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A METHOD IN GRAPH THEORY

J.A. BONDY

Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Ont., Canada

V. CHVÁTAL[1]

Centre de recherches mathématiques, Université de Montréal, Montréal, P.Q., Canada

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A unified approach to a variety of graph-theoretic problems is introduced. The k-closure $C_k(G)$ of a simple graph G of order n is the graph obtained from G by recursively joining pairs of nonadjacent vertices with degree-sum at least k. It is shown that, for many properties P, one can find a suitable value of k (depending on P and n) such that if $C_k(G)$ has P, then so does G. For instance, if P is the hamiltonian property, one may take $k = n$. Thus if $C_n(G)$ is hamiltonian, then so is G; in particular, if $n \geqslant 3$ and $C_n(G)$ is complete, then G is hamiltonian. This condition for a graph to be hamiltonian is shown to imply the well-known conditions of Chvátal and Las Vergnas. The same method, applied to other properties, yields many new theorems of a similar nature.

1. Introduction

In this paper, we present a unified approach to a variety of graph-theoretical problems. It has been inspired by the following theorem of Ore [14]: *Let G be a graph of order n, $n \geqslant 3$. If*

(1.1) $\qquad d(u) + d(v) \geqslant n$

for each pair of nonadjacent vertices, then G has a hamiltonian cycle.

Actually, Ore proved a little more. Indeed, if u and v are nonadjacent vertices satisfying (1.1) and $G + uv$ has a hamiltonian cycle, then G, too, has one. This simple idea has surprisingly far-reaching implications.

First of all, one can iterate the operation of replacing G by $G + uv$ as

[1] Present address: Department of Computer Science, Stanford University, Stanford, CA, U.S.A.

long as possible; if the resulting graph G_0 is hamiltonian then G itself is hamiltonian. In particular, if G_0 is complete then G is hamiltonian. This condition compares favourably with a number of existing sufficient conditions for a graph to be hamiltonian. Indeed, it can be shown that all the descendants of Dirac's condition [7], including those of Chvátal [5] and Las Vergnas [12], guarantee that G_0 is complete. Moreover, we have an efficient way of finding the hamiltonian cycle (whereas the original proofs in [5] and [12] do not provide any). In fact, one can find G_0 within $O(n^4)$ steps and, given a hamiltonian cycle in G_0, find one in G within an additional $O(n^3)$ steps.

Secondly, this technique applies also in many other settings. In each of these, we obtain a sufficient condition accompanied by an efficient algorithm to check it. A routine application of our Theorems 3.1 and 3.2 then yields the appropriate sufficiency theorems of Chvátal and Las Vergnas type. Some of these are already known and others are new. We present them to illustrate the versatility of our approach rather than for their own sake.

Our notation and terminology follows Berge [1] and Harary [9]. The independence (stability) number $\alpha(G)$ is as in [1]; to avoid confusion, we use "independent sets" rather than "stable sets". The graphs C_s, P_s, sK_2, $K_i + (iK_1 \cup K_{n-2i})$ etc. are as in [9]. As in [8], the smallest number of pairwise disjoint paths covering all the vertices of G is denoted by $\mu(G)$. (This invariant has also been studied by Boesch et al. [2].)

2. Stability and closure

Let P be a property defined on all graphs of order n; let k be a non-negative integer. Then P is said to be *k-stable* if whenever $G + uv$ has property P and $d_G(u) + d_G(v) \geq k$ then G itself has property P. Ore [14] proved that

(2.1) the property of containing a hamiltonian cycle is n-stable .

The proof is simple: If $G + uv$ is hamiltonian but G is not, then G has a hamiltonian path $u = u_1, u_2, \ldots, u_n = v$. If $d_G(u) + d_G(v) \geq n$, there must be some i such that u is adjacent to u_{i+1} and v is adjacent to u_i. But then G has the hamiltonian cycle $u_1 u_{i+1} u_{i+2} \ldots u_n u_i u_{i-1} \ldots u_1$.

Along similar lines, one can verify the following (for detailed proofs, see Appendix 1.)

(2.2)	"G contains a C_s $(5 \leqslant s \leqslant n)$"	... $(2n - s)$-stable,
(2.3)	"G contains a C_s $(4 \leqslant s < n,\ s$ even)"	... $(2n - s - 1)$-stable,
(2.4)	"G contains a P_s $(4 \leqslant s \leqslant n)$"	... $(n - 1)$-stable,
(2.5)	"G contains a $K_{2,s}$ $(2 \leqslant s \leqslant n - 2)$"	... $(n + s - 2)$-stable,
(2.6)	"G contains an sK_2 $(2s \leqslant n)$"	... $(2s - 1)$-stable,
(2.7)	"G contains an s-factor $(2 \leqslant s < n)$"	... $(n + 2s - 4)$-stable,
(2.8)	"G is s-connected"	... $(n + s - 2)$-stable,
(2.9)	"G is s-edge-connected"	... $(n + s - 2)$-stable,
(2.10)	"$\alpha(G) \leqslant s$"	... $(2n - 2s - 1)$-stable.

The concept of stability is also relevant to the plethora of variations on the hamiltonian theme: "s-hamiltonian" [4], "s edge-hamiltonian" [11], "s-Hamiltonian-connected" [1], etc.

(2.11)	"s-hamiltonian"	... $(n + s)$-stable,
(2.12)	"s-edge-hamiltonian"	... $(n + s)$-stable,
(2.13)	"s-Hamilton-connected"	... $(n + s + 1)$-stable,
(2.14)	"$\mu(G) \leqslant s$"	... $(n - s)$-stable.

Now, let G be a graph of order n and let k be a nonnegative integer. Among all the graphs H of order n such that $G \subset H$ and

(2.15) $\quad d_H(u) + d_H(v) < k$

for all $uv \notin E(H)$, there is a unique smallest one (for, if H_1 and H_2 have the above properties then so, too, does $H_1 \cap H_2$). We shall call this graph the *k-closure* of G and denote it by $C_k(G)$.

Obviously, $C_k(G)$ can be obtained from G by a recursive procedure which consists of joining nonadjacent vertices with degree-sum at least k. Thus we have the following.

Proposition 2.1. *If P is k-stable and $C_k(G)$ has property P then G itself has property P.*

3. Two sufficiency theorems

We note that, for $n \geqslant n(s)$, the complete graph of order n has all of the properties (2.1)–(2.14) mentioned above (with the exception of having an s-factor when ns is odd). Hence, in the light of Proposition 2.1, it is of interest to know when $C_k(G)$ is complete. More generally, if a graph G has t vertices of degree $n-1$, then G is t-connected, $\alpha(G) \leqslant \max(1, n-t)$, and so on. Therefore it is also desirable to know when $C_k(G)$ has at least t vertices of degree $n-1$. (If $t = n-1$, this question reduces to the previous one.) In this section, we present two related sufficiency theorems. They are motivated by results of Bondy [3], Chvátal [5] and Las Vergnas [12], on connectivity and hamiltonian cycles.

