General Topology and its Applications 5 (1975) 205–219

FREE *k*-GROUPS AND FREE TOPOLOGICAL GROUPS

Edward T. ORDMAN
University of Kentucky, Lexington, Ky. 40506, U.S.A.

Received 25 October 1973

0. Introduction

This is the first of two papers which will explore analogues in the category of "compactly generated" groups of free topological groups and free products of topological groups, considered as constructions in the category of all topological groups. In particular, this paper contains an existence theorem for free *k*-groups and a new existence theorem for free topological groups which is rather more constructive than previous proofs, and employs the theory of *k*-spaces (hence, compactness arguments) rather than the norms [7], pseudometrics [3], unitary groups [4] or other machinery previously used.

The free topological group on a completely regular Hausdorff space X was introduced by Markov [7] and has since been studied fairly extensively. More recently Nummela [11] has worked with the corresponding construction for "compactly generated" groups; the free *k*-group on a weak Hausdorff *k*-space. In sketching an existence proof for a free *k*-group, Nummela observed that one can define the free topology directly, rather than show its existence as the finest member of a class one member of which is constructed by the use of such techniques as cited above.

Our major object in this paper is to carry out the construction indicated by Nummela [10] and [11] with enough care to enable us to clarify the difference between the topological case and the "compactly generated" case. By constructing in general the free *k*-group on a weak Hausdorff *k*-space, we obtain also the free topological group on a compact Hausdorff space and thus the existence of free topological groups in general.

In Section 1 we provide a summary of properties of *k*-spaces which will also be used in the sequel to this paper [12]. Sections 2, 3 and 4 contain the main results. Section 5 contains further description of the topologies introduced earlier, which will be important in the sequel. The sequel will consider free products of k-groups, relations between free products and free groups, and relations between the category of topological groups and the category of k-groups. Section 6 contains details of a proof.

The author wishes to thank S.A. Morris for numerous helpful discussions and E.C. Nummela for detailed and helpful correspondence. This research was done while the author was a guest at the University of New South Wales, partially supported by a Fulbright–Hays grant.

1. Definitions and preliminaries

We will be relying heavily on machinery due to Steenrod [14], McCord [8], and Nummela [10, 11], but find it useful to vary terminology somewhat. In particular, we admit compact spaces which are not Hausdorff, and thus cannot conclude that every compact space is a k-space (compare [15]). In view of the delicacy of our arguments (many of which take place in spaces with no separation property) we state our definitions and preliminary lemmas in some detail.

The letter X will always denote a topological space. X is T_1 if one-point subsets are closed and T_2 (*Hausdorff*) if any two distinct points have disjoint neighborhoods. X is *compact* if every open cover has a finite subcover. The letter C will always denote a compact Hausdorff space. X is t_2 (*weak Hausdorff*) if for every C and every continuous map $\phi : C \to X$, $\phi(C)$ is closed in X. In particular, a compact Hausdorff subset of a t_2 space is closed.

The set of integers, in the finite-complement topology [13, p. 49] is T_1 but not t_2. The one-point compactification of the rationals [13, p. 63] is t_2 but not T_2 (for more examples, with an indication of proof, see [11, Example 1]).

Lemma 1.1. (a) *A continuous image of a compact space is compact.*

(b) *A closed subset of a compact space is compact.*

(c) *A compact subspace of a Hausdorff space is closed.*

(d) *Any subspace of a* T_2 *(resp.* t_2, T_1*) space is* T_2 *(resp.* t_2, T_1*).*

(e) T_2 *implies* t_2 *and* t_2 *implies* T_1.

(f) *If* X *is* t_2 *and* $\phi : C \to X$ *is continuous,* $F(C)$ *is compact Hausdorff.*

(g) *If* C_1 *and* C_2 *are compact Hausdorff subspaces of a* t_2 *space, then* $C_1 \cup C_2$ *is compact Hausdorff.*

Proof. (a) – (e) are routine. (f) is 2.1 of [8]. (g) follows from (f) since $C_1 \cup C_2$ is a continuous image of the union of disjoint copies of C_1 and C_2.□

A subset A of X is called *compactly closed* in X if for every C and every continuous $f : C \to X$, $\phi^{-1}(A)$ is closed in C. kX is the space with the same points as X, and A closed in kX if and only if A is compactly closed in X. X is a *k-space* if $X = kX$ (i.e., if they have the same topology). If $f : X \to Y$ is a continuous onto map of topological spaces such that $A \subset Y$ and $f^{-1}(A)$ closed in X implies A is closed in Y, then f is a *quotient* map and Y is a *quotient* of X.

The reader should be warned that our definitions of compactly closed and of k-space agree with [8] but disagree with some earlier definitions, notably [14]. Loosely, [14] considers a set compactly closed if it has closed intersection with each compact Hausdorff subset of X. We shall see in Lemma 2(h) that this coincides with our definition if X is t_2. If X fails to be t_2, however, the definitions differ. For example, by our definition, the set of integers, with the finite-complement topology, is a k-space; it fails to be a k-space in some other definitions.

