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# The Segal–Bargmann transform for noncompact symmetric spaces of the complex type

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## Abstract

We consider the generalized Segal–Bargmann transform, defined in terms of the heat operator, for a noncompact symmetric space of the complex type. For radial functions, we show that the Segal–Bargmann transform is a unitary map onto a certain  $L^2$  space of meromorphic functions. For general functions, we give an inversion formula for the Segal–Bargmann transform, involving integration against an “unwrapped” version of the heat kernel for the dual compact symmetric space. Both results involve delicate cancellations of singularities.

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*Keywords:* Segal–Bargmann transform; Symmetric space; Quantization; Analytic continuation; Heat kernel

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**1. Introduction**

The Segal–Bargmann transform for  $\mathbb{R}^d$  [Se2,Se3,Se4,Ba] is a widely used tool in mathematical physics and harmonic analysis. The transform is a unitary map  $C_t$  from  $L^2(\mathbb{R}^d)$  onto  $\mathcal{HL}^2(\mathbb{C}^d, \nu_t)$ , where  $\nu_t$  is a certain Gaussian measure on  $\mathbb{C}^d$  (depending on a positive parameter  $t$ ) and where  $\mathcal{HL}^2$  denotes the space of holomorphic functions that are square integrable with respect to the indicated measure. (See Section 2 for details.) From the point of view of harmonic analysis, one can think of the Segal–Bargmann transform as combining information about a function  $f(x)$  on  $\mathbb{R}^d$  with information about the Fourier transform  $\hat{f}(\xi)$  into a single holomorphic function  $(C_t f)(x + i\xi)$ . From the point of view of quantum mechanics for a particle moving in  $\mathbb{R}^d$ , one can think of the Segal–Bargmann transform as a unitary map between the “position Hilbert space”  $L^2(\mathbb{R}^d)$  and the “phase space Hilbert space”  $\mathcal{HL}^2(\mathbb{C}^d, \nu_t)$ . In this setting, the parameter  $t$  can be interpreted as Planck’s constant. Conceptually, the advantage of applying the Segal–Bargmann transform is that it gives a description of the state of the particle that is closer to the underlying classical mechanics, because we now have a function on the classical phase space rather than on the classical configuration space. See Section 2, [Fo,H4], for more information about the Segal–Bargmann transform for  $\mathbb{R}^d$  and its uses.

In the paper [H1], Hall introduced a generalization of the Segal–Bargmann transform in which the configuration space  $\mathbb{R}^d$  is replaced by a connected compact Lie group  $K$  and the phase space  $\mathbb{C}^d$  is replaced by the complexification  $K_{\mathbb{C}}$  of  $K$ . (See also the expository papers [H4,H6,H9].) The complex group  $K_{\mathbb{C}}$  can also be identified in a natural way with the cotangent bundle  $T^*(K)$ , which is the usual phase space associated to the configuration space  $K$ . A main result of Hall [H1] is a unitary map  $C_t$  from  $L^2(K)$  onto  $\mathcal{HL}^2(K_{\mathbb{C}}, \nu_t)$ , where  $\nu_t$  is a certain heat kernel measure on the complex group  $K_{\mathbb{C}}$ . The transform itself is given by applying the time- $t$  heat operator to a function  $f$  in  $L^2(K)$  and then analytically continuing the result from  $K$  to  $K_{\mathbb{C}}$ . The paper [H2] then gave an inversion formula for  $C_t$  in which to recover the function  $f$  on  $K$  one integrates the holomorphic function  $C_t f$  over each fiber in  $T^*(K) \cong K_{\mathbb{C}}$  with respect to a suitable heat kernel measure. See also [KTX] for a study of the Segal–Bargmann transform, defined in terms of the heat operator, on the Heisenberg group.

The motivation for the generalized Segal–Bargmann transform for  $K$  was work of Gross in stochastic analysis, specifically the Gross ergodicity theorem [Gr] for the loop group over  $K$ . See [GM,H6,H8,HS] for connections between the generalized Segal–Bargmann transform and stochastic analysis. The generalized Segal–Bargmann trans-

form has also been used in the theory of loop quantum gravity [A, Das1, Das2, Th, TW1, TW2]. It has a close connection to the canonical quantization of (1 + 1)-dimensional Yang–Mills theory [DH, H5, Wr]. It can be understood from the point of view of geometric quantization [FMMN1, FMMN2, H7]. Most recently, it has been used in studying nonabelian theta functions and the conformal blocks in WZW conformal field theory [FMN1, FMN2]. (See also [Ty].) See the paper [H6] for a survey of the generalized Segal–Bargmann transform and related notions.

In the paper [St], Stenzel extended the results of [H1, H2] from the case of compact Lie groups to the case of general compact symmetric spaces. We give here a schematic description of Stenzel’s results; see Section 5 for details. If  $X$  is a compact symmetric space, there is a natural “complexification”  $X_{\mathbb{C}}$  of  $X$ . There is a natural diffeomorphism between the cotangent bundle  $T^*(X)$  and the complexification  $X_{\mathbb{C}}$ . Under this diffeomorphism, each fiber in  $T^*(X)$  maps to a set inside  $X_{\mathbb{C}}$  that can be identified with the *dual noncompact symmetric space to  $X$* . (For example, if  $X$  is the  $d$ -sphere  $S^d$ , then each fiber in  $T^*(S^d)$  gets identified with hyperbolic  $d$ -space.) Thus the complexified symmetric space  $X_{\mathbb{C}}$  is something like a product of the compact symmetric space  $X$  and the dual noncompact symmetric space. Since each fiber in  $T^*(X) \cong X_{\mathbb{C}}$  is identified with this noncompact symmetric space, we can put on each fiber the *heat kernel measure* for that noncompact symmetric space (based at the origin in the fiber).

The Segal–Bargmann transform now consists of applying the time- $t$  heat operator to a function in  $L^2(X)$  and analytically continuing the resulting function to  $X_{\mathbb{C}}$ . The first main result is an inversion formula: to recover a function from its Segal–Bargmann transform, one simply integrates the Segal–Bargmann transform over each fiber in  $T^*(X) \cong X_{\mathbb{C}}$  with respect to the appropriate heat kernel measure. The second main result is an isometry formula: the  $L^2$  norm of the original function can be computed by integrating the absolute-value squared of the Segal–Bargmann transform, first over each fiber using the heat kernel measure and then over the base with using the Riemannian volume measure. See Theorem 10 in Section 5 for details. See Section 3.4 of [H6] for more information on the transform for general compact symmetric spaces and [H9, HM1, HM2, KR1, KR2] for more on the special case in which  $X$  is a  $d$ -sphere.

Since we now have a Segal–Bargmann transform for the Euclidean symmetric space  $\mathbb{R}^d$  and for compact symmetric spaces, it is natural to consider also the case of noncompact symmetric spaces. Indeed, since the duality relationship between compact and noncompact symmetric spaces is a symmetric one, it might seem at first glance as if one might be able to simply reverse the roles of the compact and the noncompact spaces to obtain a transform starting on a noncompact symmetric space. Unfortunately, further consideration reveals significant difficulties with this idea. First, if  $X$  is a noncompact symmetric space, then the fibers in  $T^*(X)$  are not compact and therefore cannot be identified with the compact dual to  $X$ . (For example, if  $X$  is hyperbolic  $d$ -space, then the fibers in  $T^*(X)$  are diffeomorphic to  $\mathbb{R}^d$  and not to  $S^d$ .) Second, if one applies the time- $t$  heat operator to a function on a noncompact symmetric space  $X$  and then tries to analytically continue, one encounters singularities that do not occur in the compact case.

The present paper is a first step in overcoming these difficulties. (See the end of this section for other recent work in this direction.) We consider noncompact symmetric

spaces of the “complex” type, namely, those that can be described as  $G/K$ , where  $G$  is a connected *complex* semisimple group and  $K$  is a maximal compact subgroup of  $G$ . (The simplest example is hyperbolic 3-space.) The complex case is nothing but the noncompact dual of the compact group case. For noncompact symmetric spaces of the complex type, we obtain two main results.

Our first main result is an isometry formula for the Segal–Bargmann transform on the space of radial functions. We state this briefly here; see Section 3 for details. Consider a function  $f$  in  $L^2(G/K)$  ( $G$  complex) that is “radial” in the symmetric space sense, that is, invariant under the left action of  $K$  on  $G/K$ . Let  $F = e^{t\Delta/2} f$  and consider the map

$$X \rightarrow F(e^X), \quad X \in \mathfrak{p}, \tag{1}$$

where the Lie algebra  $\mathfrak{g}$  of  $G$  is decomposed in the usual way as  $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ . We show that map (1) has a *meromorphic* (but usually not holomorphic) extension from  $\mathfrak{p}$  to  $\mathfrak{p}_{\mathbb{C}} := \mathfrak{p} + i\mathfrak{p}$ . The main result of Section 3 is that there exist a constant  $c$  and a holomorphic function  $\delta$  on  $\mathfrak{p}_{\mathbb{C}}$  such that for all radial  $f$  in  $L^2(G/K)$  we have

$$\int_{G/K} |f(x)|^2 dx = e^{ct} \int_{\mathfrak{p}_{\mathbb{C}}} \left| F(e^{X+iY}) \right|^2 |\delta(X+iY)|^2 \frac{e^{-|Y|^2/t}}{(\pi t)^{d/2}} dX dY, \quad F = e^{t\Delta/2} f. \tag{2}$$

There is a “cancellation of singularities” occurring here: although in most cases the function  $F(e^{X+iY})$  is singular at certain points, the singularities occur only at points where  $\delta(X+iY)$  is zero. Thus, the singularities in  $F(e^{X+iY})$  are canceled by the zeros in the density of the measure occurring on the right-hand side of (2). Furthermore, by considering radial functions, we are introducing a distinguished basepoint (the identity coset). Thus, in the radial case, we are able to use the complexified tangent space at the basepoint (namely,  $\mathfrak{p}_{\mathbb{C}}$ ) as our “complexification” of  $G/K$ , and we simply do not attempt to identify  $\mathfrak{p}_{\mathbb{C}}$  with  $T^*(G/K)$ . Of course, because we are treating the identity coset differently from other points, this approach is not  $G$ -invariant and is not the correct approach for the general (nonradial) case.

Our second main result is an inversion formula for the Segal–Bargmann transform of general (not necessarily radial) functions. We state this briefly here; see Section 4 for details. We continue to assume that  $G$  is a connected *complex* semisimple group and  $K$  a maximal compact subgroup. For each point  $x$  in  $G/K$ , we have the geometric exponential map  $\exp_x$  taking the tangent space  $T_x(G/K)$  into  $G/K$ . Let  $f$  be in  $L^2(G/K)$  and let  $F = e^{t\Delta/2} f$ . Then, for each  $x \in G/K$ , the function

$$X \rightarrow F(\exp_x X), \quad X \in T_x(G/K), \tag{3}$$

admits an analytic continuation to some ball around zero. For each  $x \in G/K$ , define

$$L(x, R) = e^{ct/2} \int_{\substack{Y \in T_x(G/K) \\ |Y| \leq R}} F(\exp_x iY) \delta(iY) \frac{e^{-|Y|^2/2t}}{(2\pi t)^{d/2}} dY$$

for all sufficiently small  $R$ . (Here the constant  $c$  and the function  $\delta$  are the same as in the isometry formula (2).)

Our main result is that for each  $x$  in  $G/K$ ,  $L(x, R)$  admits a real-analytic continuation in  $R$  to  $(0, \infty)$  and, if  $f$  is sufficiently regular,

$$f(x) = \lim_{R \rightarrow \infty} L(x, R).$$

We may write this informally as

$$f(x) = \text{“} \lim_{R \rightarrow \infty} \text{”} e^{ct/2} \int_{|Y| \leq R} F(\exp_x iY) \delta(iY) \frac{e^{-|Y|^2/2t}}{(2\pi t)^{d/2}} dY, \quad (4)$$

where the expression “ $\lim_{R \rightarrow \infty}$ ” means that we interpret the right-hand side of (4) literally for small  $R$  and then extend to large  $R$  by means of analytic continuation.

