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The existence of solutions for the system of vector quasi-equilibrium problems in topological order spaces

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ABSTRACT

This paper introduces two kinds of quasiconcave mappings which are different from the usual quasiconcave function. We establish a result for the existence of solutions for the system of vector quasi-equilibrium problems in the frame of topological order, by providing a maximal elements version of the well known Browder fixed points theorem.

1. Introduction

Topological vector spaces provide the usual mathematical framework in many fields. The linear feature maybe a restricted condition that seems very necessary for some problems. Avoiding the linear structure in topological semilattices, some results like the existence of a maximal element in topological spaces hold also, see [1].

In relation to a sup-semilattice, it is a partially ordered set X with the partial ordering denoted by $\leq$, in which any pair (x, x') of its elements has a least upper bound $x \vee x'$. If x and x' are elements in a partially ordered set $(X, \leq)$, and in the case where $x \leq x'$, the set $[x, x'] = \{y \in X : x \leq y \leq x'\}$ is called an order interval.

The system of vector quasi-equilibrium problems (SVQEP) is a unified way to research some nonlinear problems such as vector equilibrium problems (VEP), vector variational inequality [2], and vector complementarity problems [3] and so on. In recent decades, vector equilibrium problems [4–6] and vector quasi-equilibrium problems have been studied extensively [7–10]. Most works concerning the vector quasi-equilibrium problems concentrate on topological vector spaces. We intend to study the existence of solutions for the SVQEP avoiding the linear feature.

In this paper, we introduce two kinds of quasiconcave mappings. Examples are given to show that they are different from the usual quasiconcave function. We obtain a result for the existence of solutions for the SVQEP by providing a maximal elements version for Browder fixed points theorem in a topological sup-semilattice.

2. Preliminaries

Let $(X, \leq)$ be a sup-semilattice and $A \subseteq X$ a non-empty finite subset. Then the set $\Delta A = \cup_{x \in A}[x, \sup A]$ is well defined and has the obvious properties: $A \subseteq \Delta A$ and $\Delta A \subseteq \Delta A'$ if $A \subseteq A'$.

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Definition 2.1 ([1]). A subset $E \subseteq X$ is $\triangle$-convex, if for any non-empty finite subset $A \subseteq E$, we have $\triangle A \subseteq E$. There are many conditions of equivalence for E being a $\triangle$-convex set. We just give the following conditions:

(a) if $x, x' \in E$, then its least upper bound $x \vee x' \in E$.
(b) if $x, x' \in E$ and $x \leq x'$, then the order interval $[x, x'] \subset E$.

It is easy to prove that the intersection of two $\triangle$-convex sets is $\triangle$-convex as well.

A topological sup-semilattice is a topological space X, with a sup-semilattice as its partial ordering denoted by $\leq$, for which $f : X \times X \to X$ with $(x, x') \to x \vee x'$ is a continuous function.

Lemma 2.1 ([1]). *Let X be a topological sup-semilattice with path-connected intervals, $X_0 \subseteq X$ a non-empty subset of X, and $R \subseteq X_0 \times X$ a binary relation such that:*

(i) *for each $x \in X_0$, the set $R(x)$ is not empty and closed in $R(X_0)$;*
(ii) *there exists $x_0 \in X_0$, such that the set $R(x_0)$ is compact;*
(iii) *for any non-empty finite subset $A \subseteq X_0$, $\triangle A \subseteq \cup_{x \in A} R(x)$.*

Then the set $\cap_{x \in X_0} R(x)$ is not empty.

Assume that H is a topological vector space, θ is the zero element in H. A subset $C \subset H$ is called a cone if, for any $y \in C$ and real number $t > 0$, $ty \in C$. A cone C is convex if C is a convex set. If $C \cap -C = \{\theta\}$, we call it a pointed cone.