Theorem 3.1. *Let k, n, t be positive integers with $k \leqslant 2n-4$ and $t < n$. Let G be a graph with degree sequence $d_1 \leqslant d_2 \leqslant \ldots \leqslant d_n$. Let there be no nonnegative integer i with*

$$\begin{aligned} &k-n < i < \tfrac{1}{2}k, \\ (3.1)\qquad &d_{n-k+i} \leqslant i, \\ &d_{n-i} \leqslant k-i-1, \\ &d_{n-t+1} \leqslant k-i-1. \end{aligned}$$

Then $C_k(G)$ has at least t vertices of degree $n-1$.

Proof. We may assume $d_{n-t+1} \leqslant n-2$ (otherwise G has t vertices of degree $n-1$ and we are done). Next we may assume that

$$(3.2)\qquad d_1 > k-n.$$

Indeed, this is trivial as long as $k-n<0$. On the other hand, if $k-n \geqslant 0$ and $d_1 \leqslant k-n$ then G has at most $k-n$ vertices of degree $n-1$, so that $d_{2n-k-1} \leqslant n-2$ and (3.1) is satisfied by $i = k-n+1$.

Now, let H denote $C_k(G)$. We may assume that H is not complete (otherwise we are done). Among all the vertices of H that have degree smaller than $n-1$, choose one with largest degree and denote it by v. Among all the vertices nonadjacent to v, choose one with largest degree and denote it by u. Then (2.15) implies

$$(3.3)\qquad d_H(u) + d_H(v) \leqslant k-1.$$

Let $i = d_H(u)$. By the choice of u and v, we have $d_H(u) \leqslant d_H(v)$ and so, by (3.3), $i < \frac{1}{2}k$.

Since $d_1 \leqslant d_G(u) \leqslant d_H(u)$, (3.2) yields $i > k - n$. Besides, (3.3) implies that v is nonadjacent to at least $n - k + i$ vertices in H. By the choice of u, each of these has degree (in H) at most i. Since $d_G(w) \leqslant d_H(w)$ for every w, we conclude that $d_{n-k+i} \leqslant i$. Moreover, u is nonadjacent to exactly $n - 1 - i$ vertices in H. By (3.3) and the choice of v, each of these has degree at most $k - 1 - i$. As $d_H(u) = i \leqslant k - 1 - i$, there are at least $n - i$ vertices of H that have degree at most $k - 1 - i$. Thus we have $d_{n-i} \leqslant k - 1 - i$.

Now, to keep the hypothesis of our theorem satisfied, we must have $d_{n-t+1} \geqslant k - i$. By (3.3), the last inequality implies $d_{n-t+1} > d_H(v)$. Hence there are at least t vertices of H that have degree greater than $d_H(v)$. By the choice of v, each of them has degree $n - 1$ and the proof is finished.

Remark. If $k \leqslant 2t - 1$ or $k = n + t - 2$, then Theorem 3.1 gives the weakest monotone condition in terms of d_j's which ensures that $C_k(G)$ has at least t vertices of degree $n - 1$. More precisely, if (3.1) is satisfied for some i then there is a graph G^* with degree sequence $d_1^* \leqslant d_2^* \leqslant \ldots \leqslant d_n^*$ such that $d_j^* \geqslant d_j$ for each j, but $C_k(G^*)$ has fewer than t vertices of degree $n - 1$. Indeed, if $i \leqslant t - 1$ then we can set $G^* = K_i + (K_{k-2i} \cup (n - k + i) K_1)$ so that

$$d_j^* = \begin{cases} i, & 1 \leqslant j \leqslant n - k + i, \\ k - i - 1, & n - k + i < j \leqslant n - i, \\ n - 1, & n - i < j \leqslant n, \end{cases}$$

and $C_k(G^*) = G^*$. On the other hand, if $t \leqslant i < \frac{1}{2}k$ and $k = n + t - 2$ then we set $G^* = K_{t-1} + (K_{n-k+i} \cup K_{k-t-i+1})$ so that

$$d_j^* = \begin{cases} i, & 1 \leqslant j \leqslant n - k + i, \\ k - i - 1, & n - k + i < j \leqslant n - t + 1, \\ n - 1, & n - t + 1 < j \leqslant n. \end{cases}$$

and again $C_k(G^*) = G^*$. Outside this range of k, Theorem 3.1 ceases to be best possible in the above sense. For instance, one can easily verify that the 9-closure of a graph with a degree sequence $(d_1, d_2, \ldots, d_8)$ such that $d_i \geqslant 4$ $(1 \leqslant i \leqslant 7)$, $d_8 \geqslant 6$ must contain at least two vertices of degree seven. Moreover, this does not follow from Theorem 3.1.

Theorem 3.2. *Let k, n, t be positive integers with $k \leqslant 2n - 4$ and $t < n$. Let G be a graph with vertices $u_1, u_2, \ldots, u_n$. Let there be no i, j with*

$$\begin{array}{ll} i < j, & u_i \text{ not adjacent to } u_j, \\ d(u_i) \leqslant i + k - n, & d(u_j) \leqslant j + k - n - 1, \\ d(u_i) + d(u_j) \leqslant k - 1, & i + j \geqslant 2n - k, \\ j \geqslant n - t + 1. & \end{array}$$

Then $C_k(G)$ has at least t vertices of degree $n - 1$.

Proof. Let H denote $C_k(G)$. Suppose H contains at most $t - 1$ vertices of degree $n - 1$; as $t < n$, H is certainly not complete. Choose nonadjacent u_i, u_j with

(i) j as large as possible, and

(ii) i as large as possible subject to (i).

By (2.15), we have

(3.4) $\quad d_H(u_i) + d_H(u_j) \leqslant k - 1$.

By (i), u_j must be adjacent (in H) to all the u_s with $i < s \leqslant n$ and $s \neq j$; therefore

(3.5) $\quad d_H(u_j) \geqslant n - i - 1$.

By (ii), u_i must be adjacent (in H) to all the u_s with $j < s \leqslant n$ and so

(3.6) $\quad d_H(u_i) \geqslant n - j$.

Now (3.4), (3.5) and (3.6) imply

$$d_G(u_i) \leqslant d_H(u_i) \leqslant (k - 1) - (n - i - 1) = i + k - n,$$

$$d_G(u_j) \leqslant d_H(u_j) \leqslant (k - 1) - (n - j) = j + k - n - 1,$$

$$i + j \geqslant (n - d_H(u_j) - 1) + (n - d_H(u_i)) \geqslant (2n - 1) - (k - 1) = 2n - k.$$

Finally, (i) implies $d(u_s) = n - 1$ for each s greater than j.

Therefore $n - j \leqslant t - 1$ contradicting the hypothesis.

Remark. If G satisfies the hypothesis of Theorem 3.1 and if its vertices are labeled so that $d(u_i) = d_i$ for each i, then G satisfies the hypothesis of Theorem 3.2. This has been proved, in a more specialized setting, by Las Vergnas [13]. We present the complete proof in Appendix 2.

4. Applications to hamiltonian cycles

Combining Proposition 2.1 with (2.1) we obtain the following.