As in the isometry formula for radial functions, there is a cancellation of singularities here that allows  $L(x, R)$  to extend analytically to  $(0, \infty)$ , even though  $F(\exp_x iY)$  itself may have singularities for large  $Y$ . Because of the rotationally invariant nature of the integral in (4), the integral only “sees” the part of the function  $F(\exp_x iY)$  that is rotationally invariant. Taking the rotationally invariant part eliminates some of the singularities in  $F(\exp_x iY)$ . The remaining singularities are canceled by the zeros in the function  $\delta(iY)$ .

The measure against which we are integrating  $F(\exp_x iY)$  in (4), namely,

$$d\sigma_t(Y) = e^{ct/2} \delta(iY) \frac{e^{-|Y|^2/2t}}{(2\pi t)^{d/2}} dY$$

is closely related to the heat kernel measure on the compact symmetric space dual to  $G/K$ . Specifically, it is an “unwrapped” version of that heat kernel measure, in a precise sense described in Section 4.

The papers [H2,St] use the inversion formula for the Segal–Bargmann transform (for compact groups and compact symmetric spaces, respectively) to deduce the isometry formula. Since we now have an inversion formula for the Segal–Bargmann transform for noncompact symmetric spaces of the complex type, it is reasonable to hope to obtain an isometry formula as well, following the line of reasoning in [H2,St]. The hoped-for isometry formula in the complex case would involve integrating  $|F|^2$  over a tube of radius  $R$  (with respect to the appropriate measure) and then analytically continuing with respect to  $R$ . Since, however, there are many technicalities to attend to in carrying out this idea, we defer this project to a future paper. (See [H9] for an additional discussion of this matter.)

Meanwhile, it would be desirable to extend the results of this paper to other symmetric spaces of the noncompact type. Unfortunately, the singularities that occur in general are worse than in the complex case and are not as easily canceled out. We discuss the prospects for other symmetric spaces in Section 6.

We conclude this introduction by comparing our work here to other types of Segal–Bargmann transform for noncompact symmetric spaces. First, Ólafsson and Ørsted [OO] have introduced another sort of Segal–Bargmann transform for noncompact symmetric spaces, based on the “restriction principle.” This has been developed in [DOZ1,DOZ2] and used to study Laplace transforms and various classes of orthogonal polynomials connected to noncompact symmetric spaces. This transform does not involve the heat operator and is thus not directly comparable to the Segal–Bargmann transform in this paper.

Meanwhile, Krötz, Ólafsson, and Stanton (see [KS1,KS2,KOS]) have considered the Segal–Bargmann transform for a general symmetric space  $G/K$  of the noncompact type (not necessarily of the complex type), defined in the same way as here, in terms of the heat equation. In [KS2], Krötz and Stanton identify the maximal domain inside  $G_{\mathbb{C}}/K_{\mathbb{C}}$  to which a function of the form  $e^{t\Delta/2}f$  can be analytically continued. Then in [KOS], Krötz, Ólafsson, and Stanton give an isometry result identifying the image of  $L^2(G/K)$  under the Segal–Bargmann transform in terms of certain orbital integrals. There is also a cancellation of singularities in their approach, in that the pseudodifferential operator  $D$  in Theorem 3.3 of [KOS] is used to extend the orbital integrals into the range where the function involved becomes singular. It remains to be worked out how the results of [KOS] relate, in the complex case, to the isometry result suggested by the results we obtain in this paper.

## 2. Review of the $\mathbb{R}^d$ case

We give here a very brief review of results concerning the Segal–Bargmann transform for  $\mathbb{R}^d$ . We do this partly to put into perspective the results for noncompact symmetric spaces and partly because we will use the  $\mathbb{R}^d$  results in our analysis of the symmetric space case. See also Section 5 for a description of Stenzel’s results for the case of compact symmetric spaces.

In the  $\mathbb{R}^d$  case, we consider the “invariant” form of the Segal–Bargmann transform, which uses slightly different normalization conventions from Segal [Se4] or Bargmann [Ba]. (See [H4] or [H3] for a comparison of normalizations.) The transform is the map  $C_t$  from  $L^2(\mathbb{R}^d)$  into the space  $\mathcal{H}(\mathbb{C}^d)$  of holomorphic functions on  $\mathbb{C}^d$  given by

$$(C_t f)(z) = \int_{\mathbb{R}^d} (2\pi t)^{-d/2} e^{-(z-x)^2/2t} f(x) dx, \quad z \in \mathbb{C}^d.$$

Here  $(z-x)^2 = (z_1-x_1)^2 + \cdots + (z_d-x_d)^2$  and  $t$  is an arbitrary positive parameter. It is not hard to show that the integral is convergent for all  $z \in \mathbb{C}^d$  and the result is a holomorphic function of  $z$ .

Recognizing that the function  $(2\pi t)^{-d/2} e^{-(z-x)^2/2t}$  is (for  $z$  in  $\mathbb{R}^d$ ) the heat kernel for  $\mathbb{R}^d$ , we may also describe  $C_t f$  as

$$C_t f = \text{analytic continuation of } e^{t\Delta/2} f.$$

Here the analytic continuation is from  $\mathbb{R}^d$  to  $\mathbb{C}^d$  with  $t$  fixed. We take the Laplacian  $\Delta = \Sigma \partial^2 / \partial x_k^2$  to be a negative operator, so that  $e^{t\Delta/2}$  is the forward heat operator.

**Theorem 1** (Segal–Bargmann). *Let  $f$  be in  $L^2(\mathbb{R}^d)$  and let  $F = C_t f$ . Then we have the following results:*

1. The inversion formula. *If  $f$  is sufficiently regular we have*

$$f(x) = \int_{\mathbb{R}^d} F(x + iy) \frac{e^{-y^2/2t}}{(2\pi t)^{d/2}} dy \quad (5)$$

*with absolute convergence of the integral for all  $x$ .*

2. The isometry formula. *For all  $f$  in  $L^2(\mathbb{R}^d)$  we have*

$$\int_{\mathbb{R}^d} |f(x)|^2 dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |F(x + iy)|^2 \frac{e^{-y^2/t}}{(\pi t)^{d/2}} dy dx. \quad (6)$$

3. The surjectivity theorem. *For any holomorphic function  $F$  on  $\mathbb{C}^d$  such that the integral on the right-hand side of (6) is finite, there exists a unique  $f$  in  $L^2$  with  $F = C_t f$ .*

The reason for the “sufficiently regular” assumption in the inversion formula is to guarantee the convergence of the integral on the right-hand side of (5). It suffices to assume that  $f$  has  $n$  derivatives in  $L^2(\mathbb{R}^d)$ , with  $n > d/2$ . (See Section 2.1 of [H9].)

The isometry and surjectivity formulas are obtained by adapting results of Segal [Se4] or Bargmann [Ba] to our normalization of the transform. The inversion formula is elementary (e.g. [H9]) but does not seem to be as well known as it should be. The inversion formula is implicit in Theorem 3 of [Se1] and is essentially the same as the inversion formula for the  $S$ -transform in [Ku, Theorem 4.3]. In quantum mechanical language, the inversion formula says that the “position wave function”  $f(x)$  can be obtained from the “phase space wave function”  $F(x + iy)$  by integrating out the momentum variables (with respect to a suitable measure).

It should be noted that because  $F(x + iy)$  is holomorphic, there can be many different inversion formulas, that is, many different integrals involving  $F(x + iy)$  all of which yield the value  $f(x)$ . For example, we may think of the heat operator as a unitary map from  $L^2(\mathbb{R}^d)$  to the Hilbert space of holomorphic functions for which the right-hand side of (6) is finite. Then we may obtain one inversion formula by noting that the adjoint of a unitary map is its inverse. The resulting “inverse = adjoint” formula is sometimes described as “the” inversion formula for the Segal–Bargmann transform. Nevertheless, the inversion formula in (5) is *not* the one obtained by this method.

In light of what we are going to prove in Section 3, it is worth pointing out that we could replace “holomorphic” with “meromorphic” in the statement of Theorem 1. That is, we could describe  $F$  as the meromorphic extension of  $e^{t\Delta/2} f$  from  $\mathbb{R}^d$  to  $\mathbb{C}^d$  (if  $F$  is holomorphic then it is certainly meromorphic), and we could replace the surjectivity theorem by saying that if  $F$  is any meromorphic function for which the integral on the

right-hand side of (6) is finite arises as the meromorphic extension of  $e^{t\Delta/2}f$  for some  $f$  in  $L^2(\mathbb{R}^d)$ . After all, since the density in (6) is strictly positive everywhere, such an  $F$  would have to be locally square-integrable with respect to Lebesgue measure, and it is not hard to show that a meromorphic function with this property must actually be holomorphic. (This can be seen from the Weierstrass Preparation Theorem [GH, p. 8].) That is, under the assumption that the right-hand side of (6) is finite, meromorphic and holomorphic are equivalent.

### 3. Isometry for radial functions

In this section we describe an isometric version of the Segal–Bargmann transform for “radial” functions on a noncompact symmetric space  $X$  of the “complex type” (e.g., hyperbolic 3-space). We give two different forms of this result. The first involves integration over the complexified tangent space to the symmetric space at the basepoint. The second involves integration over the complexified tangent space to the maximal flat at the basepoint. Both results characterize the image under the Segal–Bargmann transform of the radial subspace of  $L^2(X)$  as a certain holomorphic  $L^2$  space of *meromorphic* functions. In Section 6, we discuss the prospects for extending these results to nonradial function and to other symmetric spaces of the noncompact type.

If  $f$  is a function on a noncompact symmetric space  $X = G/K$ , then we wish to define the Segal–Bargmann transform of  $f$  to be some sort of analytic continuation of the function  $F := e^{t\Delta/2}f$ . The challenge in the noncompact case is to figure out precisely what sort of analytic continuation is the right one. One could try to analytically continue to  $G_{\mathbb{C}}/K_{\mathbb{C}}$ , but examples show that  $F$  does not in general admit an analytic continuation to  $G_{\mathbb{C}}/K_{\mathbb{C}}$ . Alternatively, one could consider the maximal domain  $\Omega$  to which functions of the form  $F = e^{t\Delta/2}f$  actually have an analytic continuation. This domain was identified by Krötz and Stanton [KS2, Theorem 6.1] as the Akhiezer–Gindikin “crown domain” in  $G_{\mathbb{C}}/K_{\mathbb{C}}$ . Unfortunately, it seems that there can be no measure  $\mu$  on  $\Omega$  such that the map sending  $f$  to the analytic continuation of  $F$  is an isometry of  $L^2(G/K)$  into  $L^2(\Omega, \mu)$ . (See the discussion in [KOS, Remark 3.1].) Thus, to get an isometry result of the sort that we have in the  $\mathbb{R}^d$  case and the compact case, we must venture beyond the domain  $\Omega$  into the region where  $F$  has singularities and find a way to deal with those singularities.

In this section, we assume that the symmetric space is of the complex type and that  $f$  (and thus also  $F$ ) is radial. We then write  $F$  in exponential coordinates at the basepoint, which makes  $F$  a function on the tangent space at the basepoint. We show that  $F$  admits a *meromorphic* extension to the complexified tangent space at the basepoint. This meromorphic extension of  $F$  is then square-integrable with respect to a suitable measure; the zeros in the density of the measure cancel the singularities in  $F$ . We obtain in this way an isometry of the radial part of  $L^2(X)$  onto a certain  $L^2$  space of meromorphic functions.

In the next section, we consider the more complicated case of nonradial functions. We obtain there an inversion formula involving a more subtle type of cancellation of singularities.



The set-up is as follows. We let  $G$  be a connected *complex* semisimple group and  $K$  a maximal compact subgroup of  $G$ . Since  $G$  is complex,  $K$  will be a compact real form of  $G$ . We decompose  $\mathfrak{g}$  as  $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ , where  $\mathfrak{p} = i\mathfrak{k}$ . We then choose an inner product on  $\mathfrak{p}$  that is invariant under the adjoint action of  $K$ . We consider the manifold  $G/K$  and we think of the tangent space at the identity coset to  $G/K$  as the space  $\mathfrak{p}$ . There is then a unique  $G$ -invariant Riemannian structure on  $G/K$  whose value at the identity is the given inner product on  $\mathfrak{p}$ . Then  $G/K$  is a Riemannian symmetric space of the “complex type.”

