_j+1}}{(-l_j)_{\nu_j}(\beta+r+2|\nu^j|+|L_{j-1}|)(|L_{j-1}|+r+|\nu^j|)_{l_j-\nu_j}},$$

which combined with

$$\prod_{j=1}^{d}\bigl(|L_{j-1}|+r+\bigl|\nu^j\bigr|\bigr)_{l_j-\nu_j} = \bigl(r+|\nu|\bigr)_{|l|-|\nu|},$$

gives (5.27). □

5.2. *Case 2: $b+r=N\in\mathbb{N}_0$*

Again we have two possibilities:

5.2.1. For every $i\in\{1,2,\dots,d\}$, $l_i+1\notin\{1,2,\dots,N\}$. In this case the numerator in (5.14) is zero when $|x|=N+1$ and we have $V=V_N^d$. The corresponding orthogonal polynomials are the Hahn polynomials studied in [4]. Setting $\sigma_i=-l_i-1$ for $1\leqslant i\leqslant d$, $\sigma=(\sigma_1,\dots,\sigma_{d+1})$,

$$r = N+|\sigma|+d+1 \quad\text{and}\quad b=-\bigl(|\sigma|+d+1\bigr), \tag{5.28}$$

Eq. (5.14) gives

$$\frac{W(x)}{W(x-e_i)} = \frac{x_i+\sigma_i}{x_i}\frac{N+1-|x|}{N+1+\sigma_{d+1}-|x|}.$$

The above ratio must be positive for all $x \in V_N^d$. In particular, for $x = e_i$ and $x = Ne_i$ we see that

$$\frac{\sigma_i+1}{N+\sigma_{d+1}} > 0 \quad \text{and} \quad \frac{\sigma_i+N}{1+\sigma_{d+1}} > 0 \quad \text{for } i = 1, 2, \ldots, d.$$

From this it follows easily that the parameters $\{\sigma_i\}_{i=1}^{d+1}$ satisfy one of the following conditions:

(i) $\sigma_i > -1$ for $i = 1, 2, \ldots, d+1$;
(ii) $\sigma_i < -N$ for $i = 1, 2, \ldots, d+1$.

The weight function takes the form

$$W(x) = \prod_{i=1}^{d} \binom{x_i+\sigma_i}{x_i}\binom{N-|x|+\sigma_{d+1}}{N-|x|} \tag{5.29}$$

with $\{\sigma_i\}$ as free parameters. If $\sigma_i > -1$ for $i = 1, 2, \ldots, d+1$ or if $\sigma_i < -N$ for $i = 1, 2, \ldots, d+1$ but N is even then $W(x) > 0$ on V_N^d. If $\sigma_i < -N$ for $i = 1, 2, \ldots, d+1$ and if N is odd then $W(x) < 0$ on V_N^d, so one needs to change the sign in formula (5.29), in order to get a positive function.

The difference operator takes the form

$$\begin{aligned} D = {} & \sum_{i=1}^{d} x_i\big(N - x_i + |\sigma| - \sigma_i + d\big)\Delta_i\nabla_i - \sum_{1\leqslant i\neq j\leqslant d} x_j(x_i+\sigma_i+1)\Delta_i\nabla_j \\ & + \sum_{i=1}^{d}\big[(N-x_i)(\sigma_i+1) - x_i\big(|\sigma| - \sigma_i + d\big)\big]\Delta_i \end{aligned}$$

and the eigenvalues are $\lambda_n = -n(n+|\sigma|+d)$.

Theorem 5.4. *For $\nu \in \mathbb{N}_0^d$ and $|\nu| \leqslant N$, the polynomials*

$$\begin{aligned} Q_\nu(x;\sigma,N) = {} & \frac{(-1)^{|\nu|}}{(-N)_{|\nu|}} \prod_{j=1}^{d} \frac{(\sigma_j+1)_{\nu_j}}{(a_j+1)_{\nu_j}} \big(-N + |X_{j-1}| + |\nu^{j+1}|\big)_{\nu_j} \\ & \times Q_{\nu_j}\big(x_j; \sigma_j, a_j, N - |X_{j-1}| - |\nu^{j+1}|\big), \end{aligned} \tag{5.30}$$