Theorem 4.1. *If G is a graph of order n such that $n \geq 3$ and $C_n(G)$ is complete, then G is hamiltonian.*

We now describe an algorithm which finds $C_k(G)$ for any given value of k in $O(n^4)$ steps. The input of the algorithm is the adjacency matrix $A(i, j)$ of G; its output is the adjacency matrix of $C_k(G)$.

Step 1. For each i, set $D(i) = \Sigma A(i, j)$.
Set $M = 2$.
Step 2. Find a pair (i, j) with $i \neq j$, $A(i, j) = 0$ and $D(i) + D(j) \geq k$.
If there is none, stop.
Step 3. Replace $A(i, j)$ and $A(j, i)$ by M.
Replace $D(i)$ by $D(i) + 1$ and $D(j)$ by $D(j) + 1$.
Replace M by $M + 1$.
Go to Step 2.

Upon termination of this algorithm, we have a matrix $A(i, j)$ where $A(i, j) = 1$ iff $ij \in E(G)$ and $A(i, j) = 0$ iff $ij \notin E(C_k(G))$. Next, suppose we have a hamiltonian cycle C: $u_1 u_2 \ldots u_n u_1$ in $C_n(G)$. (Finding such a cycle is a trivial matter if $C_n(G)$ happens to be complete and $n \geq 3$.) Let m be the maximum of $A(i, j)$ taken over all n edges of C. Suppose $m > 1$; then there is a unique edge of C with $A(i, j) = m$. Without loss of generality, let this edge be $u_1 u_n$. We shall describe a way of finding a hamiltonian cycle C' in G such that the maximum of $A(i, j)$ over the edges of C' is less than m. The number of those u_i for which $0 < A(u_1, u_i) < m$ and the number of those u_j with $0 < A(u_n, u_j) < m$ sum to at least n. Hence there must be some u_s with $0 < A(u_1, u_{s+1}) < m$ and $0 < A(u_n, u_s) < m$. Such an s can be found by inspection within $O(n)$ steps; the desired hamiltonian cycle C' is

$$u_1 u_{s+1} u_{s+2} \ldots u_n u_s u_{s-1} \ldots u_1 .$$

This procedure may be repeated until a hamiltonian cycle in G is found. Since $A(i, j) \leq \binom{n}{2}$ for every i and j, the initial hamiltonian cycle C in $C_n(G)$ will be transformed into a hamiltonian cycle in G within $O(n^3)$ steps.

Therefore our Theorem 4.1 is quite useful from the operational point of view. Indeed, the first of the above algorithms decides within $O(n^4)$ steps whether the hypothesis of Theorem 4.1 is satisfied and, if this is the case, then the second procedure can be used to exhibit, within an additional $O(n^3)$ steps, a hamiltonian cycle in G.

Next, we shall compare our sufficiency condition (that is, $C_n(G) = K_n$) with various other sufficiency conditions for a graph to be hamiltonian. In particular, Theorem 4.1 compares rather favourably with a line of descendants of Dirac's theorem [7]. These include, in ascending order of strength, theorems of Ore [14], Pósa [15], Bondy [3], Chvátal [5] and Las Vergnas [12]. The last two theorems on this list are immediate corollaries of Theorem 4.1. Indeed, they can be deduced by combining Theorem 4.1 with Theorems 3.1 and 3.2.

Corollary 4.2 (Chvátal [5]). *Let G be a graph with degree sequence $d_1 \leqslant d_2 \leqslant \ldots \leqslant d_n$ such that $n \geqslant 3$ and*

$$d_i \leqslant i < \tfrac{1}{2}n \Rightarrow d_{n-i} \geqslant n - i .$$

Then G is hamiltonian.

Corollary 4.3 (Las Vergnas [12]). *Let G be a graph with vertices $u_1, u_2, \ldots, u_n$ where $n \geqslant 3$. Let there be no i, j with*

$$\begin{aligned} i &< j, & u_iu_j &\notin E(G), \\ d(u_i) &\leqslant i, & d(u_j) &\leqslant j - 1, \\ d(u_i) + d(u_j) &\leqslant n - 1, & i + j &\geqslant n . \end{aligned}$$

Then G is hamiltonian.

Hence our Theorem 4.1 applies whenever any of the descendants of Dirac's theorem does. Moreover, Theorem 4.1 is strictly stronger than all of these. Consider, for example, the graph G in Fig. 1.

We have $C_6(G) = K_6$ and so Theorem 4.1 guarantees that G is hamiltonian. However, there exists no labeling $u_1, u_2, \ldots, u_6$ of the vertices of G that satisfies the hypothesis of Las Vergnas' theorem.

At this point, one might be tempted to search for explicit conditions that are weaker than those of Las Vergnas but still imply that $C_n(G)$ is complete. We wish to stress the futility of such an undertaking. First of all, no such condition could possible cover a wider class of graphs than Theorem 4.1. Secondly, as shown above, it is easy ($O(n^4)$ steps) to decide

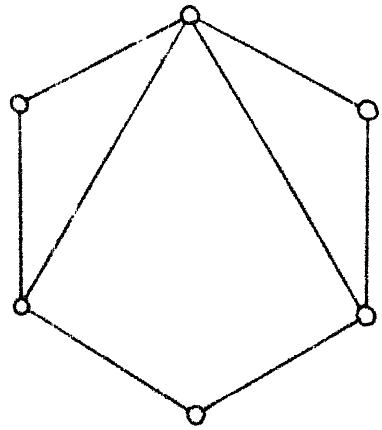

Fig. 1.

whether $C_n(G)$ is complete, whereas this may not be the case with a new "explicit" condition. The doubting reader is invited to try and apply Las Vergnas' theorem to an unlabeled graph of moderate size.

5. Other applications

It should be clear from the previous section that we have at our disposal the machinery to handle a wide variety of graph-theoretic problems. Given a k-stable property P, let $n(P)$ be an integer such that each K_n with $n \geqslant n(P)$ has property P, and let $t = t(P, n)$ be a function of n such that every graph of order n (where $n \geqslant n(P)$) with at least $t(P, n)$ vertices of degree $n - 1$ has property P.

Now, let G be a graph of order n where $n \geqslant n(P)$. The algorithm described in the previous section enables us to find, in $O(n^4)$ steps, the graph $C_k(G)$. If $C_k(G)$ happens to have at least t vertices of degree $n - 1$ then Proposition 2.1 guarantees that G has property P. In addition, Theorems 3.1 and 3.2 yield sufficiency conditions for a graph to have property P. Table 1 displays some of the products of this methodology. Its rows correspond to the properties introduced in Section 1 and its columns to Theorems 3.1 and 3.2. In all places we give the appropriate reference.

For instance, in the row corresponding to "$\mu(G) \leqslant s$" and the column corresponding to Theorem 3.1 we obtain the following.