We emphasize that the word “complex” here does *not* mean that  $G/K$  is a complex manifold but rather that  $G$  is a complex Lie group. The complex structure on  $G$  will play no direct role in any definitions or proofs; for example, we will never consider holomorphic functions on  $G$ . Nevertheless, the complex case is quite special among all symmetric spaces of the noncompact type (i.e., compared to spaces of the form  $G/K$  with  $G$  *real* semisimple and  $K$  maximal compact). What is special about the complex case is not the complex structure per se, but rather the structure of the root system for  $G/K$  in this case: it is a reduced root system in which all roots have multiplicity 2. Still, it is easier to say “complex” than to say “reduced root system with all roots having multiplicity 2”! The simplest example of a noncompact symmetric space of the complex type is hyperbolic 3-space, and this is the only hyperbolic space that is of the complex type.

We will make use of special intertwining formulas for the Laplacian that hold only in the complex case. (See the proof of Theorem 2 for a discussion of why the intertwining formulas hold only in this case.) Nevertheless, there is hope for obtaining similar but less explicit results for other symmetric spaces of the noncompact type. See Section 6 for a discussion.

We consider the geometric exponential mapping for  $G/K$  at the identity coset. This coincides with the group-theoretical exponential mapping in the sense that if we identify the tangent space at the identity coset with  $\mathfrak{p}$ , then the geometric exponential of  $X \in \mathfrak{p}$  is just the coset containing the exponential of  $X$  in the Lie-group sense. In this section, we will use the notation  $e^X$  to denote the geometric exponential at the identity coset of a vector  $X$  in  $\mathfrak{p}$ . We let  $\delta$  be the square root of the Jacobian of the exponential mapping at the identity coset. This is the positive function satisfying

$$\int_{G/K} f(x) dx = \int_{\mathfrak{p}} f(e^X) \delta(X)^2 dX, \quad (7)$$

where  $dx$  is the Riemannian volume measure on  $G/K$  and where  $dX$  is the Lebesgue measure on  $\mathfrak{p}$  (normalized by the inner product). Explicitly,  $\delta$  is the unique  $\text{Ad-}K$ -invariant function on  $\mathfrak{p}$  whose restriction to a maximal commutative subspace  $\mathfrak{a}$  is given by

$$\delta(H) = \prod_{\alpha \in R^+} \frac{\sinh \alpha(H)}{\alpha(H)}. \quad (8)$$

Here  $R$  is the set of (restricted) roots for  $G/K$  (relative to  $\mathfrak{a}$ ) and  $R^+$  is the set of positive roots relative to some fixed Weyl chamber in  $\mathfrak{a}$ . Expression (8) may be obtained by specializing results [He3, Theorem IV.4.1] for general symmetric spaces of the noncompact type to the complex case, in which *all roots have multiplicity two*. (Compare Eq. (14) in Section V.5 of [He1].)

We consider functions on  $G/K$  that are “radial” in the symmetric space sense, meaning invariant under the left action of  $K$ . (These functions are not necessarily functions of the distance from the identity coset, except in the rank-one case.) We give two isometry results, one involving integration over  $\mathfrak{p}_{\mathbb{C}} := \mathfrak{p} + i\mathfrak{p}$  and one involving integration over  $\mathfrak{a}_{\mathbb{C}} := \mathfrak{a} + i\mathfrak{a}$ .

**Theorem 2.** *Let  $f$  be a radial function in  $L^2(G/K)$  ( $G$  complex) and let  $F = e^{t\Delta_{G/K}/2} f$ . Then the function*

$$X \rightarrow F(e^X), \quad X \in \mathfrak{p} \tag{9}$$

has a meromorphic extension from  $\mathfrak{p}$  to  $\mathfrak{p}_{\mathbb{C}}$  and this meromorphic extension satisfies

$$\int_{G/K} |f(x)|^2 dx = e^{ct} \int_{\mathfrak{p}_{\mathbb{C}}} \left| F(e^{X+iY}) \right|^2 |\delta(X+iY)|^2 \frac{e^{-|Y|^2/t}}{(\pi t)^{d/2}} dY dX. \tag{10}$$

Here  $c$  is the norm-squared of half the sum (with multiplicities) of the positive roots for  $G/K$ , and  $d = \dim(G/K)$ .

Conversely, suppose  $\Phi$  is a meromorphic function on  $\mathfrak{p}_{\mathbb{C}}$  that is invariant under the adjoint action of  $K$  and that satisfies

$$e^{ct} \int_{\mathfrak{p}_{\mathbb{C}}} |\Phi(X+iY)|^2 |\delta(X+iY)|^2 \frac{e^{-|Y|^2/t}}{(\pi t)^{d/2}} dY dX < \infty. \tag{11}$$

Then there exists a unique radial function  $f$  in  $L^2(G/K)$  such that

$$\Phi(X) = (e^{t\Delta_{G/K}/2} f)(e^X)$$

for all  $X \in \mathfrak{p}$ .

On the right-hand side of (10), the expression  $F(e^{X+iY})$  means the meromorphic extension of the function  $X \rightarrow F(e^X)$ , evaluated at the point  $X+iY$ . The proof will show that  $F(e^{X+iY})\delta(X+iY)$  is holomorphic (not just meromorphic) on  $\mathfrak{p}_{\mathbb{C}}$ . This means that although  $F(e^{X+iY})$  will in most cases have singularities, these singularities can be canceled out by multiplying by  $\delta(X+iY)$ . This cancellation of singularities is the reason that the integral on the right-hand side of (10) is even *locally* finite. Note that in contrast to the  $\mathbb{R}^d$  case (where the density of the relevant measure is nowhere

zero), there exist here meromorphic functions  $F$  that are not holomorphic and yet are square-integrable with respect to the measure in (10). Theorem 2 holds also for the Euclidean symmetric space  $\mathbb{R}^d$ , where in that case  $e^{X+iY} = X + iY$ ,  $c = 0$ , and  $\delta \equiv 1$ , so that we have (6) in the case where  $f$  happens to be radial.

Observe that if  $f$  is radial, then  $F = e^{t\Delta/2}f$  is also radial. Thus  $F$  is determined by its values on a “maximal flat”  $A := \exp \mathfrak{a}$ , where  $\mathfrak{a}$  is any fixed maximal commutative subspace of  $\mathfrak{p}$ . Thus it is reasonable to hope that we could replace the right-hand side of (10) with an expression involving integration only over  $\mathfrak{a}_{\mathbb{C}}$ . Our next result is of this sort. We fix a Weyl chamber in  $\mathfrak{a}$  and let  $R^+$  be the positive roots relative to this chamber. We let  $\eta$  be the function on  $\mathfrak{a}$  given by

$$\eta(H) = \delta(H) \prod_{\alpha \in R^+} \alpha(H) = \prod_{\alpha \in R^+} \sinh \alpha(H).$$

This function has an analytic continuation to  $\mathfrak{a}_{\mathbb{C}}$ , also denoted  $\eta$ .

**Theorem 3.** *Let  $f$  be a radial function in  $L^2(G/K)$  ( $G$  complex) and let  $F = e^{t\Delta_{G/K}/2}f$ . Then the function*

$$H \rightarrow F(e^H), \quad H \in \mathfrak{a}$$

has a meromorphic extension to  $\mathfrak{a}_{\mathbb{C}}$  and this meromorphic extension satisfies

$$\int_{G/K} |f(x)|^2 dx = B e^{ct} \int_{\mathfrak{a}_{\mathbb{C}}} \left| F(e^{H+iY}) \right|^2 |\eta(H+iY)|^2 \frac{e^{-|Y|^2/t}}{(\pi t)^{r/2}} dY dH, \quad (12)$$

where  $r = \dim \mathfrak{a}$  is the rank of  $G/K$  and  $c$  is as in Theorem 2. Here  $B$  is a constant independent of  $f$  and  $t$ .

Conversely, suppose  $\Phi$  is a meromorphic function on  $\mathfrak{a}_{\mathbb{C}}$  that is invariant under the action of the Weyl group and that satisfies

$$B e^{ct} \int_{\mathfrak{a}_{\mathbb{C}}} |\Phi(H+iY)|^2 |\eta(H+iY)|^2 \frac{e^{-|Y|^2/t}}{(\pi t)^{r/2}} dY dH < \infty. \quad (13)$$

Then there exists a unique radial function  $f$  in  $L^2(G/K)$  such that

$$\Phi(H) = (e^{t\Delta_{G/K}/2} f)(e^H)$$

for all  $H \in \mathfrak{a}$ .

In the dual compact case, an analogous result was established by Florentino et al. [FMN2, Theorem 2.2] and is described in Theorem 12 in Section 5.

Note that the function  $F(e^{X+iY})$  is invariant under the adjoint action of  $K_{\mathbb{C}}$  on  $\mathfrak{p}_{\mathbb{C}}$ . Since almost every point in  $\mathfrak{p}_{\mathbb{C}}$  can be mapped into  $\mathfrak{a}_{\mathbb{C}}$  by the adjoint action of  $K_{\mathbb{C}}$ , it should be possible to show directly that the right-hand side of (12) is equal to the right-hand side of (10). Something similar to this is done in the compact group case in [FMN2, Theorem 2.3]. However, we will follow a different approach here using intertwining formulas.

**Proof of Theorem 2.** For radial functions in the complex case we have a very special “intertwining formula” relating the non-Euclidean Laplacian  $\Delta_{G/K}$  for  $G/K$  and the Euclidean Laplacian  $\Delta_{\mathfrak{p}}$  for  $\mathfrak{p}$ . Let us temporarily identify  $\mathfrak{p}$  and  $G/K$  by means of the exponential mapping, so that it makes sense to apply both  $\Delta_{G/K}$  and  $\Delta_{\mathfrak{p}}$  to the same function. Then the intertwining formula states that (for radial functions in the complex case)

$$\Delta_{G/K} f = \frac{1}{\delta} [\Delta_{\mathfrak{p}} - c](\delta f), \tag{14}$$

where  $c$  is the norm-squared of half the sum (with multiplicities) of the positive roots for  $G/K$ . (See Proposition V.5.1 in [He1] and the calculations in the complex case on p. 484.)

One way to prove identity (14) is to first verify it for spherical functions, which are known explicitly in the complex case, and then build up general radial functions from the spherical functions. A more geometric approach is to work with the bilinear form associated to the Laplacian, namely,

$$D(f, g) := \int_{G/K} f(x) \Delta g(x) dx = - \int_{G/K} \nabla f(x) \cdot \nabla g(x) dx, \tag{15}$$

where  $f$  and  $g$  are, say, smooth real-valued functions of compact support. If  $f$  and  $g$  are radial, then at each point  $\nabla f$  and  $\nabla g$  will be tangent to the maximal flat, since the tangent space to a generic  $K$ -orbit is the orthogonal complement of the tangent space to the flat. From this, it is not hard to see that the *Euclidean* gradients of  $f$  and  $g$ , viewed as functions on  $\mathfrak{p}$  by means of the exponential mapping, coincide with the non-Euclidean gradients.

Thinking of  $\nabla f$  and  $\nabla g$  as Euclidean gradients, let us multiply and divide in (15) by the Jacobian of the exponential mapping, thus turning the integral into one over  $\mathfrak{p}$  with respect to Lebesgue measure. If we then do a Euclidean integration by parts on  $\mathfrak{p}$ , we will get one term involving the Laplacian for  $\mathfrak{p}$  and one term involving derivatives of the Jacobian  $\delta^2$  of the exponential mapping. With a bit of manipulation, this leads to an expression of the same form as (14), except with the constant  $c$  replaced by the *function*  $\Omega := \Delta_{\mathfrak{p}}(\delta)/\delta$ . (See Proposition V.5.1 in [He1] or Theorem II.3.15 in [He2].)