Lemma 1.2. (a) *If A is closed in X, A is compactly closed in X.*

(b) *The topology of kX is at least as fine as that of X.*

(c) $k(kX) = kX$.

(d) *A closed subspace of a k-space is a k-space.*

(e) *A quotient of a k-space is a k-space.*

(f) *If $f : X \to Y$ is a continuous map of topological spaces, $f : kX \to kY$ is continuous.*

(g) *A compact Hausdorff space is a k-space.*

(h) *A subset of a t_2-space X is compactly closed if and only if its intersection with every compact Hausdorff subset D of X is closed (since X is t_2, D is closed in X and the intersection is closed in $D \cap X$ if and only if it is closed in X).*

(i) *If $c : A \to X$ is a closed inclusion (A is a closed subset of the topological space X), then $c : kA \to kX$ is a closed inclusion.*

Proof. All are routine or in [8]; we illustrate with (i). Let $c : A \to X$ be a closed inclusion. As $c(A)$ is closed in X, it is closed in kX; $c : kA \to kX$ is continuous by (f). Now suppose B is closed in kA, hence compactly closed in A; we must show it is closed in kX. Let $\phi : C \to X$ be continuous; $\phi : C \cap \phi^{-1}(A) \to A$ is continuous on the compact Hausdorff space

$C \cap \phi^{-1}(A)$, so $\phi^{-1}(B)$ is closed in $C \cap \phi^{-1}(A)$ and hence in C. Thus B is compactly closed in X and closed in kX, as desired. □

If X and Y are topological spaces, $X \times_t Y$ denotes the ordinary product of topological spaces. If X and Y are k-spaces, $X \times_k Y$ denotes $k(X \times_t Y)$. If $X \times_t Y = X \times_k Y$, we write $X \times Y$. If G is an (abstract) group, $p : G \times G \to G$ is the map given by $p(g, h) = gh^{-1}$. If G is a group and a topological space, G is a *topological group* if $p : G \times_t G \to G$ is continuous. If G is a group and a k-space, it is a *k-group* if $p : G \times_k G \to G$ is continuous.

Lemma 1.3. (a) *The projections $\pi_1 : X \times_k Y \to X$ and $\pi_2 : X \times_k Y \to Y$ are continuous, for X and Y k-spaces.*

(b) *If X and Y are compact Hausdorff, $X \times_k Y = X \times_t Y$.*

(c) *If X and Y are k-spaces and $f : X \to X'$, $g : Y \to Y'$ are quotient (onto) maps, $f \times g : X \times_k Y \to X' \times_k Y'$ is a quotient map.*

(d) *If X is a k-space, X is t_2 if and only if the diagonal is closed in $X \times_k X$.*

(e) *If G is a k-group and the identity $\{e\}$ is closed, G is t_2.*

(f) *If $f : A \to G$, $g : B \to G$ are continuous maps of topological spaces into topological groups, $f \times g : A \times B \to G$ given by $(f \times g)(a, b) = f(a)(g(b))^{-1}$ is continuous.*

(g) *If $f : A \to G$, $g : B \to G$ are continuous maps of k-spaces into k-groups, $f \times g : A \times_k B \to G$ given by $(f \times g)(a, b) = f(a)(g(b))^{-1}$ is continuous.*

(h) *If G is a topological group, kG is a k-group.*

Proof. (a) and (b) are routine. (c) is stated without proof as 2.2 of [8]; a proof is provided in Section 6 of this paper. (d) is 2.3 of [8]. To prove (e), note that $p : G \times_k G \to G$ is continuous, so $\{e\}$ closed implies $p^{-1}(e)$, the diagonal of $G \times_k G$, is closed; thus G is t_2 by (d). (f) and (g) are routine. To show (h), note that since $p : G \times_t G \to G$ is continuous, $p : k(G \times G) \to kG$ is continuous. However, $k(G \times_t G) = kG \times_k kG$. This may be shown directly (compare Proposition 2.8 of [1]) or by appealing to categorical arguments (e.g., k is an adjoint functor, and adjoints preserve products). □

Suppose X is the union of an expanding sequence of subspaces $X_1 \subset X_2 \subset \dots$. We say X has the *weak topology* if A is closed in X if and only if $A \cap X_n$ is closed in X_n for all n. If in addition each X_n is compact Hausdorff, X is called a k_ω-space.