Theorem 5.1. *Let n, s be positive integers; let G be a graph with degree*

Table 1

Property	k	$n(P)$	$t(P, n)$	Theorem 3.1	Theorem 3.2
Contains C_s	$2n - s$	s	$\left[\frac{s+1}{2}\right]$	$s = n$: Chvátal [5, Th. 1] $s < n$: new	$s = n$: Las Vergnas [12] $s < n$: new
Contains P_s	$n - 1$	s	$\left[\frac{s}{2}\right]$	$s = n$: Chvátal [5, Cor. 1.1] $s < n$: new	$s = n$: Las Vergnas [13, Ch. IV, Cor. 2 to Th. 1] $s < n$: new
Contains sK_2	$2s - 1$	$2s$	s	Las Vergnas [13, Ch. IV, §4, Cor. 2 to Prop. 3]	Las Vergnas [13, Ch. IV, §4. Prop. 3]
Contains s-factor ($s \geqslant 2$)	$n + 2s - 4$	$s + 1$	$\max\left(s, \left[\frac{n+1}{2}\right]\right)$	new	new
s-connected	$n + s - 2$	$s + 1$	s	Bondy [3, Th. 2]	Las Vergnas [13, Ch. IV, 4, Prop. 2]
s-hamiltonian	$n + s$	$s + 3$	$n - 1$	Chvátal [5, Cor. 1.2]	new
s-edge-hamiltonian	$n + s$	3	$n - 1$	Las Vergnas [13, Ch. IV, Cor. 1 to Th. 1]	Las Vergnas [12]
s-Hamilton-connected	$n + s + 1$	3	$n - 1$	$s = 0$: Berge [1, Ch. 10, Th. 12]	Las Vergnas [13, Ch. IV, Cor. 3 to Th. 1]
$\mu(G) \leqslant s$	$n - s$	1	$\left[\frac{n-s+1}{2}\right]$	new	new

sequence $d_1 \leqslant d_2 \leqslant \ldots \leqslant d_n$ *such that*

(5.1) $\quad d_{i+s} \leqslant i < \frac{1}{2}(n-s) \Rightarrow d_{n-i} \geqslant n-s-i\,.$

Then $\mu(G) \leqslant s$.

This theorem (as well as many others in our table) is best possible in the following sense. If (5.1) fails, then there is a graph G^* with degree sequence $d_1^* \leqslant d_2^* \leqslant \ldots \leqslant d_n^*$ such that $d_j^* \geqslant d_j$ for all j, but $\mu(G^*) > s$. Indeed, if (5.1) fails for some i then we may take $G^* = K_i + (K_{n-2i-s} \cup (i+s)K_1)$.

It may be helpful to note that, as long as $2t \geqslant k$, the situation is somewhat simplified. Indeed, $C_k(G)$ has at least t vertices of degree $n-1$ if and only if it is complete. Besides, sufficient conditions for $C_k(G)$ to be complete may be obtained from those of Theorem 3.1 (respectively Theorem 3.2) by deleting the inequality $d_{n-t+1} \leqslant k-i-1$ (respectively $j \geqslant n-t+1$).

It is sometimes of interest to know $C_k(G)$ for every value of k in the range $0 \leqslant k \leqslant 2n-3$. For instance, if $k \geqslant n-2$ and $C_k(G)$ has at least $k+2-n$ vertices of degree $n-1$ then G is $(k+2-n)$-connected. The following modification of our algorithm from Section 3 evaluates all the closures $C_k(G)$ simultaneously in $O(n^4)$ steps.

Step 1. For each i, set $D(i) = \Sigma\, A(i,j)$.
Set $k = 2n-3$ and $M = 2$.
Set $C(i,j) = A(i,j)$ for all i and j.
Replace each $A(i,j)$ by $(2n-3)\,A(i,j)$.

Step 2. If $k = 0$, stop.
Otherwise replace k by $k-1$.

Step 3. Find a pair (i,j) with $i \neq j$, $A(i,j) = 0$ and $D(i) + D(j) \geqslant k$. If there is none, go to Step 2.

Step 4. Replace $A(i,j)$ and $A(j,i)$ by k.
Replace $D(i)$ by $D(i)+1$ and $D(j)$ by $D(j)+1$.
Replace $C(i,j)$ and $C(j,i)$ by M.
Replace M by $M+1$.
Go to Step 3.

Upon termination of this algorithm, the matrix $A(i,j)$ determines all the closures of G, since ij is an edge of $C_k(G)$ if and only if $A(i,j) \geqslant k$. In addition, the matrix $C(i,j)$ indicates the order in which the edges ij were taken into each $C_k(G)$, and we have seen (in the case of hamiltonian cycles) how this information can be utilized.

6. Relative closure and bipartite graphs

Let G^* be a graph, let G be a subgraph of G^* and let k be a nonnegative integer. A property P will be said to be *k-stable relative to* G^* if whenever $G + uv$ is a subgraph of G^* and has property P and $d_G(u) + d_G(v) \geq k$ then G has property P. Among all the graphs H such that $G \subset H \subset G^*$ and $d_H(u) + d_H(v) < k$ for all $uv \in E(G^*) - E(H)$, there is a smallest one. We shall call this graph the *k-closure of G relative to* G^* and denote it by $\mathrm{C}_k(G; G^*)$. Thus $\mathrm{C}_k(G; K_n)$, where n is the order of G, is exactly the $\mathrm{C}_k(G)$ defined in Section 2.

Proposition 6.1. *If P is k-stable relative to G^* and if $\mathrm{C}_k(G; G^*)$ has property P then G itself has property P.*

We have shown in Section 2 that the property of being hamiltonian is n-stable (relative to K_n). In the context of bipartite graphs, this can be strengthened as follows.

Theorem 6.2. *The property of being hamiltonian is $(m+1)$-stable relative to $K_{m,m}$.*

Proof. Suppose that $G + uv$ is a hamiltonian subgraph of $K_{m,m}$ but G is not. Then G has a hamiltonian path

$$u = u_1, u_2, \ldots, u_{2m} = v .$$

Since $G + uv \subset K_{m,m}$, $u = u_1$ can be adjacent only to u_i's with even i while $v = u_{2m}$ can be adjacent only to u_i's with odd i. If $d_G(u) + d_G(v) \geq m + 1$ then there must be some (odd) k such that u is adjacent to u_{k+1} and v is adjacent to u_k. But then G has the hamiltonian cycle

$$u_1 u_{k+1} u_{k+2} \cdots u_{2m} u_k u_{k-1} \cdots u_1 .$$

A theory of closure relative to $K_{m,m}$ can be developed along lines parallel to our treatment of closure relative to K_n. Among the by-products of this approach one obtains theorems of Chvátal [5] and Las Vergnas [13] on hamiltonian cycles in bipartite graphs. Instead of going into the details, we only quote from [10]: "This here ... may lead to a fine old mess."

7. Edge-colorings

Let $P_1, P_2, \ldots, P_m$ be properties defined on graphs of order n; let G be a graph of order n. Then G will be said to have property $P_1 \times P_2 \times \ldots \times P_m$ if, for any coloring $G = G_1 \cup G_2 \cup \ldots \cup G_m$ of the edges of G by m colors, there is always an i such that the graph G_i defined by the ith color has property P_i. In particular, if the property P_i is "to contain a subgraph isomorphic to F_i" then

(7.1) $\quad G \to (F_1, F_2, \ldots, F_m)$

denotes the fact that G has property $P_1 \times P_2 \times \ldots \times P_m$. The smallest n such that $K_n \to (F_1, F_2, \ldots, F_m)$ is usually denoted by

(7.2) $\quad r(F_1, F_2, \ldots, F_m)$;

its existence is guaranteed by Ramsey's theorem [16]. If $F_i = F$ for all i then we write $G \to (F)_m$ and $r(F)_m$ rather than (7.1) and (7.2).