Now, up to this point, the argument is valid for an arbitrary symmetric space of the noncompact type. What is special about the complex case is that in this case [He1, p. 484], we have that  $\Delta_{\mathfrak{p}}(\delta) = c\delta$ , so that  $\Omega$  is a constant. It turns out that having

$\Delta_{\mathfrak{p}}(\delta)$  be a constant multiple of  $\delta$  is equivalent to having  $\Delta_{G/K}(\delta^{-1})$  be a constant multiple (with the opposite sign) of  $\delta^{-1}$ . It is shown in detail in [HSt, Section 2] that this last condition holds precisely when we have a reduced root system with all roots of multiplicity 2, that is, precisely in the complex case.

Meanwhile, formally exponentiating (14) would give

$$e^{t\Delta_{G/K}/2} f = \frac{1}{\delta} e^{-ct/2} e^{t\Delta_{\mathfrak{p}}/2} (\delta f). \quad (16)$$

Indeed, (16) holds for all radial functions  $f$  in  $L^2(G/K)$ , in which case  $\delta f$  is an Ad- $K$ -invariant function in  $L^2(\mathfrak{p})$ . It is not hard to prove that (16) follows from (14), once we have established that in the Hilbert space of  $L^2$  radial functions (on either  $G/K$  or  $\mathfrak{p}$ ), the Laplacian is essentially self-adjoint on  $C^\infty$  radial functions of compact support. To prove the essential self-adjointness, we start with the well-known essential self-adjointness of the Laplacian on  $C_c^\infty$ , as an operator on the full  $L^2$  space. We then note that the projection onto the radial subspace (again, on either  $G/K$  or  $\mathfrak{p}$ ) commutes with the Laplacian and preserves the space of  $C^\infty$  functions of compact support. From this, essential self-adjointness on  $C^\infty$  radial functions of compact support follows by elementary functional analysis.

Let us rewrite (16) as

$$e^{t\Delta_{\mathfrak{p}}/2} (\delta f) = e^{ct/2} \delta e^{t\Delta_{G/K}/2} f \quad (17)$$

and then apply the *Euclidean* Segal–Bargmann transform for  $\mathfrak{p}$  to the function  $\delta f$  in  $L^2(\mathfrak{p})$ . The properties of this transform tell us that  $e^{t\Delta_{\mathfrak{p}}/2} (\delta f)$  has an entire analytic continuation to  $\mathfrak{p}_{\mathbb{C}}$  and that

$$\int_{\mathfrak{p}} |\delta(X)f(X)|^2 dX = \int_{\mathfrak{p}_{\mathbb{C}}} \left| e^{t\Delta_{\mathfrak{p}}/2} (\delta f)(X+iY) \right|^2 \frac{e^{-|Y|^2/t}}{(\pi t)^{d/2}} dX dY. \quad (18)$$

Eq. (17) then tells us that  $\delta e^{t\Delta_{G/K}/2} f$  also has an analytic continuation to  $\mathfrak{p}_{\mathbb{C}}$  and that

$$\int_{\mathfrak{p}} |\delta(X)f(X)|^2 dX = e^{ct} \int_{\mathfrak{p}_{\mathbb{C}}} \left| \delta(X+iY)(e^{t\Delta_{G/K}/2} f)(X+iY) \right|^2 \frac{e^{-|Y|^2/t}}{(\pi t)^{d/2}} dX dY. \quad (19)$$

Since the function  $\delta e^{t\Delta_{G/K}/2} f$  has a *holomorphic* extension to  $\mathfrak{p}_{\mathbb{C}}$ , the function  $e^{t\Delta_{G/K}/2} f$  has a *meromorphic* extension to  $\mathfrak{p}_{\mathbb{C}}$ .

Let us now undo the identification of  $\mathfrak{p}$  with  $G/K$  in (19). The functions  $f$  and  $e^{t\Delta_{G/K}/2} f$  are radial functions on  $G/K$ . To turn these functions into functions on  $\mathfrak{p}$  we compose with the exponential mapping. So we now write  $f(e^X)$  on the left-hand side of (19) and  $(e^{t\Delta_{G/K}/2} f)(e^{X+iY})$  on the right-hand side. We then apply (7) to the

left-hand side of (19) to obtain

$$\int_{G/K} |f(x)|^2 dx = e^{ct} \int_{\mathfrak{p}_{\mathbb{C}}} \left| \delta(X + iY)(e^{t\Delta_{G/K/2}} f)(e^{X+iY}) \right|^2 \frac{e^{-|Y|^2/2}}{(\pi t)^{d/2}} dX dY.$$

This establishes the first part of the theorem.

For the second part of the theorem, suppose that  $\Phi$  is meromorphic on  $\mathfrak{p}_{\mathbb{C}}$ , radial (that is, invariant under the adjoint action of  $K$  on  $\mathfrak{p}_{\mathbb{C}}$ ), and satisfies

$$e^{ct} \int_{\mathfrak{p}_{\mathbb{C}}} |\Phi(X + iY)|^2 |\delta(X + iY)|^2 \frac{e^{-|Y|^2/t}}{(\pi t)^{d/2}} dY dX < \infty.$$

Then the function  $\Phi\delta$  is meromorphic on  $\mathfrak{p}_{\mathbb{C}}$  and square-integrable with respect to a measure with a strictly positive density. This, as pointed out in Section 2, implies that  $\Phi\delta$  is actually holomorphic on  $\mathfrak{p}_{\mathbb{C}}$ . Then by the surjectivity of the Segal–Bargmann transform for  $\mathfrak{p}$ , there exists a unique function  $g$  in  $L^2(\mathfrak{p})$  with  $e^{t\Delta_{\mathfrak{p}/2}} g = \Phi\delta$ . Since the Segal–Bargmann transform commutes with the action of  $K$ ,  $g$  must also be invariant under the adjoint action of  $K$ . If we let  $f$  be the unique function on  $G/K$  such that

$$f(e^X) = \frac{e^{ct/2} g(X)}{\delta(X)},$$

then  $f$  is radial and in  $L^2(G/K)$ . By (16) we have that  $e^{t\Delta_{G/K/2}} f = \frac{1}{\delta} e^{t\Delta_{\mathfrak{p}/2}}(g) = \Phi$  on  $\mathfrak{p}$ . This establishes the existence of the function  $f$  in the second part of the theorem. The uniqueness of this  $f$  follows from the injectivity of the operator  $e^{t\Delta_{G/K/2}}$  on  $L^2(G/K)$ .  $\square$

**Proof of Theorem 3.** The argument is similar to that in the preceding proof, except that in this case we use an “intertwining formula” that relates the non-Euclidean Laplacian on  $G/K$  to the Euclidean Laplacian on  $\mathfrak{a}$ . This formula says that (for radial functions  $f$  in the complex case) we have

$$(\Delta_{G/K} f)|_{\mathfrak{a}} = \frac{1}{\eta} [\Delta_{\mathfrak{a}} - c](\eta f_{\mathfrak{a}}), \tag{20}$$

where  $c$  is the same constant as in (14) and where  $f_{\mathfrak{a}}$  is the restriction of  $f$  to  $\mathfrak{a}$ . (See [He2, Proposition II.3.10].) An important difference between this formula and (14) above is that the function  $\eta f_{\mathfrak{a}}$  is Weyl-*anti*-invariant, whereas the function  $\delta f$  in (14) is Ad- $K$ -*invariant*. Exponentiating (20) gives that

$$e^{t\Delta_{G/K/2}} f = \frac{1}{\eta} e^{-ct/2} e^{t\Delta_{\mathfrak{a}/2}}(\eta f_{\mathfrak{a}}) \tag{21}$$

and so

$$e^{t\Delta_{\mathfrak{a}}/2}(\eta f_{\mathfrak{a}}) = e^{ct/2}\eta e^{t\Delta_{G/K}/2} f. \tag{22}$$

From properties of the Segal–Bargmann transform for  $\mathfrak{a}$  we then see that  $e^{t\Delta_{\mathfrak{a}}/2}(\eta f_{\mathfrak{a}})$  has a holomorphic extension to  $\mathfrak{a}_{\mathbb{C}}$  and that

$$\int_{\mathfrak{a}} |\eta(H)f(H)|^2 dH = \int_{\mathfrak{a}_{\mathbb{C}}} \left| e^{t\Delta_{\mathfrak{a}}/2}(\eta f_{\mathfrak{a}})(X + iY) \right|^2 \frac{e^{-|Y|^2/t}}{(\pi t)^{r/2}} dX dY, \tag{23}$$

where  $r = \dim \mathfrak{a}$ . Using (22) then gives

$$\int_{\mathfrak{a}} |\eta(H)f(H)|^2 dH = e^{ct} \int_{\mathfrak{a}_{\mathbb{C}}} \left| (e^{t\Delta_{G/K}/2} f)(X + iY)\eta(X + iY) \right|^2 \frac{e^{-|Y|^2/t}}{(\pi t)^{r/2}} dX dY.$$

We now recognize the left-hand side as being—up to an overall constant—the  $L^2$  norm of  $f$  over  $G/K$ , written using (7) and then generalized polar coordinates for  $\mathfrak{p}$  [He2, Theorem I.5.17]. We thus obtain the first part of the theorem. The unspecified constant  $B$  in Theorem 3 comes from the constant  $c$  in Theorem I.5.17 of [He2].

For the second part of the theorem, assume that  $\Phi$  is meromorphic, Weyl-invariant, and satisfies (13). Then, as in the proof of Theorem 2,  $\Phi\eta$  is holomorphic. In addition,  $\Phi\eta$  is Weyl-*anti*-invariant. There then exists a Weyl-*anti*-invariant function  $g$  in  $L^2(\mathfrak{a})$  with  $e^{t\Delta_{\mathfrak{a}}/2}g = \Phi\eta$ . We now let  $f$  be the function on  $A := \exp \mathfrak{a}$  satisfying

$$f(e^X) = \frac{e^{ct/2}g(X)}{\eta(X)}.$$

Then  $f$  is Weyl-invariant on  $A$  and has a unique radial extension to  $G/K$ . In light of the comments in the preceding paragraph, this extension of  $f$  is square-integrable over  $G/K$ . Then (21) tells us that  $e^{t\Delta_{G/K}/2}f = \Phi$ .  $\square$

#### 4. Inversion formula

In this section, we continue to consider symmetric spaces  $G/K$  of the complex type. However, we now consider functions  $f$  on  $G/K$  that are not necessarily radial. We let  $F = e^{t\Delta_{G/K}/2}f$  and we want to define the Segal–Bargmann transform as some sort of analytic continuation of  $F$ . In the radial case, we wrote  $F$  in exponential coordinates at the basepoint and then meromorphically extended  $F$  from  $\mathfrak{p}$  to  $\mathfrak{p}_{\mathbb{C}}$ . In the nonradial case, this approach is not appropriate, because we no longer have a distinguished basepoint. Instead we will analytically continue  $F$  to a neighborhood of  $G/K$  inside  $G_{\mathbb{C}}/K_{\mathbb{C}}$ .

For each  $x$  in  $G/K$ , we have the geometric exponential map  $\exp_x : T_x(G/K) \rightarrow G/K$ . It is not hard to show that this can be analytically continued to a holomorphic map, also denoted  $\exp_x$ , mapping the complexified tangent space  $T_x(G/K)_{\mathbb{C}}$  into  $G_{\mathbb{C}}/K_{\mathbb{C}}$ . We now consider tubes  $T^R(G/K)$  in the tangent bundle of  $G/K$ ,

$$T^R(G/K) = \{(x, Y) \in T(G/K) \mid |Y| < R\}.$$

Then we let  $U_R$  be the set in  $G_{\mathbb{C}}/K_{\mathbb{C}}$  given by

$$U_R = \left\{ \exp_x(iY) \mid (x, Y) \in T^R(G/K) \right\}.$$

Here,  $\exp_x(iY)$  refers to the analytic continuation of the exponential map at  $x$ . (In the  $\mathbb{R}^d$  case,  $\exp_x(iy)$  would be nothing but  $x + iy$ .)