Lemma 1.4. *Let X be a union of an expanding sequence of subspaces $X_1 \subset X_2 \subset \ldots$ and have the weak topology.*

(a) *If each X_n is closed in X, and each X_n is a t_2 k-space, then X is a t_2 k-space.*

(b) *If each X_n is compact Hausdorff, then X is a Hausdorff k-space.*

(c) *If X and Y are k_ω-spaces, $X \times_t Y$ is a k_ω-space, and hence $X \times_t Y = X \times_k Y$. If $X = \bigcup X_n$ and $Y = \bigcup Y_n$ are decompositions of X and Y as k_ω-spaces, then $X \times Y = \bigcup (X_n \times Y_n)$ is a decomposition of $X \times Y$.*

(d) *If Y is a topological space and $f: X \to Y$ is continuous on each X_n, it is continuous on X.*

(e) *If each X_n is closed in X and each X_n is T_1, and $\phi: C \to X$ is continuous, then $\phi(C) \subset X_n$ for some n.*

Proof. (a) is [8, 2.6]. (b) is essentially contained in [3, Theorem 5]: if x and y are in X_n and X_n is normal, there are neighborhoods $U_n(x)$, $V_n(y)$ in X_n with disjoint closures; an expanding union of such neighborhoods, as n increases, yields disjoint neighborhoods of x and y in X. Hence X is Hausdorff, the X_n are closed, and (a) applies. (c) is a well-known result of Milnor [9]. (d) is routine, and (e) is [14, 9.3].□

Finally, we must recall the definition of a free topological group. For simplicity we restrict ourselves to free topological groups in the sense of Graev [3]. If X is a completely regular Hausdorff topological space with basepoint e, FGX denotes the (unique) topological group containing X as a subspace and such that if $f: X \to G$ is any continuous map of the space X to a topological group G, with the basepoint $e \in X$ going to the identity $e \in G$, there is a unique continuous homomorphism $\bar{f}: FGX \to G$ extending f. The preliminaries, as in [3], are well known: to show the existence of FGX it suffices to find any topology on the free group FX generated by X with e as identity (hence, freely generated by $X - \{e\}$) which is a Hausdorff group topology and induces the original topology on the subset X; the free topological group topology is the finest such topology.

The free (Graev) k-group on X, denoted here by FKX, is similar; X must be a t_2 k-space and FKX is a t_2 k-group. The reason FGX is Hausdorff and FKX merely t_2 appears to lie in Lemma 1.3 (e); a T_1 topological group is Hausdorff, but a T_1 k-group need only be t_2. For further discussion of FKX, see [11].

2. The topology (FX, τ)

Let X be a t_2 k-space with basepoint e, and FX the abstract free group on the set X with e as identity. We shall describe a topology τ on FX. In Section 4 we will see that (FX, τ) is

(1) FKX, in general;

(2) FGX and FKX (which thus coincide) if X is compact Hausdorff.

We will go on to conclude FGX exists and is Hausdorff for all completely regular Hausdorff X.

Let X^{-1} denote a space homeomorphic to X consisting of elements x^{-1} for $x \in X$. Let X_0 denote $X \cup_e X^{-1}$, the one-point union with $e = e^{-1}$. Let $X_0^n = X_0 \times_k \ldots \times_k X_0$ (n factors; recall that if X is compact Hausdorff, this is the ordinary topological product). Imbed X_0^n in X_0^{n+1} by $X_0^n \to X_0^n \times \{e\} \subset X_0^{n+1}$ and let X^* denote the union $\bigcup_{n=1}^{\infty} X_0^n$. Let A be closed in X^* if A is relatively closed in each X_0^n ($A \cap X_0^n$ is closed in X_0^n). Clearly each X_0^n is contained as a closed subset of X^* and so by Lemma 1.4(a), X^* is a t_2 k-space. (If X is compact Hausdorff, X^* is a k_ω-space).

Now define a map $i : X^* \to FX$ by $i(x_1^{\epsilon_1}, \ldots, x_n^{\epsilon_n}) = x_1^{\epsilon_1}, \ldots, x_n^{\epsilon_n}$ ($\epsilon_i = \pm 1$). Let τ denote the quotient topology on FX; i.e. $A \subset FX$ is closed whenever $i^{-1}(A)$ is closed in X^*. It will sometimes help to let X_0^0 contain the unique empty string ( ), setting $i(\,) = e$ and making appropriate conventions on inclusions, products, and so on.

3. Properties of (FX, τ)

Lemma 3.1. *(FX, τ) has the weak topology as the union of* $\{e\} = i(X_0^0) \subset i(X_0^1) \subset i(X_0^2) \subset i(X_0^3) \subset \ldots$.

Proof. Let $A \cap i(X_0^n)$ be closed in $i(X_0^n)$ for each n. Then $i^{-1}(A \cap i(X_0^n)) \cap X_0^n$ is closed in X_0^n; but it is equal to $i^{-1}(A) \cap X_0^n$ (since an element of X_0^n mapping to A maps to $A \cap i(X_0^n)$), so each $i^{-1}(A) \cap X^n$ is closed, and A is closed in (FX, τ). □

Proposition 3.2. *Let $m \geq 1$ be fixed. Let A be a subset of $i(X_0^m) \subset (FX, \tau)$ such that $i^{-1}(A) \cap X_0^m$ is closed (in X_0^m). Then A is closed in (FX, τ).*