Theorem 7.1. *If P_i is k_i-stable for $i = 1, 2, \ldots, m$, then $P = P_1 \times P_2 \times \ldots \times P_m$ is k-stable where $k = 1 + \Sigma(k_i - 1)$.*

Proof. Assume the theorem false. Then there is a graph G with nonadjacent vertices u, v such that $d_G(u) + d_G(v) \geqslant k$, $G + uv$ has property P but G does not. Thus there is a coloring $G = G_1 \cup G_2 \cup \ldots \cup G_m$ where no G_i has property P_i. Writing $d_i(w)$ for the degree of w in G_i we have

$$\Sigma(d_i(u) + d_i(v)) \geqslant k > \Sigma(k_i - 1)$$

and so $d_i(u) + d_i(v) \geqslant k_i$ for some i. Without loss of generality we may assume $i = 1$. Since P_1 is k_1-stable, $G_1 + uv$ does not have property P_1. But then $G + uv = (G_1 + uv) \cup G_2 \cup \ldots \cup G_m$ and so $G + uv$ does not have property P, a contradiction.

Corollary 7.2. *The property $G \to (sK_2)_m$ is k-stable where $k = 1 + 2m(s-1)$.*

This follows directly from (2.6) and Theorem 7.1.

Cockayne and Lorimer [6] proved that

$$r(s_1K_2, s_2K_2, \ldots, s_mK_2) = 1 + \max s_i + \sum (s_i - 1) .$$

Burr and Graham conjectured that

$$\overline{C^{s-1}_{(m+1)s}} \to (sK_2)_m$$

and proved this with $m = 2$. Our next result settles their conjecture for $m \geq s$.

Corollary 7.3. *Let G be a graph of order at least $m(s-1)+s+1$. If every vertex of G has degree at least $m(s-1)+1$ then*

$$G \to (sK_2)_m .$$

Proof. By Corollary 7.2, the property $G \to (sK_2)_m$ is k-stable where $k = 1 + 2m(s-1)$. Thus $C_k(G)$ is complete of order at least $r(sK_2)_m$ and we have $C_k(G) \to (sK_2)_m$. By Proposition 2.1, we conclude that $G \to (sK_2)_m$.

Clearly, Corollary 7.3 can be strengthened: If $C_k(G) = K_n$ where $k = 1 + 2m(s-1)$ and $n \geq r(sK_2)_m$, then $G \to (sK_2)_m$. Therefore Theorems 3.1 and 3.2 provide weaker conditions under which $G \to (sK_2)_m$. We leave the gory details to the interested reader.

8. Stable graphs

Let us note that every k-stable property is $(k+1)$-stable and every property is $(2n-3)$-stable. In this section, we shall investigate the property of containing a subgraph isomorphic to a given graph F. If, for n large enough, this property is $(2n-4)$-stable, then F will be called *stable*. By (2.2)–(2.6), each of the graphs C_s, P_s, $K_{2,s}$ and sK_2 is stable. Similarly, one can show that C_s^m is stable whenever $s \geq 2m+2$. It is not difficult to show that the disjoint union of stable graphs is stable. However, K_s is not stable when $s \geq 3$.

Theorem 8.1. *If $F \not\supset K_3$ and each edge of F has an end of degree at most two, then F is stable.*

Proof. Suppose $F \subset G + uv$, where $d_G(u) + d_G(v) = 2n - 4$ and $n > |F|$. If $G + uv = G$, then $F \subset G$ and we are done. Otherwise, we may assume without loss of generality that $d_F(v) = 1$ or 2.

(a) $d_F(v) = 1$. Since $n > |F|$, there is a $w \in V(G) - V(F)$. Since $uw \in E(G)$, v can be replaced by w to obtain a subgraph of G isomorphic to F.

(b) $d_F(v) = 2$. Let x be the other neighbour of v in F. As before, there is a vertex $w \in V(G) - V(F)$, and uw, $vw \in E(G)$. Since $F \not\supset K_3$, $xu \notin E(F)$. Therefore the neighbours of v in F include those of x in F (except for v). One can now replace v by w and x by v to obtain a subgraph of G isomorphic to F.

Theorem 8.2. *Let F be a graph such that every vertex of F has degree at least three. Then the following conditions are equivalent:*

(i) *F is stable.*

(ii) *given any edge $w_1 w_2$ of F there are nonadjacent vertices w_3, w_4 with*

$$F - \{w_3, w_4\} \subset F - \{w_1, w_2\}\ .$$

Proof. (i) ⇒ (ii). Given an edge $w_1 w_2$ of F, set

$$H = (n - |F|) K_1 \cup (F - \{w_1, w_2\})\ .$$

Add two more vertices u, v and join each of them to all the vertices of H. Call the resulting graph G. Then $d_G(u) + d_G(v) = 2n - 4$ and $F \subset G + uv$. As F is stable, we have $F \subset G$. Besides, all but $|F|$ vertices of G have degree two. Therefore $F \subset \overline{K}_2 + (F - \{w_1, w_2\})$ and (ii) follows.

(ii) ⇒ (i). Let G be a graph such that $F \subset G + uv$ and $d_G(u) + d_G(v) = 2n - 4$. If $G + uv = G$ then $F \subset G$ and we are done. Otherwise $G = \overline{K}_2 + (F - \{u, v\})$. As $F \subset G + uv$, there are distinct vertices w_1, w_2 of F with $F - \{w_1, w_2\} \subset G - \{u, v\}$. If $w_1 w_2$ is not an edge of F, we have $F \subset G$ and so we are done. Otherwise, by (ii), there are nonadjacent vertices w_3, w_4 of F with $F - \{w_3, w_4\} \subset F - \{w_1, w_2\}$. But then

$$F \subset \overline{K}_2 + (F - \{w_3, w_4\}) \subset \overline{K}_2 + (F - \{w_1, w_2\}) \subset \overline{K}_2 + (G - \{u, v\}) = G$$

and we are done.

For example, Theorem 8.2 shows that the Petersen graph is not stable.

Problem. Characterize stable graphs.

Dr. L. Lesniak has obtained some nice refinements of Theorems 8.1 and 8.2.

9. Appendix 1

Here we present proofs of results on the stability (relative to K_n) of various properties.