It can be shown that for all sufficiently small  $R$ ,  $U_R$  is an open set in  $G_{\mathbb{C}}/K_{\mathbb{C}}$  and the map  $(x, Y) \rightarrow \exp_x(iY)$  is a diffeomorphism of  $T^R(G/K)$  onto  $U_R$ . The complex structure on  $T^R(G/K)$  obtained by identification with  $U_R$  is the “adapted complex structure” of [GS1,GS2,LS,Sz1]. Furthermore, Krötz and Stanton have shown that for any  $f$  in  $L^2(G/K)$ , the function  $F = e^{t\Delta_{G/K}/2} f$  has an analytic continuation to  $U_R$ , for all sufficiently small  $R$  [KS2, Theorem 6.1]. (These results actually hold for arbitrary symmetric spaces of the noncompact type, not necessarily of the complex type.) We think of the analytic continuation of  $F$  to  $U_R$  as the Segal–Bargmann transform of  $f$ .

Our goal in this section is to give an inversion formula that recovers  $f$  from the analytic continuation of  $F$ . In analogy to the  $\mathbb{R}^d$  case and the case of compact symmetric spaces, this should be done by integrating  $F$  over the fibers in  $U_R \cong T^R(G/K)$ . Something similar to this is done by Leichtnam et al. [LGS], in a very general setting. However, in [LGS, Theorem 0.3] there is a term involving integration over the boundary of the tube of radius  $R$ . This boundary term involves  $e^{s\Delta/2} f$ , for all  $s < t$ , and an integration with respect to  $s$ . This term is undesirable for us because we wish to think of  $t$  as fixed. In the case of compact symmetric spaces, Stenzel [St] showed that the boundary term in [LGS] could be removed by letting the radius  $R$  tend to infinity, thus leading to the inversion formula described in Section 5.

Now, our results here will not be based on the work of [LGS]. Nevertheless, Leichtnam et al. [LGS] and Stenzel [St] suggest that it is not possible to get an inversion formula of the sort we want by working with one fixed finite  $R$ ; rather, we need to let  $R$  tend to infinity. Unfortunately, (1) the map  $(x, Y) \rightarrow \exp_x(iY)$  ceases to be a diffeomorphism of  $T^R(G/K)$  with  $U_R$  for large  $R$ , and (2) the function  $F = e^{t\Delta_{G/K}/2} f$  does not in general have a holomorphic (or even meromorphic) extension to  $U_R$  for large  $R$ . For noncompact symmetric spaces of the complex type, we will nevertheless find a way to let  $R$  tend to infinity, by means of a cancellation of singularities. This leads to an inversion formula that is analogous to what we have in the compact and Euclidean cases. These results also lead to a natural conjecture of what the isometry formula should be in this setting, something we hope to address in a future paper.



#### 4.1. Inversion for radial functions at identity coset

Suppose that  $f$  is a radial function in  $L^2(G/K)$ . Then we may use the intertwining formula (17) and the inversion formula (5) in Theorem 1 to obtain the following. As in the previous section, we let  $\delta$  denote the square root of the Jacobian of the exponential mapping for  $G/K$  and we let  $c$  denote the norm-squared of half the sum (with multiplicities) of the positive roots for  $G/K$ .

**Theorem 4.** *Let  $f$  be a sufficiently regular radial function in  $L^2(G/K)$  ( $G$  complex) and let  $F = e^{t\Delta_{G/K}/2} f$ . Then*

$$f(x_0) = e^{ct/2} \int_{\mathfrak{p}} F(e^{iY}) \delta(iY) \frac{e^{-|Y|^2/2t}}{(2\pi t)^{d/2}} dY, \quad (24)$$

with absolute convergence of the integral. Here  $x_0 = e^0$  is the identity coset in  $G/K$ .

Specifically, sufficiently regular may be taken to mean that  $f$  has  $n$  derivatives in  $L^2(X)$  (with respect to the Riemannian volume measure) for some  $n > d/2$ . Note that the proof of Theorem 2 shows that the function  $X \rightarrow F(e^X)\delta(X)$  has an entire analytic continuation to  $\mathfrak{p}_{\mathbb{C}}$ . Thus the expression  $F(e^{iY})\delta(iY)$  is well defined and nonsingular on all of  $\mathfrak{p}$ .

At first glance, it may seem as if this inversion formula is not very useful, since it applies only to radial functions and then gives only the value of  $f$  at the identity coset. Nevertheless, we will see in the next subsection that this result leads to a much more general inversion formula that applies to not-necessarily-radial functions at arbitrary points.

Let us think about how this result compares to the inversion formula that holds for the compact symmetric space  $U/K$  that is dual to  $G/K$  (where, since  $G/K$  is of the complex type,  $U/K$  is isometric to a compact Lie group). In (24), the meromorphically continued function  $F(e^{iY})$  is being integrated against the signed measure given by

$$d\sigma_t(Y) := e^{ct/2} \delta(iY) \frac{e^{-|Y|^2/2t}}{(2\pi t)^{d/2}} dY, \quad Y \in \mathfrak{p}. \quad (25)$$

By analogy with the compact case (Theorem 10 in the special form of Theorem 11), we would expect that the (signed) measure  $\sigma_t$  should be the heat kernel measure at the identity coset for the compact symmetric space  $U/K$  dual to  $G/K$ , written in exponential coordinates. Clearly, this cannot be precisely true, first, because one does not have global exponential coordinates on the compact symmetric space and, second, because the density of the measure in (25) assumes negative values, whereas the heat kernel measure is a positive measure.

Nevertheless, the signed measure in (25) turns out to be very closely related to the heat kernel measure for  $U/K$ . Specifically, the *push-forward* of the measure (25) under the exponential mapping for  $U/K$  is precisely the heat kernel measure (at the identity

coset) for  $U/K$ . Thus (25) itself may be thought of as an “unwrapped” version of the heat kernel for  $U/K$ , where we think of the exponential map as “wrapping” the tangent space (in a many-to-one way) around  $U/K$ . What is going on is that the heat kernel at a point  $x$  in  $U/K$  may be expressed as a sum of contributions from all of the geodesics connecting the identity coset to  $x$ . The quantity in (25) is what we obtain by breaking apart those contributions, thus obtaining something on the space of geodesics, that is, on the tangent space at the identity coset. Although some geodesics make a negative contribution to the heat kernel, the heat kernel itself (obtained by summing over all geodesics) is positive at every point.

**Theorem 5.** *We may identify  $\mathfrak{p}$  with the tangent space at the identity coset to  $U/K$  in such a way that the following holds: The push-forward of the signed measure  $\sigma_t$  in (25) under the exponential mapping for  $U/K$  coincides with the heat kernel measure for  $U/K$  at the identity coset.*

Let us now recall the construction [He3, Section V.2] of  $U/K$  and explain how  $\mathfrak{p}$  is identified with the tangent space to  $U/K$  at the identity coset. Let  $G_{\mathbb{C}}$  be the unique simply connected Lie group whose Lie algebra is  $\mathfrak{g}_{\mathbb{C}}$ . Let  $\tilde{G}$  be the connected Lie subgroup of  $G_{\mathbb{C}}$  whose Lie algebra is  $\mathfrak{g}$ . For notational simplicity, let us assume that the inclusion of  $\mathfrak{g}$  into  $\mathfrak{g}_{\mathbb{C}}$  induces an isomorphism of  $G$  with  $\tilde{G}$ . (Every symmetric space of the noncompact type can be realized as  $G/K$  with  $G$  having this property.) Let  $U$  be the connected Lie subgroup of  $G_{\mathbb{C}}$  whose Lie algebra is  $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$ . Then the connected Lie subgroup of  $U$  with Lie algebra  $\mathfrak{k}$  is simply the group  $K$ .

We consider the quotient manifold  $U/K$  and we identify the tangent space at the identity coset in  $U/K$  with  $\mathfrak{p}_* := i\mathfrak{p}$ . If we use the multiplication by  $i$  map to identify  $\mathfrak{p}$  with  $\mathfrak{p}_*$ , then we may transport the inner product on  $\mathfrak{p}$  to  $\mathfrak{p}_*$ . There is then a unique  $U$ -invariant Riemannian metric on  $U/K$  coinciding with this inner product at the identity coset. With this Riemannian metric,  $U/K$  becomes a simply connected symmetric space of the compact type, and is called the “dual” of the symmetric space  $G/K$  of the noncompact type. The duality construction is valid starting with any symmetric space of the noncompact type, producing a symmetric space of the compact type (and a very similar procedure goes from compact type to noncompact type). If one begins with a noncompact symmetric space of the complex type, the dual compact symmetric space will be isometric to a compact Lie group with a bi-invariant measure.

**Proof of Theorem 4.** Let us again identify  $G/K$  with  $\mathfrak{p}$  by means of the exponential mapping at the identity coset. Suppose  $f$  is a radial function square-integrable with respect to the Riemannian volume measure for  $G/K$ . Then  $\delta f$  is a radial function square-integrable with respect to the Lebesgue measure for  $\mathfrak{p}$ . According to (17) in the previous section, we have

$$e^{t\Delta_{\mathfrak{p}}/2}(\delta f) = e^{ct/2}\delta e^{t\Delta_{G/K}/2}f. \tag{26}$$

If  $\delta f$  is “sufficiently regular,” then we may apply the inversion formula for the Euclidean Segal–Bargmann transform ((5) in Theorem 1) to the function  $\delta f$ . Noting that

$\delta(0) = 1$ , applying the inversion at the origin gives

$$f(0) = (\delta f)(0) = e^{ct/2} \int_{\mathfrak{p}} F(iY)\delta(iY) \frac{e^{-|Y|^2/2t}}{(2\pi t)^{d/2}} dY,$$

with absolute convergence of the integral, where  $F$  is the meromorphic extension of  $e^{t\Delta_{G/K}/2} f$ . To undo the identification of  $G/K$  with  $\mathfrak{p}$ , we simply replace  $f(0)$  with  $f(e^0)$  and  $F(Y)$  with  $F(e^iY)$ . This establishes Theorem 4, provided that  $\delta f$  is “sufficiently regular.”

To address the regularity condition, we recall the intertwining formula (14). From this formula it is not hard to show that if  $f$  is radial and in the domain of  $(cI - \Delta_{G/K})^{n/2}$  for some  $n$ , then  $\delta f$  is in the domain of  $(cI - \Delta_{\mathfrak{p}})^{n/2}$ . However, the domain of  $(cI - \Delta_{G/K})^{n/2}$  is precisely the Sobolev space of functions on  $G/K$  having  $n$  derivatives in  $L^2$ . Thus if  $f$  is in this Sobolev space with  $n > d/2$ ,  $\delta f$  will be in the corresponding Sobolev space on  $\mathfrak{p}$  and  $\delta f$  will indeed be “sufficiently regular” in the sense of [H9, Section 2.1].  $\square$

**Proof of Theorem 5.** We make use of the formula for the heat kernel function (at the identity) on a compact Lie group, as originally obtained by Èskin [E] and rediscovered by Urakawa [U]. We continue to use symmetric space notation for  $U/K$ , rather than switching to group notation. Nevertheless, the following formula is valid *only* in the case that  $U/K$  is isometric to a compact Lie group (which is precisely when  $G/K$  is of the complex type). We think of  $\mathfrak{p}_* := i\mathfrak{p}$  as the tangent space to  $U/K$  at the identity coset and we write  $e^Y$  for the exponential (in the geometric sense) of  $Y \in \mathfrak{p}_*$ . For any maximal commutative subspace  $\mathfrak{a}$  of  $\mathfrak{p}$ , the space  $\mathfrak{a}_* := i\mathfrak{a}$  is a maximal commutative subspace of  $\mathfrak{p}_*$  (and every maximal commutative subspace of  $\mathfrak{p}_*$  arises in this way). Given a fixed such subspace  $\mathfrak{a}_*$ , the set  $A_* = \exp(\mathfrak{a}_*)$  is a maximal flat in  $U/K$  and  $A_*$  is isometric to a flat Euclidean torus. Let  $\Gamma \subset \mathfrak{a}_*$  denote the kernel of the exponential mapping for  $\mathfrak{a}_*$ , so that  $\Gamma$  is a lattice in  $\mathfrak{a}_*$ .