Proof. We must show that $i^{-1}(A)$ is closed in X^*, i.e., that $i^{-1}(A) \cap X_0^n$ is closed in X_0^n for all n. If $0 \leqslant n \leqslant m$, $i^{-1}(A) \cap X_0^n = i^{-1}(A) \cap (X_0^m \cap X_0^n) = (i^{-1}(A) \cap X_0^m) \cap X_0^n$, the intersection of two closed sets in X^*. Thus $i^{-1}(A) \cap X_0^n$ is closed in X^*, and thus in X_0^n, for $n \leqslant m$, and in particular for $n = m$ and $n = m - 1$. Now suppose $i^{-1}(A) \cap X_0^n$ and $i^{-1}(A) \cap X_0^{n+1}$ are closed, $n + 2 > m$, and proceed by induction. $i^{-1}(A) \cap X_0^{n+2}$ is the union of (1) finitely many copies of $i^{-1}(A) \cap X_0^{n+1}$; a typical injection

$$f_2 : i^{-1}(A) \cap X_0^{n+1} \to i^{-1}(A) \cap X_0^{n+2}$$

is $f_2(a_1^{\epsilon_1}, a_2^{\epsilon_2}, ..., a_{n+1}^{\epsilon_{n+1}}) = (a_1^{\epsilon_1}, e, a_2^{\epsilon_2}, ..., a_{n+1}^{\epsilon_{n+1}})$ (there are $n + 2$ places where e can be inserted and the $n + 2$ images are clearly closed in X_0^{n+2}); together with the union of (2) finitely many copies of the closed set $(i^{-1}(A) \cap X_0^n) \times_k X_0$; a typical injection is

$$g_3 : (i^{-1}(A) \cap X_0^n) \times_k X_0 \to i^{-1}(A) \cap X_0^{n+2}$$

given by $g_3((a_1^{\epsilon_1}, ..., a_n^{\epsilon_n}), x^{\pm 1}) = (a_1^{\epsilon_1}, a_2^{\epsilon_2}, x^{\pm 1}, x^{\mp 1}, a_3^{\epsilon_3}, ..., a_n^{\epsilon_n})$. There are $n + 1$ places where $(x^{\pm 1}, x^{\mp 1})$ can be inserted; that the image in each case is closed follows easily from Lemma 1.3(d) and the fact that the product of closed sets is closed in the topological product and hence in the k-product. (We remark that in the case when X is compact Hausdorff, it is routine that each set in (1) and (2) is compact and hence closed in X_0^{n+2}.)

This union ((1) and (2)) includes all of $i^{-1}(A) \cap X_0^{n+2}$ since any word representing an element of A can be reduced to a word of at most m ($< n + 2$) letters. Thus each $i^{-1}(A) \cap X_0^n$ is closed, and A is closed in the quotient (FX, τ). □

Corollary 3.3. *(FX, τ) is T_1; each $i(X_0^m)$ is closed in (FX, τ); each map $i : X_0^m \to i(X_0^m)$ is a quotient map.*

Proof. $\{e\}$ is closed in (FX, τ) since $i^{-1}(e) \cap X_0^1 = \{e\}$ is closed in X_0^1. If $w \neq e$, write w as $x_1^{\epsilon_1}, ..., x_m^{\epsilon_m}$ where no x_i is e, each $\epsilon_i = \pm 1$, and if $x_i = x_{i+1}$ then $\epsilon_i = \epsilon_{i+1}$. Then $w \in i(X_0^m)$, and $i^{-1}(w) \cap X_0^m = \{(x_1^{\epsilon_1}, ..., x_m^{\epsilon_m})\}$, which is closed in X_0^m, so $\{w\}$ is closed in (FX, τ). $i(X_0^m)$ is closed in (FX, τ) since $i^{-1} i(X_0^m) \cap X_0^m) = X_0^m$ is closed in X_0^m. $i : X_0^m \to i(X_0^m)$ is a quotient map since if $A \subset i(X_0^m)$ and $i^{-1}(A)$ is closed in X_0^m, A is closed in (FX, τ) and thus in $i(X_0^m)$. □

Proposition 3.4. *The topology induced on the subset $i(X_0^n) \times i(X_0^k)$ of $(FX, \tau) \times_k (FX, \tau)$ is $i(X_0^n) \times_k i(X_0^k)$.*

Proof. $i(X_0^n)$ is a quotient of a k-space, hence a k-space. X^* is a t_2 k-space by Lemma 1.4(b), so its quotient FX is a k-space. As $i(X_0^n)$ is a closed subspace of (FX, τ),

$$r : i(X_0^n) \times_t i(X_0^k) \to (FX, \tau) \times_t (FX, \tau)$$

is a closed inclusion, so by Lemma 1.2(i), so is

$$r : i(X_0^n) \times_k i(X_0^k) \to (FX, \tau) \times_k (FX, \tau). \square$$

Recall $p : FX \times FX \to FX$ is the algebraic map $p(g, h) = gh^{-1}$. We first consider its restriction to $i(X_0^n) \times i(X_0^k)$.