Lemma 2.2 ([11]). *Let C be a convex and pointed cone with $\operatorname{int} C \neq \emptyset$ in a Hausdorff locally convex topological vector space H. For any $y_1, y_2 \in \operatorname{int} C$, there exists an $\alpha \in (0, 1)$ such that $y_2 - \alpha y_1 \in \operatorname{int} C$.*

Lemma 2.3 ([12]). *Let C be a convex cone with $\operatorname{int} C \neq \emptyset$ in a Hausdorff locally convex topological vector space H, then $\operatorname{int} C$ is also a convex cone.*

Definition 2.2. Let X be a topological sup-semilattice, H a topological vector space with a cone $C \subset H$. The vector valued function $\varphi : X \to H$ is $C_\triangle$-quasiconcave if, for any non-empty two points subset $A = \{x_1, x_2\} \subset X$ and $y \in H$, $\varphi(A) \subset y + C \Rightarrow \varphi(\triangle A) \subset y + C$.

Lemma 2.4. *Let X be a topological sup-semilattice, H a Hausdorff locally convex topological vector space, and $C \subset H$ a closed, convex and pointed cone. If the vector valued function $\varphi : X \to H$ is $C_\triangle$-quasiconcave, then the set $A = \{x : \varphi(x) \in \operatorname{int} C\}$ is $\triangle$-convex.*

Proof. For any $x_1, x_2 \in A$, we have $\varphi(x_1) \in \operatorname{int} C$ and $\varphi(x_2) \in \operatorname{int} C$. From Lemma 2.2, there exists an $\alpha \in (0, 1)$, such that $\varphi(x_2) - \alpha\varphi(x_1) \in \operatorname{int} C$, that is $\varphi(x_2) \in \operatorname{int} C + \alpha\varphi(x_1) \subset \alpha\varphi(x_1) + C$. Because $1 - \alpha > 0$ and from Lemma 2.3, we know that $(1-\alpha)\varphi(x_1) \in \operatorname{int} C$, then $\varphi(x_1) \in \alpha\varphi(x_1) + \operatorname{int} C \subset \alpha\varphi(x_1) + C$. Let B be the set $\{x_1, x_2\}$. We obtain that $\varphi(B) \subset \alpha\varphi(x_1) + C$. Since φ is $C_\triangle$-quasiconcave, $\varphi(\triangle B) \subset \alpha\varphi(x_1) + C \subseteq \operatorname{int} C$. Because $\sup B = x_1 \vee x_2$, $\triangle B = [x_1, x_1 \vee x_2] \cup [x_2, x_1 \vee x_2]$, then $\varphi(x_1 \vee x_2) \in \operatorname{int} C$, therefore, $x_1 \vee x_2 \in A$. If $x_1 \leq x_2$, $[x_1, x_2] \subseteq \triangle B$, consequently, $\varphi([x_1, x_2]) \subseteq \operatorname{int} C$, it follows that $[x_1, x_2] \subset A$. Therefore, A is a $\triangle$-convex set. □

Definition 2.3. Let X be a sup-semilattice, H a topological vector space, with a cone $C \subset H$. $\varphi : X \to H$ is said to be $C_\triangle$-quasiconcave-like if, for any $x_1, x_2 \in X$, $\varphi(\triangle \{x_1, x_2\}) \in \varphi(x_1) + C$ or $\varphi(\triangle \{x_1, x_2\}) \in \varphi(x_2) + C$.

Remark 2.1. (a) If φ is $C_\triangle$-quasiconcave-like function mapping from a sup-semilattice X to a topological vector space H, where C is a closed, convex and pointed cone in H, then the set $\{x : \varphi(x) \in \operatorname{int} C\}$ is a $\triangle$-convex subset of X.
(b) Let $\varphi : \mathbb{R} \to \mathbb{R}$ ($\mathbb{R}$denotes the real numbers) and the partial order on $\mathbb{R}$ be "$\leq$" (less than or equal to), then Definitions 2.2 and 2.3 coincide, and they coincide with the usual quasiconcave function, that is for any $x_1, x_2, y \in \mathbb{R}$, $\varphi(x_1) \leq y$ and $\varphi(x_2) \leq y \Rightarrow \varphi(\lambda x_1 + (1 - \lambda)x_2) \leq y$, $\forall \lambda \in [0, 1]$.

The $C_\triangle$-quasiconcave, $C_\triangle$-quasiconcave-like and usual quasiconcave functions are independent of each other. To illustrate this point, we give the following example.