Theorem 9.1. *Let n, s be positive integers with $4 \leqslant s \leqslant n$. Then the property of containing C_s is $(2n-s)$-stable.*

Proof. If $G + uv$ contains a C_s but G does not, then G contains a path $u_1 u_2 \dots u_s$ with $u_1 = u$, $u_s = v$. Let H be the subgraph of G induced by $\{u_1, u_2, \dots, u_s\}$. Then $H + uv$ is hamiltonian but H is not. Therefore

$$d_G(u) + d_G(v) \leqslant 2(n-s) + d_H(u) + d_H(v) < 2(n-s) + s = 2n - s\,.$$

Fig. 2 (with $s = 9$) and its obvious generalizations show that the bound $2n - s$ in Theorem 9.1 cannot be improved as long as s is odd. If s is even and equal to n then $K_{n/2,n/2}$ shows that $2n - s$ is best possible again. However, for s even and strictly smaller than n we obtain a slight refinement of Theorem 9.1.

Theorem 9.2. *Let n, s be positive integers such that s is even and $4 \leqslant s < n$. Then the property of containing C_s is $(2n-s-1)$-stable.*

Proof. If $G + uv$ contains a C_s but G does not then G contains a path $u_1 u_2 \dots u_s$ with $u_1 = u$, $u_s = v$. Let H be the subgraph of G induced by

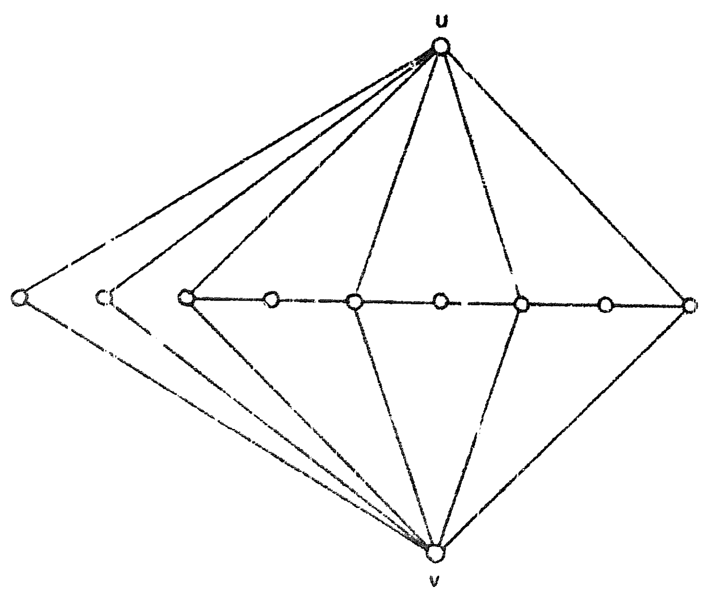

Fig. 2.

$\{u_1, u_2, ..., u_s\}$. As in the proof of Theorem 9.1, we have $d_H(u) + d_H(v) < s$. If u and v have no common neighbour outside H then we are done as

$$d_G(u) + d_G(v) \leq n - s + d_H(u) + d_H(v) < n \leq 2n - s - 1 .$$

Thus we may assume that u and v have a common neighbour w outside H. Now, set

$$A = \{i: 2 \leq i \leq s,\ u \text{ adjacent to } u_i\} ,$$

$$B = \{i: 2 \leq i \leq s,\ v \text{ adjacent to } u_{i-1}\} .$$

We have $d_H(u) = |A|$ and $d_H(v) = |B|$. If $d_H(u) + d_H(v) < s - 1$ then we are done as

$$d_G(u) + d_G(v) \leq 2(n - s) + d_H(u) + d_H(v) < 2n - s - 1 .$$

Thus we may assume that

(9.1) $\quad |A| + |B| \geq s - 1$.

Besides, we have

(9.2) $\quad A \cap B = \emptyset$;

indeed, if $i \in A \cap B$ then $u_1 u_i u_{i+1} \dots u_s u_{i-1} \dots u_1$ is a C_s in G. Now, (9.1) and (9.2) imply

(9.3) $\quad A \cup B = \{2, 3, ..., s\}$.

Clearly, we have $3 \notin A$; otherwise $u_1 u_3 u_4 \dots u_s w u_1$ is a C_s in G. Similarly, we have $s - 1 \notin B$; otherwise $u_1 u_2 \dots u_{s-2} u_s w u_1$ is a C_s in G. Hence (9.3) implies that $3 \in B$ and $s - 1 \in A$; that is, u_s is adjacent to u_2 and u_1 is adjacent to u_{s-1}.

Next, let us note that

(9.4) $\quad j \in A \Rightarrow j + 1 \notin A$;

otherwise $u_1 u_{j+1} u_{j+2} \dots u_n u_2 u_3 \dots u_j u_1$ is a C_s in G. Similarly,

(9.5) $\quad j \in B \Rightarrow j + 1 \notin B$;

otherwise $u_s u_j u_{j+1} \dots u_{s-1} u_1 u_2 \dots u_{j-1} u_s$ is a C_s in G.

Now, (9.3), (9.4), (9.5) and $3 \in B$ imply that every odd j with $j \leq s$ belongs to B. In particular, we have $s - 1 \in B$ contradicting $s - 1 \notin B$ established above.

Fig. 3 (with $s = 10$) and its obvious generalizations show that the

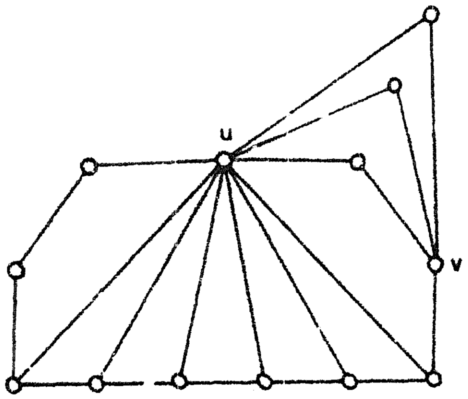

Fig. 3.

bound $2n - s - 1$ in Theorem 9.2 cannot be improved as long as $s \geqslant 6$. For C_4, we easily obtain a much better result.

Theorem 9.3. *Let n, s be positive integers with $s \leqslant n - 2$. Then the property of containing $K_{2,s}$ is $(n + s - 2)$-stable.*

Proof. If $G + uv$ contains $K_{2,s}$ but G does not, then u and v have less than s common neighbours in G. Therefore

$$d_G(u) + d_G(v) < (n - 2) + s\,.$$

To see that the bound $n + s - 2$ in Theorem 9.3 cannot be improved, consider a graph G with three special vertices u, v, w where (i) u is adjacent to every vertex but itself and v, and (ii) exactly $s - 1$ neighbours of u are adjacent to both v and w.