Proposition 3.5. $p : i(X_0^n) \times_k i(X_0^k) \to i(X_0^{n+k})$ *is continuous.*

Proof. Define $p' : X_0^n \times_k X_0^k \to X_0^{n+k}$ by $p'((g_1^{e_1}, ..., g_n^{e_n}), (h_1^{\nu_1}, ..., h_k^{\nu_k})) = (g_1^{e_1}, ..., g_n^{e_n}, h_k^{-\nu_k}, ..., h_1^{-\nu_1})$. This is a homeomorphism. In the commutative diagram

$$(1)\qquad \begin{array}{ccc} X_0^n \times_k X_0^k & \xrightarrow{\;p'\;} & X_0^{n+k} \\ \big\downarrow{\scriptstyle i'} & & \big\downarrow{\scriptstyle i} \\ i(X_0^n) \times_k i(X_0^k) & \xrightarrow{\;p\;} & i(X_0^{n+k}) \end{array}$$

the vertical map on each side is a quotient map ($i' = i \times i$ is a quotient map by Lemma 1.3 (c)), so p is continuous. (If B is closed in $i(X_0^{n+k})$, $p'^{-1} i^{-1}(B) = i'^{-1} p^{-1}(B)$ is closed, so $p^{-1}(B)$ is closed). $\square$

Proposition 3.6. $p : (FX, \tau) \times_k (FX, \tau) \to (FX, \tau)$ *is continuous.*

Proof. Let $B \subset (FX, \tau)$ be closed; we must show $p^{-1}(B)$ compactly closed. If C is compact Hausdorff and $\phi : C \to (FX, \tau) \times_k (FX, \tau)$ is continuous, $\pi_1 \circ \phi(C) \subset i(X_0^n)$ for some n by Lemma 1.4(e). Similarly $\pi_2 \circ \phi(C) \subset i(X_0^k)$ for some k; so $\phi(C) \subset i(X_0^n) \times_k i(X_0^k)$. Now, as p restricted to $i(X_0^n) \times_k i(X_0^k)$ is continuous, $p^{-1}(B) \cap [i(X_0^n) \times_k i(X_0^k)]$ is closed: so

$$\phi^{-1} p^{-1}(B) = \phi^{-1}(p^{-1}(B) \cap \phi(C)) = \phi^{-1}(p^{-1}(B) \cap [i(X_0^n) \times_k i(X_0^k)])$$

is closed. As this is true for all ϕ, $p^{-1}(B)$ is compactly closed and thus closed in the k-space $(FX, \tau) \times_k (FX, \tau)$.

4. Existence theorems

Theorem 4.1. *Let X be a t_2 k-space and (FX, τ) the free group on X (with $e \in X$ as identity) with the topology τ as constructed in Section 2. Then*

(1) *(FX, τ) is a t_2 k-group;*

(2) *(FX, τ) is the (Graev) free k-group FKX; hence FKX exists, is t_2, is algebraically FX, and contains X as a closed subset.*

Proof. By Proposition 3.6, (FX, τ) is a k-group. It is T_1 by Corollary 3.3 and hence t_2 by Lemma 1.3(e). By Corollary 3.3, $i : X_0^1 \to i(X_0^1) \subset (FX, \tau)$ is a quotient map. Since i is one-to-one on X_0^1, it is a homeomorphism of X_0^1 onto the closed subset $i(X_0^1)$ of (FX, τ). Considering X as a closed subset of $X_0 = X_0^1$, X is homeomorphic to the closed subset $i(X) \subset (FX, \tau)$ as desired. All that remains to be shown is that if $f : X \to G$ is a continuous map of X to a k-group G, with $f(e) = e$, then the unique algebraic extension $\hat{f} : (FX, \tau) \to G$ is continuous. Clearly the natural extension of f to a map $X_0 \to G$ is continuous, as is each $X_0^n \to G$ (by an obvious extension of Lemma 1.3(g)). Thus the extension $f^* : X^* \to G$ is continuous (Lemma 1.4(d)) and since $i : X^* \to (FX, \tau)$ is a quotient and $\hat{f}i = f^*$, $\hat{f}$ is continuous. □

To avoid unduly complicating the proof of Theorem 4.3, Theorem 4.2 is stated only for compact Hausdorff spaces. In the next section we will extend it to k_ω-spaces.

Theorem 4.2. *Let X be a compact Hausdorff space and (FX, τ) the free group on X (with $e \in X$ as identity) with the topology τ introduced in Section 2. Then*

(1) *(FX, τ) is a Hausdorff topological group and a k_ω-space;*

(2) *(FX, τ) is the (Graev) free topological group FGX; hence FGX exists, is Hausdorff, is algebraically FX, and contains X as a closed subset.*

Proof. By Theorem 4.1, (FX, τ) is a t_2 k-space. Since each X_0^n is compact Hausdorff, each $i(X_0^n)$ is compact Hausdorff by Lemma 1.1(f). Hence by Lemma 3.1, (FX, τ) is a k_ω-space, and by Lemma 1.4(b), it is Hausdorff. Now by Lemma 1.4(c), $(FX, \tau) \times_k (FX, \tau) = (FX, \tau) \times_t (FX, \tau)$ and by Proposition 3.6, (FX, τ) is a topological group. The proof that it is FGX is identical to the corresponding part of Theorem 4.1, but relies on Lemma 1.3(f) instead of 1.3(g). □

We now are able to conclude FGX exists for all completely regular Hausdorff spaces.