Example 2.1. Let $X = [0, 1] \times [0, 1]$. We set $x^1 \leq x^2$ denoting that $x^2 \in x^1 + \mathbb{R}^2_+$, $\forall x^1, x^2 \in X$, where $\mathbb{R}^2_+ = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \geq 0, y_2 \geq 0\}$. It is obvious that $(X, \leq)$ is a sup-semilattice, in which $x^1 \vee x^2 = (\max(x^1_1, x^2_1), \max(x^1_2, x^2_2))$, $\forall x^i = (x^i_1, x^i_2) \in X$, $i = 1, 2$.

(a) Let $\varphi : X \to \mathbb{R}$ and $C = -\mathbb{R}_+ = \{y \in \mathbb{R} : y \leq 0\}$ such that $\varphi(x) \equiv c$, then φ is $C_\triangle$-quasiconcave and $C_\triangle$-quasiconcave-like, and also a usual quasiconcave function (see Remark 2.1).
(b) Let $\varphi : X \to \mathbb{R}$ and $C = -\mathbb{R}_+$ such that $\varphi(x) = \max(x_1, x_2)$, $\forall x = (x_1, x_2) \in X$, then φ is $C_\triangle$-quasiconcave and $C_\triangle$-quasiconcave-like.
(c) Let $\varphi : X \to X$ and $C = \mathbb{R}^2_+$ such that $\varphi(x) = x$, $\forall x \in X$. Then φ is $C_\triangle$-quasiconcave, and it is not $C_\triangle$-quasiconcave-like.

(d) Let $\varphi : X \to \mathbb{R}$ and $C = -\mathbb{R}_+$ such that $\varphi(x) = (1 - x_1)(1 - x_2), \forall x = (x_1, x_2) \in X$. Then φ is $C_\triangle$-quasiconcave and $C_\triangle$-quasiconcave-like, but it is not a usual quasiconcave function.

We just give the proof of (d). For any $x^i = (x_1^i, x_2^i) \in X, i = 1, 2$, denote $\min(x_1^1, x_1^2)$, $\max(x_1^1, x_1^2)$, $\min(x_2^1, x_2^2)$, and $\max(x_2^1, x_2^2)$ by a, b, c and d respectively. If $x^1 \le x^2$ or $x^2 \le x^1$,

$$\triangle\{x^1, x^2\} = \{x : \lambda x^1 + (1 - \lambda)x^2\}, \quad \forall \lambda \in [0, 1].$$

If $x^1 \not\le x^2$ and $x^2 \not\le x^1$,

$$\triangle\{x^1, x^2\} = \{x : x_2 = d, a \le x_1 \le b\} \cup \{x : x_1 = b, c \le x_2 \le d\}.$$

Clearly, we have $\varphi(x^1) \le y, \varphi(x^2) \le y \Rightarrow \varphi(x) \le y, \forall x \in \triangle \{x^1, x^2\}, \forall y \in \mathbb{R}$. However, $\varphi(1, 0) \le 0, \varphi(0, 1) \le 0$, while $\varphi(1/2, 1/2) = 1/4$, that is, φ is definitely not a usual quasiconcave function.

Now we give the notion of the system of vector quasi-equilibrium problems that we will study in this paper.

Let I be an index set. For each $i \in I$, K_i is a topological sup-semilattice and Y_i is a topological vector space. Denote $K = \prod_{i \in I} K_i$, $K_{-i} = \prod_{j \in I \setminus i} K_j$. $C_i \subset Y_i$ is a closed, convex and pointed cone with $\operatorname{int} C_i \ne \emptyset$. For each $i \in I$, $\varphi_i : K_i \times K_{-i} \times K_i \to Y_i$ is a vector valued function and G_i is a multivalued map from K_{-i} to 2^{K_i}. The system of vector quasi-equilibrium problems $(K_i, Y_i, C_i, \varphi_i, G_i)_{i \in I}$ is to find $\bar{x} \in K$, such that

$$\forall i \in I, \quad \bar{x}_i \in G_i(\bar{x}_{-i}) : \varphi(\bar{x}, y_i) \notin -\operatorname{int} C_i, \ \forall y_i \in G_i(\bar{x}_{-i}).$$

If $|I| = 1$, denote K_i, Y_i and C_i by K, Y and C for short, respectively. Let $\varphi : K \to Y$, then the SVQEP turns to be a vector equilibrium problem (K, Y, C, φ). That is to find $\bar{x} \in K$, such that

$$\varphi(\bar{x}, y) \notin -\operatorname{int} C, \quad \forall y \in K.$$

3. The existence of solutions for the SVQEP

Firstly, we give a maximal element version of the Browder fixed point theorem in a topological sup-semilattice.