where $a_j = |\sigma^{j+1}| + 2|\nu^{j+1}| + d - j$, satisfy the difference equation

$$DQ_\nu = \lambda_{|\nu|} Q_\nu$$

and the orthogonal relation

$$\sum_{|x|\leqslant N} Q_\nu(x;\sigma,N)Q_\mu(x;\sigma,N)\prod_{i=1}^{d}\binom{x_i+\sigma_i}{x_i}\binom{N-|x|+\sigma_{d+1}}{N-|x|}=A_\nu\delta_{\nu,\mu}, \tag{5.31}$$

where A_ν *is given by*

$$A_\nu=\frac{(-1)^{|\nu|}(|\sigma|+d+2|\nu|+1)_{N-|\nu|}}{(-N)_{|\nu|}N!}\prod_{j=1}^{d}\frac{(\sigma_j+a_j+\nu_j+1)_{\nu_j}(\sigma_j+1)_{\nu_j}\nu_j!}{(a_j+1)_{\nu_j}}. \tag{5.32}$$

These formulas are essentially contained in [4]. They can be deduced from (4.7) as in the proof of Theorem 5.2. Explicit biorthogonal (not mutually orthogonal) Hahn polynomials were also found in [8].

5.2.2. There is a nonempty set $S\subset\{1,2,\ldots,d\}$ and $l_i+1\in\{1,2,\ldots,N\}$ for $i\in S$. In this case the set V is

$$V_{N,S}^d=V_N^d\cap\{x\colon\ x_i\leqslant l_i \text{ for } i\in S\}. \tag{5.33}$$

If $S=\{1,2,\ldots,d\}$ we can also assume that $l_1+\cdots+l_d>N$, otherwise it becomes the parallelepiped case discussed Theorem 5.3.

The weight $W(x)$ is again given by (5.29), but V_N^d is replaced by $V_{N,S}^d$. The corresponding polynomials are the same as in Theorem 5.4, with the restriction $\nu\in V_{N,S}^d$.

Theorem 5.5. *For* $\nu\in V_{N,S}^d$ *the polynomials* $Q_\nu(x;\sigma,N)$ *defined by* (5.30) *satisfy the difference equation*

$$DQ_\nu=\lambda_{|\nu|}Q_\nu$$

and the orthogonal relation

$$\sum_{x\in V} Q_\nu(x;\sigma,N)Q_\mu(x;\sigma,N)\prod_{i=1}^{d}\binom{x_i+\sigma_i}{x_i}\binom{N-|x|+\sigma_{d+1}}{N-|x|}=A_\nu\delta_{\nu,\mu},$$

where A_ν *is given by* (5.32).

Remark 5.6. Notice that (5.25) in the parallelepiped case V_l^d gives a polynomial of total degree $|\nu|$. We can use the generators G_l^d in Example 3.1 to write the same polynomial using only the monomials x^μ with $\mu\in V_l^d$. Similarly, in the case of Theorem 5.5 we can use the generators $\{(-x_i)_{l_i+1}\colon\ i\in S\}$ to express Q_ν in terms of the monomials x^μ with $\mu\in V_{N,S}^d$.

Remark 5.7. In Theorems 5.3–5.5 we have $r_k=|\Lambda_k(V)|<\dim(\Pi_k^d/\Pi_{k-1}^d)$ for some k's and therefore the equation $Du(x)=\lambda u(x)$ will hold a priori only if we consider $u(x)$ as an element of $\mathbb{R}[V]$. However, one can show that the corresponding polynomials can be obtained as a limit from the polynomials in Theorem 5.2. This fact can be used to show that the equation $Du(x) = \lambda u(x)$ actually holds in the space $\mathbb{R}[x_1, x_2, \dots, x_d]$.

5.3. Summary

If $a = -1$ we have essentially one difference equation with coefficients given in (5.2). By specifying the free parameters, however, we end up with four different types of solutions given in Theorems 5.2–5.5, respectively.