We now let  $\rho_t$  denote the fundamental solution at the identity coset to the heat equation  $\partial u / \partial t = \frac{1}{2} \Delta u$  on  $U/K$ . The heat kernel formula asserts that for any maximal commutative subspace  $\mathfrak{a}_*$  of  $\mathfrak{p}_*$  we have

$$\rho_t(e^H) = \frac{e^{ct/2}}{(2\pi t)^{d/2}} \sum_{\gamma \in \Gamma} j^{-1/2}(H + \gamma) e^{-|H + \gamma|^2/2t}, \quad H \in \mathfrak{a}_*. \tag{27}$$

The function  $\rho_t$  is the heat kernel *function*, that is, the density of the heat kernel *measure* (at the identity coset) with respect to the (un-normalized) Riemannian volume measure on  $U/K$ .

In this formula,  $d = \dim(U/K)$  and  $c$  is the norm squared of half the sum (with multiplicities) of the positive roots for  $U/K$ . Since (it is easily seen) the roots and multiplicities for  $U/K$  are the same as for  $G/K$ , this definition of  $c$  agrees with the one made earlier in this section. Meanwhile, the function  $j$  is the Jacobian of the

exponential mapping for  $U/K$ ,  $j^{1/2}$  is the unique smooth square root of  $j$  that is positive near the origin, and  $j^{-1/2}$  is the reciprocal of  $j^{1/2}$ . Explicitly, for  $H$  in  $\mathfrak{a}_*$  we have

$$j^{1/2}(H) = \prod_{\alpha \in R^+} \frac{\sin \alpha(H)}{\alpha(H)}, \tag{28}$$

where  $R^+$  is a set of positive roots for  $U/K$ . Note that  $j^{1/2}$  takes on both positive and negative values; the nonnegative square root of  $j$  is not a smooth function. Properly, formula (27) is valid only for  $H$  such that  $j(H)$  is nonzero, in which case  $j(H + \gamma)$  will be nonzero for all  $\gamma \in \Gamma$ . However, since  $\rho_t$  is continuous, we may then extend the right-hand side by continuity to all  $H \in \mathfrak{a}_*$ .

Since the roots for  $U/K$  are the same as for  $G/K$  (under the obvious identification of  $\mathfrak{p}_*$  with  $\mathfrak{p}$ ), comparing formula (8) with (28) gives that

$$j^{1/2}(Y) = \delta(iY) \tag{29}$$

for all  $Y$  in  $\mathfrak{p} \cong \mathfrak{p}_*$ .

Formula (27) is not quite what is given in [E] or [U], but can be deduced from those papers. Our formula differs from the one in Urakawa by some factors of 2 having to do with group notation versus symmetric space notation, some additional factors of 2 having to do with different normalizations of the heat equation, and an overall constant coming from different normalizations of the measure on  $U/K$ .

Now, a “generic” point in  $U/K$  (in a sense to be specified later) is contained in a unique maximal flat  $A_*$ . If  $x$  is contained in a unique maximal flat  $A_*$  and if  $e^Y = x$  for some  $Y$  in  $\mathfrak{p}_*$ , then we must have  $Y \in \mathfrak{a}_*$ . (If  $Y$  were not in  $\mathfrak{a}_*$ , then  $Y$  would be contained in some maximal commutative subspace  $\mathfrak{b}_* \neq \mathfrak{a}_*$  and then  $x$  would be in the maximal flat  $B_* \neq A_*$ .) Fix such a point  $x$  and pick one  $H$  in  $\mathfrak{a}_*$  with  $e^H = x$ . Then the elements of the form  $Y = H + \gamma$ , with  $\gamma$  in  $\Gamma$ , represent all the points in  $\mathfrak{p}_*$  with  $e^Y = x$ . This means that for a generic point  $x = e^H$ , the sum in (27) may be thought of as a sum over all the geodesics connecting the identity coset to  $x$ . If we also make use of (29), we may rewrite (27) as

$$\rho_t(x) = \frac{e^{ct/2}}{(2\pi t)^{d/2}} \sum_{\{Y \in \mathfrak{p}_* | e^Y = x\}} \delta^{-1}(iY) e^{-|Y|^2/2t}, \tag{30}$$

whenever  $x$  in  $U/K$  is contained in a unique maximal flat.

We are now in a position to understand why Theorem 5 holds. If we push forward the signed measure in  $\sigma_t$  in (25), we will get a factor of  $1/j(Y)$  ( $= 1/\delta^2(iY)$ ) from the change of variables formula, which will change the  $\delta$  in (25) to  $\delta^{-1}$ . The density of the pushed-forward measure at a generic point  $x$  in  $U/K$  will then be a sum over  $\{Y | e^Y = x\}$  of the density in (25) multiplied by  $1/\delta(iY)$ , which is precisely what we have in (30). This is what Theorem 5 asserts.

To make the argument in the preceding paragraphs into a real proof, we need to attend to a few technicalities, including an appropriate notion of “generic.” We call an element  $Y$  of  $\mathfrak{p}_*$  *singular* if there exist a maximal commutative subspace  $\mathfrak{a}$  containing  $Y$ , a root  $\alpha$  for  $\mathfrak{a}$ , and an integer  $n$  such that  $\alpha(Y) = n\pi$ ; we call  $Y$  *regular* otherwise. We call an element  $x$  of  $U/K$  *singular* if  $x$  can be expressed as  $x = e^Y$  for some singular element  $Y \in \mathfrak{p}_*$ ; we call  $x$  *regular* otherwise. It can be shown that  $e^Y$  is regular whenever  $Y$  is regular (this is not immediately evident from the definitions). In both  $\mathfrak{p}_*$  and  $U/K$ , the singular elements form a closed set of measure zero. Thus in pushing forward the signed measure  $\sigma_t$ , we may simply ignore the singular points and regard the exponential mapping as taking the open set of regular elements in  $\mathfrak{p}_*$  onto the open set of regular elements in  $U/K$ . (See Sections VII.2 and VII.5 of [He3].)

If  $x$  is regular and  $x = e^Y$ , then (by definition)  $Y$  is regular and it follows that  $j(Y)$  is nonzero. Furthermore, if  $x$  is regular then (it can be shown)  $x$  is contained in a unique maximal flat. Thus (30) is valid for all regular elements. Furthermore, it is easily seen that the function  $j(Y) = \delta(iY)$  has constant sign on each connected component of the set of regular elements in  $\mathfrak{p}_*$ . Finally, we note that the exponential mapping is a local diffeomorphism near each regular element of  $\mathfrak{p}_*$ , since the Jacobian of the exponential mapping is nonzero at regular points. From all of this, it is not hard to use a partition of unity to show that the argument given above is correct.  $\square$

#### 4.2. Inversion for general functions

At each point  $x$  in  $G/K$ , we have the geometric exponential mapping,  $\exp_x$ , mapping the tangent space  $T_x(G/K)$  into  $G/K$ . We have also the square root of the Jacobian of the exponential mapping for  $\exp_x$ , denoted  $\delta_x$ . Now, the action of  $G$  gives a linear isometric identification of  $T_x(G/K)$  with  $T_{x_0}(G/K) \cong \mathfrak{p}$ . This identification is unique up to the adjoint action of  $K$  on  $\mathfrak{p}$ . Under any such identification, the function  $\delta_x$  will coincide with the function  $\delta = \delta_{x_0}$  considered in the previous section. Thus, in a slight abuse of notation, we let  $\delta$  stand for the square root of the Jacobian of  $\exp_x$  at any point  $x$ . For example, in the case of three-dimensional hyperbolic space (with the usual normalization of the metric), we have  $\delta(X) = \sinh |X| / |X|$  (for all  $x$ ). For any  $x$ , the function  $\delta$  has an entire analytic continuation to the complexified tangent space at  $x$ .

**Theorem 6.** *Let  $f$  be in  $L^2(G/K)$  ( $G$  complex) and let  $F = e^{t\Delta_{G/K}/2} f$ . Then define*

$$L(x, R) = e^{ct/2} \int_{|Y| \leq R} F(\exp_x(iY)) \delta(iY) \frac{e^{-|Y|^2/2t}}{(2\pi t)^{d/2}} dY, \quad (31)$$

for all sufficiently small  $R$ .

Then for each  $x$ ,  $L(x, R)$  admits a real-analytic continuation in  $R$  to  $(0, \infty)$ . Furthermore, if  $f$  is sufficiently regular, then

$$f(x) = \lim_{R \rightarrow \infty} L(x, R) \quad (32)$$

for all  $x$  in  $G/K$ . Thus we may write, informally,

$$f(x) = \lim_{R \rightarrow \infty} e^{ct/2} \int_{|Y| \leq R} F(\exp_x iY) \delta(iY) \frac{e^{-|Y|^2/2t}}{(2\pi t)^{d/2}} dY, \tag{33}$$

with the understanding that the right-hand side is to be interpreted literally for small  $R$  and by analytic continuation in  $R$  for large  $R$ .

As in the radial case, “sufficiently regular” may be interpreted to mean that  $f$  has  $n$  derivatives in  $L^2(G/K)$ , for some  $n$  with  $n > d/2$ .

Formula (33) should be thought of as the noncompact dual to the compact group formula (37) in Theorem 11. Specifically (as in (29)),  $\delta(iY)$  is nothing but the square root of the Jacobian of the exponential mapping for the dual compact symmetric space  $U/K$ , so that this factor in (33) is dual to the factor of  $j(Y)^{1/2}$  in (37). The positive constant  $c$  has the same value in (33) as in (37) (because the roots and multiplicities for  $G/K$  and  $U/K$  are the same); the change from  $e^{-ct/2}$  in (37) to  $e^{ct/2}$  in (33) is part of the duality. (For example, the exponential factors are related to the scalar curvature, which is negative in  $G/K$  and positive in  $U/K$ .)

Let us think about why  $L(x, R)$  admits an analytic continuation in  $R$ , despite the singularities that develop in  $F(\exp_x(iY))$  when  $Y$  is not small. The key observation is that the signed measure in the definition of  $L(x, R)$  (denoted  $\sigma_r$  in (25)) is radial. Thus the integral in (31) only “sees” the part of  $F(\exp_x(iY))$  that is radial as a function of  $Y$ . Taking the radial part of  $F(\exp_x(iY))$  eliminates many of the singularities. The singularities that remain in the radial part of  $F(\exp_x(iY))$  are then of a “universal” nature, coming essentially from the singularities in the analytically continued spherical functions for  $G/K$ . These remaining singularities are canceled by the zeros in the function  $\delta(iY)$ . See Section 5 of the expository paper [H9] for further discussion of the cancellation of singularities.

**Proof.** For any  $x$  in  $G/K$ , we let  $K_x$  denote the subgroup of  $G$  that stabilizes  $x$ . (This group is conjugate in  $G$  to  $K$ .) For any continuous function  $\phi$  on  $G/K$ , we let  $\phi^{(x)}$  denote the “radial part of  $\phi$  relative to  $x$ ,” given by

$$\phi^{(x)}(y) = \int_{K_x} \phi(k \cdot y) dk,$$

where  $dk$  is the normalized Haar measure on  $K_x$ .

We wish to reduce the inversion formula in Theorem 6 to the radial case in Theorem 4. Of course, there is nothing special about the identity coset in Theorem 4; the same result applies to functions that are radial with respect to any point  $x$  in  $G/K$ . Now, note that

$$f^{(x)}(x) = f(x)$$

and that (since the heat operator commutes with the action of  $K_x$ )

$$e^{t\Delta_{G/K/2}}(f^{(x)}) = (e^{t\Delta_{G/K/2}} f)^{(x)} = F^{(x)}.$$

Furthermore, if  $f$  is sufficiently regular, then so is  $f^{(x)}$ .

Thus, by Theorem 4 (extended to functions that are radial around  $x$ ) we have

$$\begin{aligned} f(x) &= f^{(x)}(x) \\ &= \int_{T_x(G/K)} e^{t\Delta_{G/K/2}}(f^{(x)})(\exp_x(iY))\delta(iY) \frac{e^{-|Y|^2/2t}}{(2\pi t)^{d/2}} dY \\ &= \int_{T_x(G/K)} F^{(x)}(\exp_x(iY))\delta(iY) \frac{e^{-|Y|^2/2t}}{(2\pi t)^{d/2}} dY. \end{aligned} \tag{34}$$

Note that the function  $X \rightarrow F^{(x)}(\exp_x(X))\delta(X)$  has an entire analytic continuation to  $T_x(G/K)_{\mathbb{C}}$  and therefore  $F^{(x)}(\exp_x(iY))\delta(iY)$  is nonsingular for all  $Y$ .