Theorem 3.1. *Let X be a compact topological sup-semilattice with path connected interval, $S : X \to 2^X$ a multivalued map on X with the conditions:* (i) $\forall x \in X, S(x)$ *is* $\triangle$*-convex;* (ii) $\forall y \in X, S^{-1}(y) = \{x \in X : y \in S(x)\} \subset X$ *is open in* X*;* (iii) $\forall x \in X, x \notin S(x)$. *Then there exists an $\bar{x} \in X$, such that $S(\bar{x}) = \emptyset$.*

Proof. Assume that $S(x)$ is non-empty for any $x \in X$. Let $R = \{(x, y) \in X \times X : y \in X \setminus S^{-1}(x)\}$ be a binary relationship on X. Clearly, for any $x \in X$, the set $R(x) = \{y \in X : (x, y) \in R\}$ is non-empty and closed. For any finite subset $A \subseteq X$, if there is $\triangle A \subseteq \cup_{x \in A} R(x)$, we have $\cap_{x \in X} R(x) \ne \emptyset$ by Lemma 2.1. In fact, we know that $X \subseteq \cup_{x \in X} S^{-1}(x)$, then,

$$\cap_{x \in X}(X \setminus S^{-1}(x)) = \emptyset.$$

It follows that $\cap_{x \in X} R(x) = \emptyset$, a contradiction. Therefore, there exists a finite set $A_0 \subseteq X$ denoted by $\{x_1, x_2, \ldots, x_n\}$, such that $\triangle A_0 \nsubseteq \cup_{x \in A_0} R(x)$. Furthermore, we can find at least one point $z_0 \in \triangle A_0$, however,

$$z_0 \notin \cup_{x \in A_0} R(x) = \cup_{x \in A_0}(X \setminus S^{-1}(x)).$$

Then we obtain that $z_0 \in S^{-1}(x_i), \ i = 1, 2, \ldots, n$, then $x_i \in S(z_0), \ i = 1, 2, \ldots, n$, that is $A_0 \subseteq S(z_0)$. Because $S(z_0)$ is a $\triangle$-convex set, we have $\triangle(A_0) \subseteq S(z_0)$. As a result, it holds that $z_0 \in S(z_0)$, a contradiction to condition (iii). Therefore, the assumption of $S(x) \ne \emptyset$ for any $x \in X$ is not true. Consequently, there exists an $\bar{x} \in X$, such that $S(\bar{x}) = \emptyset$. □

Theorem 3.2. *Let $(K_i, Y_i, C_i, \varphi_i, G_i)_{i \in I}$ be a system of vector quasi-equilibrium problems. For each $i \in I$, K_i is a compact topological sup-semilattice with path connected intervals, Y_i is a Hausdorff locally convex topological vector space, and G_i is non-empty $\triangle$-convex value mapping. For any $y_i \in K_i$, the set $G_i^{-1}(y_i) = \{x_{-i} \in K_{-i} : y_i \in G_i(x_{-i})\}$ is open in K_{-i}. If the SVQEP satisfies that:*

(i) $\forall i \in I, \forall x \in K, \varphi_i(x_i, x_{-i}, \cdot)$ *is* $-C_{i\triangle}$*-quasiconcave;*
(ii) $\forall i \in I, \forall x \in K, \{z_i \in K_i : \varphi_i(z_i, x_{-i}, x_i) \notin -\operatorname{int} C_i\}$ *is closed in* $G_i(x_{-i})$*;*
(iii) $\forall i \in I, \forall x \in K,$ *if* $x_i \in G_i(x_{-i}), \varphi(x_i, x_{-i}, x_i) \notin -\operatorname{int} C_i$*;*
(iv) $\forall i \in I, \forall x \in K, \varphi_i(\cdot, x_{-i}, x_i)$ *is* $C_{i\triangle}$*-quasiconcave-like;*
(v) $\forall i \in I, \forall z_i, y_i \in K_i, \{x_{-i} \in K_{-i} : \varphi_i(z_i, x_{-i}, y_i) \in -\operatorname{int} C_i\}$ *is closed.*

Then there is a solution for this SVQEP.