As an example, let us consider the case $d = 2$. The weight functions and the corresponding sets on which they live are listed below:

$$\text{(i)}\quad W(x) = \frac{(\sigma_1+1)_{x_1}(\sigma_2+1)_{x_2}}{x_1!x_2!}\frac{(\beta+1)_{|x|}}{(-|\sigma|-d-r+1)_{|x|}},\quad V = \mathbb{N}_0^2;$$

$$\text{(ii)}\quad W(x) = \frac{(-l_1)_{x_1}(-l_2)_{x_2}}{x_1!x_2!}\frac{(\beta+1)_{|x|}}{(-l-r)_{|x|}},\quad V = V_l^2;$$

$$\text{(iii)}\quad W(x) = \frac{(\sigma_1+1)_{x_1}(\sigma_2+1)_{x_2}}{x_1!x_2!}\frac{(\sigma_3+1)_{N-|x|}}{(N-|x|)!},\quad V = V_N^2;$$

$$\text{(iv)}\quad W(x) = \frac{(-l_1)_{x_1}}{x_1!}\frac{(-l_2)_{x_2}}{x_2!}\frac{(\sigma_3+1)_{N-|x|}}{(N-|x|)!},\quad V = V_{N,S}^2,\ \text{where } S = \{1,2\};$$

$$\text{(v)}\quad W(x) = \frac{(-l_1)_{x_1}}{x_1!}\frac{(\sigma_2+1)_{x_2}}{x_2!}\frac{(\sigma_3+1)_{N-|x|}}{(N-|x|)!},\quad V = V_{N,S}^2,\ \text{where } S = \{1\}.$$

In the last case, one can also exchange x_1 and x_2 to get another case. It should be mentioned that Eq. (5.2) was considered in [12], but only the case (iii) was identified there.

6. Multivariable case with linear eigenvalue: $a = 0$ and $b = -1$

When $a = 0$ and $b = -1$, Eqs. (3.15)–(3.18) become

$$\alpha_{i,j} = -l_{i,j}x_j \quad \text{for } i \neq j, \tag{6.1a}$$

$$B_i = -x_i + s_i, \tag{6.1b}$$

$$\beta_i = \sum_{k\neq i} l_{i,k}x_k + (l_{i,i}-1)x_i + s_i, \tag{6.1c}$$

$$\gamma_i = x_i\sum_{k=1}^{d} l_{k,i}. \tag{6.1d}$$

First, notice that we cannot have $\gamma_i = 0$. Indeed, if we assume that $\gamma_i = 0$ then in order to have a self-adjoint operator we need $\beta_i = 0$, which implies that $l_{i,k} = 0$ for $k \neq i$. But then $\alpha_{i,k} = 0$ for $k \neq i$ and therefore $\alpha_{k,i} = 0$ for $k \neq i$, which simply means that the operator D is independent of x_i and $E_i^{\pm 1}$. Thus we have an operator acting in a $(d-1)$-dimensional space, trivially extended, by adding x_i as explained in Remark 3.6.

Below we assume that $\gamma_i \neq 0$ for all i. Notice that γ_i depends only on x_i, so in (2.12b) we can cancel $\gamma_i\gamma_j$ and we get

$$\beta_i(x-e_i)\beta_j(x-e_i-e_j) = \beta_j(x-e_j)\beta_i(x-e_i-e_j). \tag{6.2}$$

But $\beta_i(x-e_i)=\beta_i(x)+1-l_{i,i}$ and $\beta_i(x-e_i-e_j)=\beta_i(x)+1-l_{i,i}-l_{i,j}$ and plugging these in (6.2) we get

$$l_{j,i}\big(\beta_i(x)+1-l_{i,i}\big)=l_{i,j}\big(\beta_j(x)+1-l_{j,j}\big), \tag{6.3}$$

for $i\neq j$. Comparing the coefficients of x_k in the last formula we see that

$$l_{j,i}l_{i,k}=l_{i,j}l_{j,k} \quad \text{for } k\neq i,\, j, \tag{6.4}$$

$$l_{j,i}l_{i,j}=l_{i,j}(l_{j,j}-1). \tag{6.5}$$

These two equations have essentially two different types of solutions, which we discuss in two subsections.