Now, the action of  $K_x$  commutes with  $\exp_x$  and with analytic continuation from  $T_x(G/K)$  to  $T_x(G/K)_{\mathbb{C}}$ . Thus

$$F^{(x)}(\exp_x(iY)) = \int_{K_x} F(\exp_x(i\text{Ad}_k(Y))) dk.$$

From this and the fact that  $\delta(iY)$  and  $|Y|^2$  are radial functions of  $Y$ , we obtain the following: We may replace  $F(\exp_x(iY))$  in (31) with  $F^{(x)}(\exp_x(iY))$  without affecting the value of the integral. This establishes the existence of the analytic continuation in  $R$  of  $L(x, R)$ : The analytic continuation is given by

$$L(x, R) = e^{ct/2} \int_{|Y| \leq R} F^{(x)}(\exp_x(iY))\delta(iY) \frac{e^{-|Y|^2/2t}}{(2\pi t)^{d/2}} dY$$

for all  $R$ . (This expression is easily seen to be analytic in  $R$ .) Letting  $R$  tend to infinity gives the inversion formula (32), by (34).  $\square$

### 5. Review of the compact case

In order to put our results for noncompact symmetric spaces of the complex type into perspective, we review here the main results from the compact case. We describe first the results of Stenzel [St] for general compact symmetric spaces. Then we describe how those results simplify in the case of a compact Lie group, recovering results

of [H1,H2]. Finally, we describe a recent result of Florentino et al. [FMN2] for radial functions in the compact group case. Our isometry formula for radial functions in the complex case (especially Theorem 3) should be compared to the result of Florentino et al. [FMN2], as described in our Section 5.3. Our inversion formula for general functions (Theorem 6) should be compared to the inversion formula in the compact group case, as described in (37) of Theorem 11.

For additional information on the Segal–Bargmann transform for compact groups and compact symmetric spaces, see the expository papers [H6,H9]. See also [HM1,HM2] for more on the special case of spheres.

We make use here of standard results about compact symmetric spaces (see, for example, [He3]) as well as results from Section 2 of [St] (or Section 8 of [LGS]).

### 5.1. The general compact case

We consider a compact symmetric space  $X$ , assumed for simplicity to be simply connected. Suppose that  $U$  is a compact, simply connected Lie group (necessarily semisimple) and that  $\sigma$  is an involution of  $U$ . Let  $K$  be the subgroup of  $U$  consisting of the elements fixed by  $\sigma$ . Then  $K$  is automatically a closed, connected subgroup of  $U$ . Consider the quotient manifold  $X := U/K$ , together with any Riemannian metric on  $U/K$  that is invariant under the action of  $U$ . Then  $X$  is a simply connected compact symmetric space, and every simply connected compact symmetric space arises in this way. We will assume (without loss of generality) that  $U$  acts in a locally effective way on  $X$ , that is, that the set of  $u \in U$  for which  $u$  acts trivially on  $X$  is discrete. Under this assumption, the  $U$  and  $\sigma$  are unique up to isomorphism for a given  $X$ , and  $U$  is isomorphic to the universal cover of the identity component of the isometry group of  $X$ .

We consider the complexification of the group  $U$ , denoted  $U_{\mathbb{C}}$ . Since we assume  $U$  is simply connected,  $U_{\mathbb{C}}$  is just the unique simply connected group whose Lie algebra is  $\mathfrak{u}_{\mathbb{C}} := \mathfrak{u} + i\mathfrak{u}$  (where  $\mathfrak{u}$  is the Lie algebra of  $U$ ), and  $U$  sits inside  $U_{\mathbb{C}}$  as a maximal compact subgroup. We also let  $K_{\mathbb{C}}$  denote the connected Lie subgroup of  $U_{\mathbb{C}}$  whose Lie algebra is  $\mathfrak{k}_{\mathbb{C}} := \mathfrak{k} + i\mathfrak{k}$  (where  $\mathfrak{k}$  is the Lie algebra of  $K$ ). Then  $K_{\mathbb{C}}$  is always a closed subgroup of  $U_{\mathbb{C}}$ . We may introduce the “complexification” of  $U/K$ , namely, the complex manifold

$$X_{\mathbb{C}} := U_{\mathbb{C}}/K_{\mathbb{C}}.$$

It can be shown that  $K_{\mathbb{C}} \cap U = K$ ; as a result, the inclusion of  $U$  into  $U_{\mathbb{C}}$  induces an inclusion of  $U/K$  into  $U_{\mathbb{C}}/K_{\mathbb{C}}$ .

We write  $g \cdot x$  for the action of an element  $g$  in  $U_{\mathbb{C}}$  on a point  $x$  in  $U_{\mathbb{C}}/K_{\mathbb{C}}$  and we let  $x_0$  denote the identity coset in  $U/K \subset U_{\mathbb{C}}/K_{\mathbb{C}}$ .

**Definition 7.** The Segal–Bargmann transform for  $U/K$  is the map

$$C_t : L^2(U/K) \rightarrow \mathcal{H}(U_{\mathbb{C}}/K_{\mathbb{C}})$$



given by

$$C_t f = \text{analytic continuation of } e^{t\Delta/2} f.$$

Here  $e^{t\Delta/2}$  is the time- $t$  forward heat operator and the analytic continuation is from  $U/K$  to  $U_{\mathbb{C}}/K_{\mathbb{C}}$  with  $t$  fixed.

It follows from [H1, Section 4] (applied to  $K$ -invariant functions on  $U$ ) that for any  $f$  in  $L^2(U/K)$  (with respect to the Riemannian volume measure),  $e^{t\Delta/2} f$  has a unique analytic continuation from  $U/K$  to  $U_{\mathbb{C}}/K_{\mathbb{C}}$ .

At each point  $x$  in  $U/K$ , we have the geometric exponential map

$$\exp_x : T_x(U/K) \rightarrow U/K.$$

(If  $\gamma$  is the unique geodesic with  $\gamma(0) = x$  and  $\dot{\gamma}(0) = Y$ , then  $\exp_x(Y) = \gamma(1)$ .) For each  $x$ , the map  $\exp_x$  can be analytically continued to a holomorphic map of the complexified tangent space  $T_x(U/K)_{\mathbb{C}}$  into  $U_{\mathbb{C}}/K_{\mathbb{C}}$ .

**Proposition 8** (*Identification of  $T(X)$  with  $X_{\mathbb{C}}$* ). *The map  $\Phi : T(U/K) \rightarrow U_{\mathbb{C}}/K_{\mathbb{C}}$  given by*

$$\Phi(x, Y) = \exp_x(iY), \quad x \in U/K, \quad Y \in T_x(U/K)$$

*is a diffeomorphism. On right-hand side of the above formula,  $\exp_x(iY)$  refers to the analytic continuation of geometric exponential map.*

From the point of view of quantization, we should really identify  $U_{\mathbb{C}}/K_{\mathbb{C}}$  with the cotangent bundle  $T^*(U/K)$ . However, since  $U/K$  is a Riemannian manifold we naturally and permanently identify  $T^*(U/K)$  with the tangent bundle  $T(U/K)$ . In the  $\mathbb{R}^d$  case,  $\exp_x(iy)$  would be nothing but  $x + iy$ .

The Lie algebra  $\mathfrak{u}$  of  $U$  decomposes as  $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$ , where  $\mathfrak{p}$  is the  $-1$  eigenspace for the action of the involution  $\sigma$  on  $\mathfrak{u}$ . For any  $x$  in  $U/K$  we define

$$\begin{aligned} K_x &= \text{Ad}_u(K), \\ \mathfrak{k}_x &= \text{Ad}_u(\mathfrak{k}), \\ \mathfrak{p}_x &= \text{Ad}_u(\mathfrak{p}), \end{aligned}$$

where  $u$  is any element of  $U$  such that  $u \cdot x_0 = x$ . We identify  $\mathfrak{p} = \mathfrak{p}_{x_0}$  with the tangent space to  $U/K$  at  $x_0$ ; more generally, we identify  $\mathfrak{p}_x$  with the tangent space at  $x$  to  $U/K$ . With this identification, we have

$$\exp_x(Y) = e^Y \cdot x, \quad x \in U/K, \quad Y \in \mathfrak{p}_x,$$

where  $e^Y \in U$  is the exponential of  $Y$  in the Lie group sense.

Now, for each  $x \in U/K$ , define a subspace  $\mathfrak{g}_x$  of  $\mathfrak{u}_\mathbb{C}$  by

$$\mathfrak{g}_x = \mathfrak{k}_x + i\mathfrak{p}_x.$$

Then  $\mathfrak{g}_x$  is a Lie subalgebra of  $\mathfrak{u}_\mathbb{C}$ . We let  $G_x$  denote the connected Lie subgroup of  $U_\mathbb{C}$  whose Lie algebra is  $\mathfrak{g}_x$ . Note that  $e^{iY}$  belongs to  $G_x$  for any  $Y$  in  $\mathfrak{p}_x$ . Thus, the image under  $\Phi$  of  $T_x(U/K)$  is contained in the  $G_x$ -orbit of  $x$ . In fact,  $\Phi(T_x(U/K))$  is precisely the  $G_x$ -orbit of  $x$ , and the stabilizer in  $G_x$  of  $x$  is precisely  $K_x$ . We record this result in the following.

**Proposition 9** (Identification of the fibers). *For any  $x \in U/K$ , the image inside  $U_\mathbb{C}/K_\mathbb{C}$  of  $T_x(U/K) \cong \mathfrak{p}_x$  under  $\Phi$  is precisely the orbit of  $x$  under  $G_x$ . Thus the image of  $T_x(U/K)$  may be identified naturally with  $G_x/K_x$ .*

Now, each  $G_x$  is conjugate under the action of  $U$  to  $G := G_{x_0}$ . Thus each quotient space  $G_x/K_x$  may be identified with  $G/K$ . This identification depends on the choice of an element  $u$  of  $U$  mapping  $x_0$  to  $x$  and is therefore unique only up to the action of  $K$  on  $G/K$ . The space  $G/K$ , with an appropriately chosen  $G$ -invariant Riemannian metric, is the *dual noncompact symmetric space* to  $U/K$ . Thus we see that the map  $\Phi$  leads naturally to an identification (unique up to the action of  $K$ ) of each fiber in  $T(U/K)$  with the noncompact symmetric space  $G/K$ .

Another way to think about the appearance of the geometry of  $G/K$  in the problem is from the following result of Leichtnam, Golse, and Stenzel. If we analytically continue the metric tensor from  $U/K$  to  $U_\mathbb{C}/K_\mathbb{C}$  and then restrict to the image of  $T_x(U/K)$  under  $\Phi$ . The result is the *negative* of a Riemannian metric and the image of  $T_x(U/K)$ , with the resulting Riemannian metric, is isometric to  $G/K$ . (See [LGS, Proposition 1.17 and Theorem 8.5].)

On each fiber  $T_x(U/K) \cong G/K$  we may then introduce the *heat kernel measure* (at the identity coset). This measure is given by the Riemannian volume measure for  $G/K$  multiplied by the *heat kernel function*, denoted  $v_t$ . Under the identification of  $T_x(U/K)$  with  $G/K$ , the Riemannian volume measure on  $G/K$  corresponds to Lebesgue measure on  $T_x(U/K)$  multiplied by an explicitly computable Jacobian function  $j$ . Thus the heat kernel measure on  $T_x(U/K)$  is the measure  $v_t(Y)j(Y) dY$ , where  $dY$  denotes Lebesgue measure.

We are now ready to state the main results of Stenzel’s paper [St].

**Theorem 10** (Stenzel). *Let  $f$  be in  $L^2(U/K)$  and let  $F = e^{t\Delta/2} f$ . Then we have the following results:*

1. *The inversion formula. If  $f$  is sufficiently regular we have*

$$f(x) = \int_{T_x(U/K)} F(\exp_x(iY))v_t(Y)j(Y) dY, \tag{35}$$

*with absolute convergence of the integral for all  $x$ .*

2. The isometry formula. For all  $f$  in  $L^2(U/K)$  we have

$$\int_{U/K} |f(x)|^2 dx = \int_{U/K} \int_{T_x(U/K)} |F(\exp_x(iY))|^2 v_{2t}(2Y) j(2Y) 2^d dY dx, \quad (36)$$

where  $d = \dim(U/K)$ .