Theorem 9.4. *Let n, s be positive integers with $4 \leqslant s \leqslant n$. Then the property of containing P_s is $(n - 1)$-stable.*

Proof. Let $G + uv$ contain a path $u_1 u_2 \ldots u_s$; assume that G itself contains no P_s. Let H be the subgraph of G induced by $\{u_1, u_2, \ldots, u_s\}$. Then $(H + K_1) + uv$ is hamiltonian but $H + K_1$ is not. Therefore

$$(d_H(u) + 1) + (d_H(v) + 1) < s + 1\,.$$

Besides, u and v have no common neighbours outside H (otherwise G contains a P_{s+1}) and so

$$d_G(u) + d_G(v) \leqslant n - s + d_H(u) + d_H(v) < n - 1\,.$$

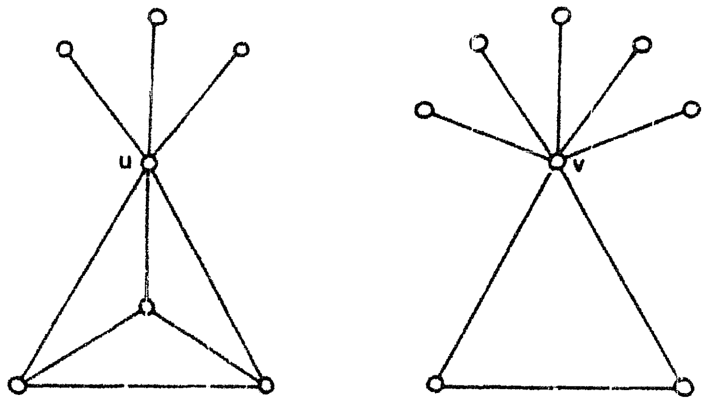

Fig. 4.

Fig. 4 (with $s = 7$) and its obvious generalizations show that the bound $n - 1$ in Theorem 9.4 cannot be improved.

Theorem 9.5. *Let n, s be positive integers with $s \leq \frac{1}{2}n$. Then the property of containing an sK_2 is $(2s - 1)$-stable.*

Proof. If $G + uv$ has an sK_2 but G does not then G includes $2(s - 1)$ vertices

$$u_1, u_2, \ldots, u_{s-1}, v_1, v_2, \ldots, v_{s-1} \tag{9.6}$$

all distinct from u and v, with u_i adjacent to v_i for each i. As G does not contain an sK_2, each neighbour of u and each neighbour of v comes from (9.6). For the same reason, no three edges of G can have one endpoint in the same set $\{u_i, v_i\}$ and the other in $\{u, v\}$. Hence

$$d_G(u) + d_G(v) \leq 2(s - 1) .$$

The graph $K_{2s-1} \cup \overline{K_{n-2s+1}}$ shows that the bound $2s - 1$ in Theorem 9.5 cannot be improved.

Theorem 9.6. *Let n, s be positive integers with $2 \leq s < n$. Then the property of having an s-factor is $(n + 2s - 4)$-stable.*

Proof. If $G + uv$ has an s-factor but G does not, then G has a spanning subgraph F such that $d_F(u) = d_F(v) = s - 1$ and $d_F(w) = s$ whenever $w \neq u, v$. Let A be the set of vertices that are adjacent to u in G but not in F. Similarly, let B be the set of vertices adjacent to v in G but not in F. Then no edge of F has one endpoint w_i in A and the other w_j in B. (Otherwise $F + uw_i + vw_j - w_iw_j$ is an s-factor of G.) Therefore

(9.7) $\quad |\{w_i, w_j\} \cap A| + |\{w_i, w_j\} \cap B|$

is at most two for each edge w_iw_j of F. Besides, if w_i or w_j is u or v then (9.7) is at most one.

The graph F has exactly $\frac{1}{2}ns - 1$ edges; $2(s-1)$ of these are of the second kind. If N denotes the sum of (9.7) over all the edges of F then we have

$$N \leqslant 2(\tfrac{1}{2}ns - 1) - 2(s-1) = (n-2)s\,.$$

For the moment, let us assume that $N = (n-2)s$; let uw be an edge of F. As (9.7) with $\{w_i, w_j\} = \{u, w\}$ equals one, we must have $w \in B - A$. Similarly, there is a w_1 such that vw_1 is an edge of F; we have $w_1 \in A - B$.

As $d_F(w_1) = s \geqslant 2$, there is another vertex w_2 such that w_1w_2 is an edge of F. Since $w_1 \in A$, we have $w_2 \neq u$ and so (9.7) with $\{w_i, w_j\} = \{w_1, w_2\}$ equals two. But then necessarily $w_2 \in A$ and

$$F + uw_1 - w_1w_2 + w_2u - uw + wv$$

is an s-factor of G. To avoid this contradiction, we must have $N < (n-2)s$. On the other hand, $A \cup B$ includes neither u nor v and so $N = s(|A| + |B|)$. Comparing the last two inequalities we obtain $|A| + |B| < n - 2$ and so

$$d_G(u) + d_G(v) = d_F(u) + |A| + d_F(v) + |B| < 2(s-1) + n - 2\,.$$

If $n \geqslant 3s + 3$ then the bound $n + 2s - 4$ in Theorem 9.6 cannot be improved. Indeed, we shall show this by means of an example. Take the complete bipartite graph $K_{s-1,s-1}$ (with bipartition $P \cup Q$). Take a graph H with a distinguished vertex v such that $d_H(v) = s - 2$ but all the other vertices of H have degree s. (If s is even, this can be done with $s + 2$ vertices: if s is odd, with $s + 3$ vertices.) Add two new vertices, u and w. Join v and w to each other and to all the vertices in Q. Join u to all the present vertices except v and w. Call the resulting graph G. Obviously, $G + uv$ has an s-factor; besides, $d_G(u) = n - 3$ and $d_G(v) = 2s - 2$. However, G has no s-factor. Indeed, each vertex in P has degree s and so does w. Besides, each vertex in Q is adjacent to all the s vertices of $P \cup \{w\}$. Thus no vertex from Q can be adjacent, in an s-factor of G, to v. But then only $s - 1$ neighbours of v are available.

Theorem 9.7. *Let n, s be positive integers with $s \leqslant n - 2$. Then the property of being s-connected is $(n+s-2)$-stable.*

Proof. Suppose $G + uv$ is s-connected but G is not. Then there is a set T of $s-1$ vertices of G such that u and v belong to distinct components of $G - T$. So the common neighbours of u and v can come only from T and we have

$$d_G(u) + d_G(v) \leqslant n - 2 + |T| < n + s - 2 .$$

The graph $K_{s-1} + (K_{n-s} \cup K_1)$ shows that the bound $n + s - 2$ in Theorem 9.7 cannot be improved.

Theorem 9.8. *Let n, s be positive integers with $s \leqslant n - 2$. Then the property of being s-edge-connected is $(n+s-2)$-stable.*

Proof. Suppose $G + uv$ is s-edge-connected but G is not. Then there is a set R of $s-1$ edges of G such that u and v belong to distinct components of $G - R$. Therefore G contains at most $s-1$ edge-disjoint paths from u to v. In particular, u and v have at most $s-1$ common neighbours and so $d_G(u) + d_G(v) < n - 2 + s$.

Again, the graph $K_{s-1} + (K_{n-s} \cup K_1)$ shows that the bound $n + s - 2$ is best possible.

Theorem 9.9. *Let n, s be positive integers with $s \leqslant n$. Then the property "$\alpha(G) \leqslant s$" is $(2n - 2s - 1)$-stable.*

Proof. If $\alpha(G + uv) \leqslant s$ but $\alpha(G) > s$ then there is a set W of $s-1$ vertices of G such that $u, v \notin W$ and $W \cup \{u, v\}$ is independent in G. But then $d_G(u) \leqslant n - 2 - |W|$ and $d_G(v) \leqslant n - 2 - |W|$, so that

$$d_G(u) + d_G(v) \leqslant 2(n - s - 1) .$$

The graph $K_{n-s-1} + \bar{K}_{s+1}$ shows that the bound $2n - 2s - 1$ in Theorem 9.9 is best possible.