6.1. *Case 1:* $l_{i,j}\neq 0$ *for all* $1\leqslant i\neq j\leqslant d$

Proposition 6.1. *Assume that* $l_{i,j}\neq 0$ *for all* $1\leqslant i\neq j\leqslant d$. *Then the most general solution of the compatibility conditions* (2.12) *is given by*

$$\alpha_{i,j}=-l_i x_j \quad \textit{for } i\neq j, \tag{6.6a}$$

$$B_i=-x_i+l_i s, \tag{6.6b}$$

$$\beta_i=l_i\left(\sum_{k=1}^{d}x_k+s\right), \tag{6.6c}$$

$$\gamma_i=x_i\left(\sum_{k=1}^{d}l_k+1\right), \tag{6.6d}$$

where s *and* $\{l_i\}_{i=1}^d$ *are free parameters.*

Proof. From Eq. (6.5) it follows that $l_{j,i}=l_{j,j}-1$. Denote $l_j:=l_{j,j}-1$. Then we have $l_{j,i}=l_j$ for all $i\neq j$. Using now (6.3) we see that $l_j s_i=l_i s_j$. Thus $s_i=l_i s$, which leads to formulas (6.6).

Conversely, it is straightforward to see that if we define $\alpha_{i,j}$, β_i and γ_i as in (6.6a), (6.6c) and (6.6d), then the compatibility conditions in Corollary 2.3 are satisfied, i.e. the above formulas give the most general solution in the case $l_{i,j}\neq 0$. □

Below we determine the weight functions and the corresponding orthogonal polynomials. For every i we have

$$\frac{W(x)}{W(x-e_i)}=\frac{l_i}{|l|+1}\frac{|x|+s-1}{x_i}$$

and therefore

$$\frac{W(e_i)}{W(0)}=\frac{l_i s}{|l|+1}>0,$$

which shows that l_i must have the same signs. There are two possible cases.

6.1.1. $l_i = -p_i < 0$ for all i. In this case,

$$\frac{W(x)}{W(x-e_i)} = \frac{p_i}{1-|p|}\frac{1-s-|x|}{x_i}.$$

If the denominator does not vanish, then for $|x|$ large the second ratio will be negative and therefore we must have $1-|p|<0$, i.e. $|p|>1$. But then if we denote $c_i = p_i/(|p|-1)$ we will have $|c|>1$ and up to a constant factor the weight is $W(x)=(s)_{|x|}\prod_{i=1}^{d} c_i^{x_i}/x_i!$, which leads to divergent series.

Hence the only possibility here is $(-s)=N\in\mathbb{N}_0$. This forces $|p|<1$ and $V=V_N^d$. The difference operator is then

$$D = \sum_{1\leqslant i,j\leqslant d}(\delta_{i,j}-p_i)x_j\Delta_i\nabla_j + \sum_{i=1}^{d}(p_iN-x_i)\Delta_i, \tag{6.7}$$

and the eigenvalues are $\lambda_n=-n$. The orthogonal polynomials are the Krawtchouk polynomials on V_N^d. Recall that Krawtchouk polynomial in one variable is denoted by $K_n(x;p,N)$.

Theorem 6.2. *Let $0<p_i<1$, $1\leqslant i\leqslant d$, and $|p|<1$. For $\nu\in\mathbb{N}_0^d$, $|\nu|\leqslant N$, the polynomials*

$$\begin{aligned} K_\nu(x;p,N) = {}& \frac{(-1)^{|\nu|}}{(-N)_{|\nu|}}\prod_{j=1}^{d}\frac{p_j^{\nu_j}}{(1-p_1-\cdots-p_j)^{\nu_j}}\left(-N+|X_{j-1}|+\left|\nu^{j+1}\right|\right)_{\nu_j} \\ &\times K_{\nu_j}\left(x_j;\frac{p_j}{1-p_1-\cdots-p_{j-1}},N-|X_{j-1}|-\left|\nu^{j+1}\right|\right) \end{aligned} \tag{6.8}$$