Proof. Let $B(x) = \times_{i \in I} B_i(x_{-i}), \forall x \in K$, where for each $i \in I$, $B_i(x_{-i}) = \{z_i \in G_i(x_{-i}) : \varphi_i(z_i, x_{-i}, y_i) \notin -\operatorname{int} C_i, \forall y_i \in G_i(x_{-i})\}$. We will complete our proof in the following three steps (a), (b) and (c). For each $i \in I$ and any $x \in K$,

(a) We set $Q_i(z_i) = \{y_i \in G_i(x_{-i}) : \varphi_i(z_i, x_{-i}, y_i) \in -\text{int}\, C_i\}$, $\forall z_i \in G_i(x_{-i})$. Thus, Q_i is a multivalued map on $G_i(x_{-i})$. Then for any $z_i \in G_i(x_{-i})$, $Q_i(z_i) = G_i(x_{-i}) \cap Y_i(z_i, x_{-i})$, where $Y_i(z_i, x_{-i}) = \{y_i \in K_i : \varphi_i(z_i, x_{-i}, y_i) \in -\text{int}\, C_i\}$. From (i), $\varphi_i(z_i, x_{-i}, \cdot)$ is $-C_{i\triangle}$-quasiconcave, then $Y_i(z_i, x_{-i})$ is $\triangle$-convex by Lemma 2.4. Because $G_i(x_{-i})$ is also $\triangle$-convex, $Q_i(z_i)$ is $\triangle$-convex.

For each $y_i \in G_i(x_{-i})$, we have

$$Q_i^{-1}(y_i) = \{z_i \in G_i(x_{-i}) : y_i \in Q_i(z_i)\} = \{z_i \in G_i(x_{-i}) : \varphi_i(z_i, x_{-i}, y_i) \in -\text{int}\, C_i\}.$$

From (ii), $Q_i^{-1}(y_i)$ is open in $G_i(x_{-i})$.

In addition, if $x_i \in G_i(x_{-i})$, $\varphi_i(x_i, x_{-i}, x_i) \notin -\text{int}\, C_i$ by the condition (iii), that is $x_i \notin Q_i(x_i)$.

Because of Theorem 3.1, there exists an $x_i^* \in G_i(x_{-i})$, such that $Q_i(x_i^*) = \emptyset$, that is $\varphi_i(x_i^*, x_{-i}, y_i) \notin -\text{int}\, C_i$, $\forall y_i \in G_i(x_{-i})$. Consequently, $B_i(x_{-i})$ is not empty, and $B(x)$ is also.

(b) For any $z_i^1, z_i^2 \in B_i(x_{-i})$, we have $z_i^j \in G_i(x_{-i})$ and $\varphi_i(z_i^j, x_{-i}, y_i) \notin -\text{int}\, C_i, \forall y_i \in G_i(x_{-i})$, $j = 1, 2$. From the condition (iv), $\varphi_i(\cdot, x_{-i}, y_i)$ is $C_{i\triangle}$-quasiconcave-like. Then for any $z_i \in \triangle \{z_i^1, z_i^2\}$, $y_i \in G_i(x_{-i})$, without loss of generality, we assume that $\varphi_i(z_i, x_{-i}, y_i) \in \varphi_i(z_i^1, x_{-i}, y_i) + C_i$, that is $\varphi_i(z_i^1, x_{-i}, y_i) \in \varphi_i(z_i, x_{-i}, y_i) - C_i$. If there is $y_i^0 \in G_i(x_{-i})$, such that $\varphi_i(z_i, x_{-i}, y_i^0) \in -\text{int}\, C_i$, then