Theorem 4.3. *Let X be a completely regular Hausdorff space. Then FGX exists, is algebraically FX, is Hausdorff, and contains X as a closed subset.*

Proof. There is an inclusion $f: X \to \beta X$, where βX is the Stone–Čech compactification of X [2]. Form $FG\beta X$ as in Theorem 4.2, and extend f algebraically to an injection $f: FX \to FG\beta X$. Topologize FX as a subset of $FG\beta X$. Clearly this is a Hausdorff group topology on FX containing X in its original topology as a closed subset ($X = i(\beta X) \cap FX$ in $FG\beta X$); the finest such topology is the topology of FGX. □

We remark that even in very simple cases, there is no reason to expect the topology induced on FGX by $FG\beta X$ to be fine enough to be the free topological group topology; compare the example in [5] where it is shown that the subgroup of $FG[0, 1]$ generated by $(0, 1) \subset [0, 1]$ is not $FG(0, 1)$.

It would be of interest to know whether a Hausdorff k-space always generates a Hausdorff free k-group. We cannot answer this, but observe:

Corollary 4.4. *If X is a completely regular Hausdorff k-space, then FKX is Hausdorff.*

Proof. FGX exists by Theorem 4.3 and is Hausdorff. By Lemma 1.3(h), $kFGX$ is a k-group topology on FX which contains X as a closed subspace by Lemma 1.2(i). The finest such topology is that of FKX; since it is finer than the topology of FGX, it is Hausdorff. □

B.V.S. Thomas has recently [16] improved Theorem 4.3, clarifying what happens if the hypotheses on X are weakened. FGX still exists but contains X as a closed subspace if and only if X is completely regular T_1, and is Hausdorff if and only if X is functionally Hausdorff.

5. Further observations

If X is compact Hausdorff, FGX and FKX coincide. We shall see that they also coincide if X is a k_ω-space.

If C is a subset of X containing the identity e, then C_0, C_0^n and so on are defined as X_0, X_0^n and so on were in Section 2. If X is a t_2 k-space and C compact Hausdorff, C_0^n is a compact Hausdorff subspace of X_0^n (this may be proven by taking topological products and applying Lemma 1.2(i)).

Lemma 5.1. *Let X be a t_2 k-space and FKX its free k-group. A subset A of FKX is closed in FKX if and only if $A \cap i(C_0^n)$ is closed in $i(C_0^n)$ ($i^{-1}(A)$ is closed in C_0^n) for every n and every compact Hausdorff $C \subset X$ containing e.*

Proof. If A is closed in FKX, it is closed in each $i(X_0^n)$ and hence in each compact $i(C_0^n) \subset i(X_0^n)$. Conversely, let A be closed in each $i(C_0^n)$; we must show $i^{-1}(A)$ is compactly closed in each X_0^n. By Lemma 1.2(h), it will suffice to show $i^{-1}(A)$ is closed in each compact Hausdorff $D \subset X_0^n$. The projections of D onto each of the n factors of X_0^n are compact Hausdorff spaces $D_1, ..., D_n$. For each D_i, let

$$D_i^+ = D_i \cap X, \qquad D_i^- = \{x \in X \mid x^{-1} \in D_i\}$$

(D_i^- is homeomorphic to $D_i \cap X^{-1}$). Let

$$C = D_1^+ \cup D_1^{-1} \cup ... \cup D_n^+ \cup D_n^- \,.$$

Clearly C is compact Hausdorff (by Lemma 1.1(g), since each D_i^+ and D_i^- is) and C_0 contains $D_1 \cup ... \cup D_n$, so $D \subset C_0^n$. But $i^{-1}(A)$ is relatively closed in C_0^n, so surely it is relatively closed in $D \subset C_0^n$, completing the proof. □

Theorem 5.2. *If X is a k_ω-space, then FKX is the (Graev) free topological group FGX and is a k_ω-space.*

Proof. X is the ascending union of an ascending sequence of compact Hausdorff subspaces X_n, $n \geq 1$, with this weak topology. Set ${}_nX = X_n \cup \{e\}$; then since X is Hausdorff (Lemma 1.4(b)), ${}_nX$ is compact Hausdorff and in fact X is the weak union of the ${}_nX$, $n \geq 1$. (For details of the manipulation of k_ω-spaces, see [6].) We shall show that FKX is the weak union of the compact Hausdorff subspaces $i(({}_nX)_0^n)$, $n \geq 1$.

If A is closed in FKX, it clearly has closed intersection with each $i(({}_nX)_0^n)$. Now suppose A has closed intersection with each $i(({}_nX)_0^n)$; by Lemma 5.1, it will suffice to show A has closed intersection with $i(C_0^m)$

for every $m \geqslant 1$ and every compact Hausdorff $C \subset X$. Fix m and C. By Lemma 1.4(e), $C \subset {}_pX$ for some $p \geqslant 1$. Letting $n = \max(m, p)$, C_0^n is a subspace of $({}_nX)_0^n$, so $A \cap i(C_0^m) = A \cap i(({}_nX)_0^n) \cap i(C_0^m)$ is closed as desired.