satisfy the difference equation

$$D\psi_\nu = \lambda_{|\nu|}\psi_\nu$$

and the orthogonal relation

$$\begin{aligned} &\sum_{|x|\leqslant N} K_\nu(x;p,N)K_\mu(x;p,N)\prod_{i=1}^{d+1}\frac{p_i^{x_i}}{x_i!} \\ &\quad = \frac{(-1)^{|\nu|}}{(-N)_{|\nu|}N!}\prod_{j=1}^{d}\frac{\nu_j!\,p_j^{\nu_j}}{(1-p_1-\cdots-p_j)^{\nu_j-\nu_{j+1}}}\delta_{\nu,\mu}, \end{aligned} \tag{6.9}$$

where $x_{d+1}=N-|x|$, $p_{d+1}=1-|p|$ and $\nu_{d+1}=0$.

In fact, these orthogonal polynomials can be considered as a limit of the Hahn polynomials (5.30) (see [9,10]). Indeed, using the well known relation

$$\lim_{t\to\infty} Q_n\big(x;pt,(1-p)t,N\big) = K_n(x;p,N)$$

in one variable, it is not hard to see that

$$\lim_{t\to\infty} Q_\nu\big(x; p_1 t, \ldots, p_d t, (1-p_1-\cdots-p_d)t, N\big) = K_\nu(x; p, N).$$

The orthogonality of $K_\nu(x; p, N)$ follows from (5.31) under the limit. We refer the reader to [5] for other properties and applications of Krawtchouk polynomials.

6.1.2. $l_i > 0$ *for all* i. In this case $s > 0$. Denote $c_i = l_i/(1+|l|)$. Then $c_i < 1$ and $|c| < 1$, the weight function is

$$W(x) = \frac{(s)_{|x|}}{x!} c^x = (s)_{|x|} \prod_{i=1}^{d} \frac{c_i^{x_i}}{x_i!}$$

and $V = \mathbb{N}_0^d$. The difference operator takes the form

$$D = \sum_{1\leqslant i,j\leqslant d} \left(\delta_{i,j} + \frac{c_i}{1-|c|}\right) x_j \Delta_i \nabla_j + \sum_{i=1}^{d} \left(-x_i + \frac{c_i}{1-|c|} s\right) \Delta_i. \tag{6.10}$$

The orthogonal polynomials are the Meixner polynomials on $\mathbb{N}_0^d$ but they are different from product Meixner polynomials. Recall that Meixner polynomial in one variable is denoted by $M_n(x; \beta, c)$. We write $|C_j| = c_j + c_{j+1} + \cdots + c_d$ and define $C_{d+1} = 0$.

Theorem 6.3. *Let* $0 < c_i < 1$, $1 \leqslant i \leqslant d$, *and* $|c| < 1$. *For* $\nu \in \mathbb{N}_0^d$, *the polynomials*

$$M_\nu(x; s, c) = \prod_{j=1}^{d} (\delta_j)_{\nu_j} M_{\nu_j}\left(x_j; \delta_j, \frac{c_j}{1-|C_{j+1}|}\right) \tag{6.11}$$

where $\delta_j = s + |\nu^{j+1}| + |X_{j-1}|$, *satisfy the difference equation*

$$D\psi_\nu = \lambda_{|\nu|}\psi_\nu$$

and the orthogonal relation

$$\sum_{x\in\mathbb{N}_0^d} M_\nu(x; s, c) M_\mu(x; s, c) (s)_{|x|} \prod_{i=1}^{d} \frac{c_i^{x_i}}{x_i!} = \frac{(s)_{|\nu|}}{(1-|c|)^s} \prod_{j=1}^{d} \nu_j! \left(\frac{c_j}{1-|C_{j+1}|}\right)^{-\nu_j} \delta_{\nu,\mu}. \tag{6.12}$$