$$\varphi_i(z_i^1, x_{-i}, y_i^0) \in \varphi_i(z_i, x_{-i}, y_i^0) - C_i \subset -\text{int}\, C_i - C_i \subseteq -\text{int}\, C_i,$$

a contradiction to $z_i^1 \in B_i(x_{-i})$. Therefore, $z_i \in G_i(x_{-i})$ and $\varphi_i(z_i, x_{-i}, y_i) \notin -\text{int}\, C_i$, $\forall y_i \in G_i(x_{-i})$, that is $z_i \in B_i(x_{-i})$, then $\triangle \{z_i^1, z_i^2\} \subseteq B_i(x_{-i})$, it follows that $B_i(x_{-i})$ is $\triangle$-convex.

(c) For each $z \in K$, denote

$$D_i(z_i) = \cap_{y_i \in G_i(x_{-i})} \{x_{-i} \in K_{-i} : \varphi_i(z_i, x_{-i}, y_i) \notin -\text{int}\, C_i\},$$

then we have

$$B_i^{-1}(z_i) = \{x_{-i} \in K_{-i} : z_i \in B_i(x_{-i})\} = \{x_{-i} \in K_{-i} : z_i \in G_i(x_{-i})\} \cap D_i(z_i).$$

The set $\{x_{-i} \in K_{-i} : \varphi_i(z_i, x_{-i}, y_i) \in -\text{int}\, C_i\}$ is closed according to (v), thus, D_i is open in K_{-i}. In addition, $G_i^{-1}(z_i) = \{x_{-i} \in K_{-i} : z_i \in G_i(x_{-i})\}$ is open. Consequently, we obtain $B_i^{-1}(z_i)$ is open, $B^{-1}(z)$ is also.

Up to now, $B(x)$ is non-empty and $\triangle$-convex; $B^{-1}(x)$ is open for any $x \in K$. If $x \notin B(x)$, $\forall x \in K$, then there exists an $\tilde{x}$ such that $B(\tilde{x}) = \emptyset$ by Theorem 3.1. This contradicts to (a) ($B(x)$ is non-empty). Therefore, we can find an $\bar{x} \in K$ such that $\bar{x} \in B(\bar{x})$. Obviously, $\bar{x}$ is a solution of the SVQEP. □

From Step (a) in the Theorem 3.2, we can easily obtain the existence of solutions for VEP as follows.

Corollary 3.1. *Let (K, Y, C, φ) be a vector equilibrium problem, where K is a compact topological sup-semilattice with path connected intervals, Y is a Hausdorff locally convex topological vector space. If the VEP satisfies the following conditions:*

(i) $\forall x \in K$, $\varphi(x, \cdot)$ *is* $-C_{i\triangle}$ *quasiconcave or* $-C_{i\triangle}$ *quasiconcave-like;*
(ii) $\forall y \in K$, $\{x : \varphi(x, y) \notin -\text{int}\, C\}$ *is closed in* K*;*
(iii) $\forall x \in K$, $\varphi(x, x) \notin -\text{int}\, C$.

Then this VEP has a solution.

Example 3.1. Let $K = [0, 1] \times [0, 1]$, $C = \mathbb{R}_+$. The $(K, \leq)$ is a sup-semilattice, in which $x^1 \leq x^2$ means that $x^2 \in x^1 + \mathbb{R}_+^2$, $\forall x^1, x^2 \in K$.

(a) For any $x = (x_1, x_2)$, $y = (y_1, y_2) \in K$, the function $\varphi_\alpha(x, y) = (1-\alpha y_1)(1-\alpha y_2) - (1-x_1)(1-x_2)$, where $\alpha \in [0, 1]$, satisfies all the conditions in Corollary 3.1. For φ_1, we can find $\bar{x} = (1, 1)$ is the unique solution for the VEP, $(K, \mathbb{R}, C, \varphi_1)$.
(b) For any $x = (x_1, x_2)$, $y = (y_1, y_2) \in K$, let $\varphi(x, y) = (1 + x_1 - y_1)(1 + x_2 - y_2)$, then the function φ meets all the conditions in Corollary 3.1. The set of solutions for the $(K, \mathbb{R}, C, \varphi)$ is the overall K.

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