Theorem 9.10. *Let n, s be positive integers with $s \leqslant n - 3$. Then the property of being s-hamiltonian is $(n+s)$-stable.*

Proof. Suppose that, for some set W of at most s vertices of G, $(G+uv)-W$ is hamiltonian but $H = G - W$ is not. Then

$$d_G(u) + d_G(v) \leq (d_H(u) + |W|) + (d_H(v) + |W|)$$
$$< (n - |W|) + 2|W| = n + |W| \leq n + s .$$

The graph $K_{s+1} + (K_{n-s-2} \cup K_1)$ shows that the bound $n+s$ in Theorem 9.10 is best possible.

Theorem 9.11. *Let n, s be positive integers with $s \leq n-3$. Then the property of being s-edge-hamiltonian is $(n+s)$-stable.*

Proof. Suppose that $G + uv$ is s-edge-hamiltonian but G is not. Then there is a set F of s edges that form pairwise disjoint paths in G such that $G + uv$ has a hamiltonian cycle containing all of F, but G does not. Consider the graph H obtained from G by subdividing each edge in F into two. Then $H + uv$ is hamiltonian but H is not. Therefore

$$d_G(u) + d_G(v) = d_H(u) + d_H(v) < n + s .$$

Again, the graph $K_{s+1} + (K_{n-s-2} \cup K_1)$ shows that the bound $n+s$ in Theorem 9.11 cannot be improved.

Theorem 9.12. *Let n, s be positive integers with $s \leq n-4$. Then the property of being s-Hamilton-connected is $(n+s+1)$-stable.*

Proof. Suppose that $G + uv$ is s-Hamilton-connected but G is not. Then there is a hamiltonian path $u_1 u_2 \ldots u_n$ in $G + uv$ and a set J of s subscripts such that

(i) $u = u_m$, $v = u_{m+1}$ for some m with $m \notin J$,

(ii) G has no hamiltonian path with endpoints u and v which contains all the edges $u_i u_{i+1}$ with $i \in J$.

Let

$$A = \{1 \leq i \leq m: u_i \text{ adjacent to } u \text{ in } G\} ,$$
$$B = \{1 \leq i \leq m: u_{i+1} \text{ adjacent to } v \text{ in } G\},$$
$$C = \{m+1 \leq i \leq n: u_i \text{ adjacent to } u \text{ in } G\} ,$$
$$D = \{m+1 \leq i \leq n: u_{i+1} \text{ adjacent to } v \text{ in } G\} .$$

We claim that $A \cap B \subset J$; if there is an i with $i \in (A \cap B) - J$ then

$$u_1 u_2 \ldots u_i u_m u_{m-1} \ldots u_{i+1} u_{m+1} u_{m+2} \ldots u_n$$

contradicts (ii). Similarly, we have $C \cap D \subset J$ for otherwise

$$u_1 u_2 \ldots u_m u_i u_{i-1} \ldots u_{m+1} u_{i+1} u_{i+2} \ldots u_n$$

contradicts (ii). Now, we have

$$\begin{aligned} d(u) + d(v) &\leqslant (|A| + |C|) + (|B| + |D|) \\ &= |A \cup B| + |C \cup D| + (|A \cap B| + |C \cap D|) \\ &\leqslant m + (n - m) + s = n + s\,. \end{aligned}$$

The graph $K_{s+2} + (K_{n-s-3} \cup K_1)$ shows that the bound $n + s + 1$ in Theorem 9.12 cannot be improved.

Theorem 9.13. *Let n, s be positive integers with $s < n$. Then the property "$\mu(G) \leqslant s$" is $(n-s)$-stable.*

Proof. Suppose that $\mu(G + uv) \leqslant s$ but $\mu(G) > s$; set $H = G + K_s$. Then $H + uv$ is hamiltonian but H is not. Therefore

$$d_G(u) + d_G(v) = (d_H(u) - s) + (d_H(v) - s) < (n + s) - 2s\,.$$

The graph $\overline{K_s} \cup K_{n-s}$ shows that the bound $n - s$ in Theorem 9.13 is best possible.

10. Appendix 2

Here we give a proof of the statement in the remark following Theorem 3.2.

Theorem 10.1. *Let k, n, t be positive integers with $k \leqslant 2n - 4$. Let G be a graph with degree sequence $d_1 \leqslant d_2 \leqslant \ldots \leqslant d_n$. Let there be no non-negative s with*

$$k - n < s < \tfrac{1}{2}k\,, \qquad d_{n-k+s} \leqslant s\,,$$
$$d_{n-s} \leqslant k - s - 1\,, \qquad d_{n-t+1} \leqslant k - s - 1\,.$$

Then there are no i, j with

$$
(10.1)\qquad \begin{array}{lll}
1 \le i < j \le n, & d_i \le i + k - n, & \\
d_j \le j + k - n - 1, & d_i + d_j \le k - 1, & \\
i + j \ge 2n - k, & j \ge n - t + 1, & d_j \le n - 2.
\end{array}
$$

Proof. Let the conclusion fail. There are i, j with (10.1); choose them with i as small as possible. First of all, we shall show that

$$(10.2)\qquad i < n - \tfrac{1}{2}k.$$

Indeed, if $i \ge n - \frac{1}{2}k$ then $(i-1) + j \ge (i-1) + (i+1) \ge 2n - k$ and $i \ge 2$. By minimality of i, we must have $d_{i-1} > (i-1) + k - n$ and so $d_i = i + k - n \ge \frac{1}{2}k$. But then $d_i + d_j \ge 2d_i \ge k$ contradicting (10.1).

Now, set $s = i + k - n$. Then s is nonnegative (as $d_i \le s$); $i \ge 1$ and (10.2) imply $k - n < s < \frac{1}{2}k$. Besides, $d_i \le i + k - n$ reads $d_{n-k+s} \le s$. Let us also note that G has at most d_i vertices of degree $n - 1$ and so $d_{n-s} \le n - 2$.

Next, we shall prove that $i \ge 2$. Indeed, if $i = 1$ then $k - s - 1 = n - 2$ and, to keep the hypothesis satisfied, we must have $d_{n-t+1} = n - 1$. But then $d_j = n - 1$ contradicting (10.1).

Finally, let us note that $j \ge \max(n - s, n - t + 1)$ and so $d_j \ge \max(d_{n-s}, d_{n-t+1}) \ge k - s$. Therefore

$$j \ge d_j - k + n + 1 \ge n - s + 1 = 2n - i - k + 1$$

and so $(i-1) + j \ge 2n - k$. By minimality of i, we must have $d_{i-1} > (i-1) + k - n$, so that $d_i = i + k - n$. But then

$$d_i + d_j \ge i + k - n + k - s = k$$

contradicting (10.1).

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