Hence FKX is a k_ω-space, and thus $FKX \times_k FKX = FKX \times_t FKX$, so FKX is a topological group. The proof that maps $X \to G$ extend to maps $FKX \to G$ is as in Theorems 4.1 and 4.2. □

Propositions 5.3 and 5.4 will be useful in the sequel [12] to this paper. They, together with Lemma 5.1, result from correspondence with E.C. Nummela.

Proposition 5.3. *Let X be a t_2 k-space and C be a compact Hausdorff subspace containing the basepoint. Then the subgroup of FKX generated by C is FKC and is closed in FKX. Further, a subset A of FKX is closed if and only if it has closed intersection with each such FKC.*

Proof. Since the inclusion map $C \to X$ is continuous, so is the extension $FKC \to FKX$. We must show that if A is a closed subset of FKC, it is closed in FKX. Since A is closed in FKC, $i^{-1}(A) \cap C_0^n$ is closed in C_0^n (hence compact Hausdorff) for each n; note that it is immaterial whether we are talking about $i^{-1}(A)$ using the map $i: C_0^n \to FKC$, or using the restriction of the map $i: X_0^n \to FKX$ to $C_0^n \subset X_0^n$, since the product topology on C_0^n and the subspace topology on $C_0^n \subset X_0^n$ coincide (this is obviously true if X_0^n is taken to be an n-fold topological product, so it is true for the k-space product by Lemma 2(i)). Hence $A \cap i(C_0^n)$ is compact Hausdorff and thus closed in FKX. Let D be any compact Hausdorff subset of X containing e. Then

$$A \cap i(D_0^n) = (A \cap i(C_0^n)) \cap i(D_0^n)$$

is the intersection of two closed subsets of FKX and is closed. Thus by Lemma 5.1, A is closed in FKX as desired. Hence $FKC \to FKX$ is a homeomorphism onto a closed subgroup.

Now suppose $A \subset FKX$ is such that $A \cap FKC$ is closed in FKC for every compact Hausdorff $C \subset X$ containing e. Then $A \cap i(C_0^n)$ is closed for all C and n, so A is closed in FKX by Lemma 5.1, completing the proof. □

Proposition 5.4. *If X is a completely regular Hausdorff space and C is a compact subspace containing the basepoint, then the subgroup of FGX generated by C is FGC and is closed in FGX.*

Proof. As above, the map $FGC \xrightarrow{f} FGX$ is continuous. Further, the composite

$$FGC \xrightarrow{f} FGX \xrightarrow{g} FG\beta X$$

coincides with the map

$$FGC = FKC \xrightarrow{gf} FK\beta X = FG\beta X$$

which is a closed inclusion by Proposition 5.3. Thus if A is closed in FGC, $f(A) = g^{-1}(gf(A)) = g^{-1}(gf(A))$ is closed in FGX as desired. □

The situations of Propositions 5.3 and 5.4 are more distinct than may appear; it is no accident that Proposition 5.3 required Lemma 5.1 and Proposition 5.4 did not. We will see in the sequel [12] that Lemma 5.1, and the last statement of Proposition 5.3, may be false for FGX even when X is a k-space. In fact, it will suffice to produce a completely regular Hausdorff k-space X for which FGX is not a k-space; for if FGX had its topology determined by the compact subsets $i(C_0^n)$ of Lemma 5.1, or by the k_ω-space subsets $FKC = FGC$ of Proposition 5.3, it would necessarily be a k-space.

If X is a t_2 k-space, $X = \text{colim } C$ where C runs through the compact Hausdorff subsets of X. As the functor $FK(C \to FKC)$ is left adjoint to the forgetful functor, it preserves colimits. (This was pointed out to me by Nummela). Hence $FKX = \text{colim } FKC$, and Proposition 5.3 tells us that in this case this is not only a colimit in the category of k-groups but a straightforward topological colimit. If X is also a completely regular Hausdorff space, we similarly have $FGX = \text{colim } FGC$ (with $FGC = FKC$) but the colimit is now in the category of topological groups and fails in general to be a colimit of topological spaces.

6. Proof of Lemma 1.3(c).

Lemma 1.3(c) is stated without proof in [8] and proven (for the case when all the spaces involved are Hausdorff) in [14, Theorem 4.4]. We here sketch the proof of the general case (without separation axioms).

Lemma 1.3(c). *If X and Y are k-spaces and $f: X \to X'$, $g: Y \to Y'$ are quotient (onto) maps, then $f \times g : X \times_k Y \to X' \times_k Y'$ is a quotient map.*

Sketch of Proof. *Step* 1. It is observed in [14] that it is sufficient to consider the special case when $Y = Y'$ and g is the identity map $1 : Y \to Y$. The second paragraph of the proof in [14] handles the special case when $Y = Y'$ and Y and X' are compact Hausdorff; X is a k-space, but no separation axiom is needed for X.