Proof. These relations can be derived inductively using the orthogonality (4.26) of the Meixner polynomials of one variable. It is clear from (4.24) that M_ν defined by (6.11) are indeed polynomials of x. We can write the weight function as a product

$$W(x) = \frac{c_d^{x_d} (s + |X_{d-1}|)_{x_d}}{x_d!} W'(X_{d-1})$$

where

$$W'(X_{d-1}) = (s)_{|X_{d-1}|} \prod_{i=1}^{d-1} \frac{c_i^{x_i}}{x_i!}.$$

Using (4.26) the sum over x_d in (6.12) becomes

$$\begin{aligned} I_{\nu_d,\mu_d} &= \big(s+|X_{d-1}|\big)_{\nu_d}\big(s+|X_{d-1}|\big)_{\mu_d} \sum_{x_d=0}^{\infty} M_{\nu_d}(x_d) M_{\mu_d}(x_d) \frac{c_d^{x_d}(s+|X_{d-1}|)_{x_d}}{x_d!} \\ &= \delta_{\nu_d,\mu_d} \frac{c_d^{-\nu_d}\nu_d!}{(1-c_d)^s} \frac{(s+|X_{d-1}|)_{\nu_d}}{(1-c_d)^{|X_{d-1}|}}, \end{aligned}$$

where $M_m(x_d) = M_m(x_d, s+|X_{d-1}|, c_d)$.

Combining this with W' we obtain the new weight function

$$W_{d-1}(X_{d-1}) = \frac{c_d^{-\nu_d}\nu_d!(s)_{\nu_d}}{(1-c_d)^s}(s+\nu_d)_{|X_{d-1}|} \prod_{i=1}^{d-1} \frac{1}{x_i!}\left(\frac{c_i}{1-c_d}\right)^{x_i}.$$

Apart from a constant multiple, the weight function W_{d-1} has exactly the same structure of $W(x)$ with s replaced by $s+\nu_d$, and c_i replaced by $c_i/(1-c_d)$ for $i = 1, 2, \ldots, d-1$. The proof now follows by induction. The constant $\langle M_\nu, M_\nu\rangle$ is given by

$$\prod_{j=1}^{d} \frac{(\frac{c_j}{1-|C_{j+1}|})^{-\nu_j}\nu_j!(s+|\nu^{j+1}|)_{\nu_j}}{(1-\frac{c_j}{1-|C_{j+1}|})^s},$$

which leads to (6.12). □

Remark 6.4. It is well known that the Meixner polynomials are the limit of the Hahn polynomials,

$$\lim_{N\to\infty} Q_n\left(x; b-1, N\frac{1-c}{c}, N\right) = M_n(x; b, c).$$

There is an analogous relation between the Meixner polynomials (6.11) of several variables and the orthogonal polynomials on the parallelepiped in (5.25), at least when parameters c have rational values. Indeed, let $\phi_\nu(x; \beta, r, l)$ denote the polynomials defined in (5.25), which are orthogonal with respect to the weight function in (5.24). In these polynomials we set $\beta = s-1$ and

$$|L_{j-1}| + r = \left(\frac{1-|C_j|}{c_j}\right) l_j, \quad 1 \leqslant j \leqslant d.$$

Using induction, it is easy to see that $l_j/c_j = l_{j-1}/c_{j-1}$, which then implies that

$$l_j = \big(|l|+r\big)c_j, \quad \text{or} \quad l_j = \frac{c_j}{1-|c|} r, \quad 1 \leqslant j \leqslant d.$$

If c_j are rational numbers, we can choose $r = (1 - |c|)N$, $N \in \mathbb{N}_0$, for certain N, so that l_j are integers. Upon taking $r \to \infty$, which shows $l_j \to \infty$, it follows from (4.26) that

$$\lim_{l\to\infty} \phi_\nu(x; s-1, r, l) = M_\nu(x; s, c). \tag{6.13}$$

6.2. *Case 2: $l_{i,j} = 0$ for some $1 \leqslant i \neq j \leqslant d$*

Recall that $\alpha_{i,j} = 0$ for $i \neq j$ if and only if $\alpha_{j,i} = 0$. This essentially means that $l_{i,j}$ and $l_{j,i}$ are simultaneously zero or nonzero for $i \neq j$.