Step 2. If C is a compact Hausdorff space and B is any k-space, then $C \times_t B = C \times_k B$ [1, Theorem 2.11].

Step 3. We now show that for arbitrary k-spaces X, X', and Y such that $f : X \to X'$ is an (onto) quotient, $f \times 1 = X \times_k Y \to X' \times_k Y$ is a quotient. Let $A \subset X' \times_k Y$ be such that $(f \times 1)^{-1}(A)$ is closed in $X \times_k Y$. It will suffice to show that A is compactly closed in $X' \times_t Y$. Given $\phi : C \to X' \times_t Y$, define $\pi_1\phi : C \to X'$, $\pi_2\phi : C \to Y$, and $\hat{\phi} = \pi_1\phi \times \pi_2\phi : C \times C \to X' \times_t Y$. Let $d : C \to C \times C$ be the diagonal map $d(c) = (c, c)$; then $\hat{\phi}d = \phi$. We must show $\phi^{-1}(A)$ closed in C; to do this, it will suffice to show $\hat{\phi}^{-1}(A)$ closed in $C \times C$.

Let $\hat{C}$ be the subspace of $C \times_k X$ given by $\hat{C} = \{(c, x) \mid \pi_1\phi(c) = f(x)\}$ and let $\rho_1 : \hat{C} \to C$ and $\rho_2 : \hat{C} \to X$ be the projections. Then ρ_1 is a quotient map ($C \times_k X = C \times_t X$ by Step 2, and ρ_1 is onto since f is onto), and (2) commutes:

$$\begin{array}{ccc} \hat{C} \times_k C & \xrightarrow{\rho_1 \times 1} & C \times C \\ \Big\downarrow{\scriptstyle \rho_2 \times \pi_2\phi} & & \Big\downarrow{\scriptstyle \hat{\phi} = \pi_1\phi \times \pi_2\phi} \\ X \times_k Y & \xrightarrow{f \times 1} & X' \times_k Y \end{array} \tag{2}$$

Now since $(f \times 1)^{-1}(A)$ is closed in $X \times_k Y$,

$$(\rho_2 \times \pi_2\phi)^{-1}(f \times 1)^{-1}(A) = (\rho_1 \times 1)^{-1}\hat{\phi}^{-1}(A),$$

is closed in $\hat{C} \times_k C$. Since $\rho_1 \times 1$ is a quotient map by Step 1, $\hat{\phi}^{-1}(A)$ is closed, completing the proof.

References

[1] R. Brown, Ten topologies for $X \times Y$, Quart. J. Math. Oxford (2) 14 (1963) 303–319.

[2] L. Gillman and M. Jerison, Rings of Continuous Functions (Van Nostrand, Princeton, N.J., 1960).

[3] M.I. Graev, Free topological groups, Izv. Akad. Nauk SSSR Ser. Mat. 12 (1948) 279–324 (in Russian; English transl.: Am. Math. Soc. Transl. No. 35 (1951). Reprint, Am. Math. Soc. Transl. (1) 8 (1962) 395–364).

[4] E. Hewitt and K.A. Ross, Abstract Harmonic Analysis, Vol. 1, Die Grundlehren der Mathematischen Wissenschaften, Band 115 (Springer, Berlin, 1963).
[5] D.C. Hunt and S.A. Morris, Commutator subgroups of free topological groups, Second International Conference on Group Theory, Canberra 1973.
[6] J. Mack, S.A. Morris and E.T. Ordman, Free topological groups and the projective dimension of a locally compact abelian group, Proc. Am. Math. Soc. 40 (1973) 303–308.
[7] A.A. Markov, On free topological groups, C.R. (Doklady) Acad. Sci. URSS (N.S.) 31 (1941) 299–301. Bull. Acad. Sci. URSS Ser. Mat. [Izv. Akad. Nauk SSSR] 9 (1945) 3–64 (in Russian with English summary; English transl.: Am. Math. Soc. Transl. 30 (1950) 11–88. Reprint Am. Math. Soc. Transl. (1) 8 (1962) 195–272).
[8] M.C. McCord, Classifying spaces and infinite symmetric products, Trans. Am. Math. Soc. 146 (1969) 273–298.
[9] J. Milnor, Construction of universal bundles, I, Ann. Math. 63 (1956) 272–284.
[10] E.C. Nummela, Homological algebra of *K*-modules.
[11] E.C. Nummela, *K*-groups generated by *K*-spaces.
[12] E.T. Ordman, Free products in the category of *k*-groups.
[13] L. Steen and J. Seebach, Jr., Counterexamples in Topology (Holt, Rinehart and Winston, New York, 1970).
[14] N.E. Steenrod, A convenient category of topological spaces, Mich. Math. J. 14 (1967) 133–152.
[15] R.C. Steinlage, On Ascoli theorems and the product of *k*-spaces, Kyungpook Math. J. 12 (1972) 145–151.
[16] B.V.S. Thomas, Free topological groups, Gen. Topology Appl. 4 (1974) 51–72.