Lemma 6.5. *If $j \neq k$ and $l_{j,k} = 0$ then for every $i \neq j, k$ we have $l_{i,j} = l_{j,i} = 0$ or $l_{i,k} = l_{k,i} = 0$.*

Proof. Follows immediately from (6.4). □

Theorem 6.6. *Assume that $l_{i,j} = 0$ for some $1 \leqslant i \neq j \leqslant d$. Define $I = \{i\} \cup \{m\colon l_{m,i} \neq 0\}$ and $J = \{1, 2, \dots, d\} \setminus I \supset \{j\}$. Then $D = D_I + D_J$, where D_I is an admissible operator in the variables $\{x_m\colon m \in I\}$ and D_J is an admissible operator in $\{x_k\colon k \in J\}$.*

Proof. First we show that for $m \in I$ and $k \in J$ we have $l_{m,k} = l_{k,m} = 0$. Indeed if $m = i$, then $l_{k,i} = 0$ by the definition of J. If $m \neq i$ then $l_{m,i} \neq 0$, but $l_{k,i} = 0$. The previous lemma shows that $l_{k,m} = l_{m,k} = 0$.

Thus if $m \in I$ and $k \in J$ we see at once that:

- $\alpha_{m,k} = \alpha_{k,m} = 0$;
- β_m contains only the variables from I;
- β_k contains only the variables from J.

The decomposition of D follows immediately from the above observations. □

Remark 6.7. Theorem 6.6 says that if $l_{i,j} = 0$ for some $i \neq j$, then the operator D splits as a sum of 2 operators of independent variables. Conversely let D_I be an admissible operator in the variables $x' = \{x_m\colon m \in I\}$ and D_J be an admissible operator in $x'' = \{x_k\colon k \in J\}$ with $I \cap J = \emptyset$ and $I \cup J = \{1, 2, \dots, d\}$. Let us denote by $p^I_\mu(x')$ and $p^J_\nu(x'')$ the polynomials satisfying

$$D_I\big(p^I_\mu\big) = -|\mu| p^I_\mu \quad \text{and} \quad D_J\big(p^J_\nu\big) = -|\nu| p^J_\nu.$$

Then if put $D = D_I + D_J$ and $p_{\mu,\nu}(x) = p^I_\mu(x')p^J_\nu(x'')$, where $x = (x', x'') \in \mathbb{R}^d$, we have

$$Dp_{\mu,\nu}(x) = -\big(|\mu| + |\nu|\big) p_{\mu,\nu}(x).$$

Thus, in this case, the eigenfunctions of the difference operator are product of orthogonal polynomials of fewer variables, which satisfy difference equations of lower dimension with linear eigenvalues. For $d = 1$ these are the polynomials of Charlier, Krawtchouk and Meixner given in Section 4.2. In higher dimensions, there are also the Krawtchouk polynomials of several variables given in Theorem 6.2 and the Meixner polynomials of several variables given in Theorem 6.3.

Clearly there are many product polynomials of this type and the number increases drastically as the dimension grows. As an example, we list all cases for $d = 3$ below. To list the different types we use the abbreviation of C, K, M for Charlier, Meixner, and Krawtchouk polynomial of one variable, respectively, and use K_2 and M_2 to denote the Krawtchouk polynomials of two variables on V_N^2 and Meixner polynomials of two variables on $\mathbb{N}_0^2$. A product polynomial is denoted by its components. As an example, CCM stands for a product of the type Charlier–Charlier–Meixner.

Example 6.8 *($d = 3$).* There are sixteen product types, which we list according to their domains of orthogonality:

(1) $\mathbb{N}_0^3$: CCC, MMM, CCM, CMM, CM_2, MM_2.
(2) $\mathbb{N}_0^2 \times V_N^1$: CCK, MMK, CMK, KM_2.
(3) $\mathbb{N}_0 \times V_{N_1}^1 \times V_{N_2}^1$: CKK, MKK.
(4) $V_{N_1}^1 \times V_{N_2}^1 \times V_{N_3}^1$: KKK.
(5) $V_N^2 \times \mathbb{N}_0$: $\mathrm{K}_2\mathrm{C}$, $\mathrm{K}_2\mathrm{M}$.
(6) $V_{N_1}^2 \times V_{N_2}^1$: $\mathrm{K}_2\mathrm{K}$.

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