3. The surjectivity theorem. For any holomorphic function  $F$  on  $U_{\mathbb{C}}/K_{\mathbb{C}} \cong T(U/K)$  such that the integral on the right-hand side of (36) is finite, there exists a unique  $f$  in  $L^2(U/K)$  with  $F = C_t f$ .

Note that in (35) we have  $v_t(Y)j(Y)$ , whereas in (36) we have  $v_{2t}(2Y)j(2Y)$ . The smoothness assumption on  $f$  in the inversion formula is necessary to guarantee the convergence in the inversion formula (35). (The optimal smoothness conditions are not known in general; Stenzel actually assumes that  $f$  is  $C^\infty$ .) As in the  $\mathbb{R}^n$  case, the inversion formula in (35) is *not* the one obtained by viewing the heat operator as a unitary map (as in the isometry formula) and then taking the adjoint.

The special case of Theorem 10 in which  $U/K$  is a compact Lie group was established in [H1,H2]. (The compact group case is the one in which  $U$  is  $H \times H$  and  $K$  is the diagonal copy of  $H$  inside  $H \times H$ , where  $H$  is a simply connected compact Lie group.) See also [HM1,KR2] for an elementary proof of the isometry formula in the case of  $X = S^d$ .

The proof of the inversion formula hinges on the duality between the compact symmetric space  $U/K$  and noncompact symmetric space  $G/K$ . Specifically, for a holomorphic function  $F$  on  $U_{\mathbb{C}}/K_{\mathbb{C}} \cong T(U/K)$  we have that applying the Laplacian for  $G_x/K_x$  in each fiber and then restricting to the base gives the negative of the result of first restricting  $F$  to the base and then applying the Laplacian for  $U/K$ . So, roughly, the Laplacian in the fiber is the negative of the Laplacian on the base, on holomorphic functions. (Compare the result in  $\mathbb{C}$  that  $d^2/dy^2$  is the negative of  $d^2/dx^2$  when applied to a holomorphic function.) The argument is then that applying the *forward* heat equation in the fibers (by integrating against the heat kernel) has the effect of computing the *backward* heat equation in the base. The proof of the isometry formula may then be reduced to the inversion formula; in the process of this reduction, the change from  $v_t(Y)j(Y)$  to  $v_{2t}(2Y)j(2Y)$  occurs naturally.

## 5.2. The compact group case

Although the Jacobian function  $j$  is explicitly computable for any symmetric space, the heat kernel  $v_t$  is not. Nevertheless, if  $X$  is isometric to a simply connected compact Lie group with a bi-invariant metric, then the dual noncompact symmetric space is of the complex type and in this case there is an explicit formula for  $v_t$  due to Gangolli [Ga, Proposition 3.2]. Expressed in terms of the heat kernel *measure*, this formula

becomes

$$v_t(Y)j(Y) dY = e^{-ct/2} j(Y)^{1/2} \frac{e^{-|Y|^2/2t}}{(2\pi t)^{d/2}} dY,$$

where  $dY$  is Lebesgue measure on the fiber,  $d = \dim(U/K) = \dim(G/K)$ , and  $c$  is the norm-squared of half the sum of the positive roots for  $X$  (thinking of  $X$  as a symmetric space and counting the roots with their multiplicities). In the expression for the heat kernel function, we would have  $j(Y)^{-1/2}$  instead of  $j(Y)^{1/2}$ . Thus we obtain the following.

**Theorem 11.** *In the compact group case, the inversion formula take the form*

$$f(x) = e^{-ct/2} \int_{T_x(U/K)} F(\exp_x(iY)) j(Y)^{1/2} \frac{e^{-|Y|^2/2t}}{(2\pi t)^{d/2}} dY \tag{37}$$

and the isometry formula takes the form

$$\int_{U/K} |f(x)|^2 dx = e^{-ct} \int_{U/K} \int_{T_x(U/K)} |F(\exp_x(iY))|^2 j(2Y)^{1/2} \frac{e^{-|Y|^2/t}}{(\pi t)^{d/2}} dY dx. \tag{38}$$

As in the general case, (37) holds for sufficiently regular  $f$  in  $L^2(U/K)$  and (38) holds for all  $f$  in  $L^2(U/K)$ .

If we specialize further to the case in which  $X$  is the unit sphere  $S^3$  inside  $\mathbb{R}^4$  (so that  $X$  is isometric to the compact group  $SU(2)$ ) and put in the explicit expression for  $j(Y)$ , the inversion formula becomes

$$f(x) = e^{-t/2} \int_{T_x(S^3)} F(\exp_x(iY)) \frac{\sinh |Y|}{|Y|} \frac{e^{-|Y|^2/2t}}{(2\pi t)^{3/2}} dY, \tag{39}$$

and this isometry formula becomes

$$\int_{S^3} |f(x)|^2 dx = e^{-t} \int_{S^3} \int_{T_x(S^3)} |F(\exp_x(iY))|^2 \frac{\sinh |2Y|}{|2Y|} \frac{e^{-|Y|^2/t}}{(\pi t)^{3/2}} dY. \tag{40}$$

### 5.3. Radial functions in the compact group case

In the compact group case, Florentino, Mourão, and Nunes have obtained a special form of the isometry theorem for radial functions. In this case, the radial functions (in the symmetric space sense) are simply the class functions on the compact group. Our Theorem 3 is just the noncompact dual to Theorem 2.2 of [FMN2]. There does not

appear to be an analog of our Theorem 2 in the compact group case, because there the exponential mapping is not a global diffeomorphism.

We continue to use symmetric space notation rather than switching to compact group notation. Let  $\mathfrak{a}$  be a maximal commutative subspace of  $\mathfrak{p}$  and let  $A = \exp_{x_0}(\mathfrak{a})$ . Then  $A$  is a “maximal flat” in  $X$  and is isometric to a flat Euclidean torus. Every point  $x$  in  $U/K$  can be mapped by the left action of  $K$  into  $A$ . Thus a radial function is determined by its values on  $A$ .

Because  $\mathfrak{a}$  is commutative, we can simultaneously identify the tangent space at every point in  $A$  with  $\mathfrak{a}$ . We now define the “complexification”  $A_{\mathbb{C}}$  of  $A$  to be the image under  $\Phi$  of  $T(A) \subset T(X)$ , where  $\Phi$  is the map in Proposition 8. That is to say, we define

$$A_{\mathbb{C}} = \{ \exp_a(iY) \in X_{\mathbb{C}} \mid a \in A, Y \in \mathfrak{a} \}.$$

The restriction of  $\Phi$  to  $T(A)$  is a diffeomorphism of  $T(A)$  with  $A_{\mathbb{C}}$ . (If we identify  $X$  with a compact Lie group  $H$ , then  $A$  is a maximal torus  $T$  inside  $H$  and  $A_{\mathbb{C}}$  is the complexification of  $T$  inside  $H_{\mathbb{C}}$ .)

It is convenient to multiply the Riemannian volume measures on  $X$  and  $A$  by normalizing factors, so that the total volume of each manifold is equal to 1. If we used instead the un-normalized Riemannian volume measures, there would be an additional normalization constant in Theorem 12, as in Theorem 3. We now let  $\eta$  be the Weyl denominator function on  $A$ . This is the smooth, real-valued function, unique up to an overall sign, with the property that

$$\int_X f(x) dx = \frac{1}{|W|} \int_A f(a) \eta(a)^2 da,$$

for all continuous radial functions  $f$  on  $X$ . Here  $|W|$  is the order of the Weyl group for  $X$ , and  $dx$  and  $da$  are the normalized volume measures on  $X$  and  $A$ , respectively. The function  $\eta$  has an entire analytic continuation from  $A$  to  $A_{\mathbb{C}}$ , also denoted  $\eta$ .

We are now ready to state Theorem 2.2 of [FMN2], using slightly different notation.

**Theorem 12** (Florentino, Mourão, and Nunes). *Suppose  $X$  is isometric to a compact Lie group with a bi-invariant metric. If  $f$  is any radial function in  $L^2(X)$ , let  $F$  denote the analytic continuation to  $X_{\mathbb{C}}$  of  $e^{t\Delta/2} f$ . Then*

$$\int_X |f(x)|^2 dx = \frac{e^{-ct}}{|W|} \int_A \int_{\mathfrak{a}} |F(\exp_a(iY))|^2 |\eta(\exp_a(iY))|^2 \frac{e^{-|Y|^2/t}}{(\pi t)^{r/2}} dY da. \tag{41}$$

Here  $r$  is the dimension of  $\mathfrak{a}$ , the constant  $c$  is the same as in (37) and (38),  $|W|$  is the order of the Weyl group, and  $dx$  and  $da$  are the normalized Riemannian volume measures on  $X$  and  $A$ , respectively.

Furthermore, if  $F$  is any Weyl-invariant holomorphic function on  $A_{\mathbb{C}}$  for which the integral on the right-hand side of (41) is finite, then there exists a unique radial function  $f$  in  $L^2(X)$  such that  $F = e^{t\Delta/2} f$  on  $A$ .

Consider, for example, the case in which  $X$  is the unit sphere  $S^3$  in  $\mathbb{R}^4$ , in which case  $X_{\mathbb{C}}$  is the complexified sphere

$$S^3_{\mathbb{C}} := \left\{ z \in \mathbb{C}^4 \mid z_1^2 + z_2^2 + z_3^2 + z_4^2 = 1 \right\}.$$

Fix the basepoint  $x_0 := (1, 0, 0, 0)$ . In that case, a “radial” function on  $S^3$  is one that is invariant under the rotations that fix  $x_0$ . If we take  $\mathfrak{a}$  to be the one-dimensional subspace of  $T_{x_0}(S^3)$  spanned by the vector  $e_2 = (0, 1, 0, 0)$ , then  $A$  is the set

$$A = \{ (\cos \theta, \sin \theta, 0, 0) \mid \theta \in \mathbb{R} \} \tag{42}$$

and  $A_{\mathbb{C}}$  is the set of points in  $S^3_{\mathbb{C}}$  of the same form as in (42), except with  $\theta$  in  $\mathbb{C}$ . In the  $S^3$  case,  $|W| = 2$ ,  $c = 1$ , the Weyl denominator is  $2 \sin \theta$ , and the normalized measure on  $A$  is  $d\theta/2\pi$ . Thus (41) becomes

$$\begin{aligned} \int_{S^3} |f(x)|^2 dx &= \frac{e^{-t}}{2} \int_0^{2\pi} \int_{-\infty}^{\infty} |F[(\cos(\theta + iy), \sin(\theta + iy), 0, 0)]|^2 \\ &\quad \times |2 \sin(\theta + iy)|^2 \frac{e^{-y^2/t}}{(\pi t)^{1/2}} dy \frac{d\theta}{2\pi}. \end{aligned} \tag{43}$$

### 6. Concluding remarks

In this paper we have established an isometry formula (in two different versions) for the Segal–Bargmann transform of radial functions and an inversion formula for the Segal–Bargmann transform of general functions, both in the case of a noncompact symmetric space of the complex type. Both the isometry formula and the inversion formula require a cancellation of singularities, but otherwise they closely parallel the results from the compact group case. Specifically, Theorem 3 in the complex case is very similar to Theorem 12 in the compact group case and Theorem 6 in the complex case is very similar to the inversion formula in Theorem 11 in the compact group case. Besides the cancellation of singularities, the main difference between the formulas in the two cases is the interchange of hyperbolic sine with ordinary sine. It is natural, then, to look ahead and consider the prospects for obtaining results in the noncompact setting paralleling *all* of the results we have for compact symmetric spaces. This would entail extending the isometry result to nonradial functions and then extending both the isometry and the inversion results to other noncompact symmetric spaces, beyond those

of the complex type. In [H2] in the compact group case and in [St] in the general compact symmetric space case, the inversion formula is proved first and the isometry formula obtained from it. As a result, we fully expect that the inversion formula we prove here will lead to an isometry formula for not-necessarily radial functions in the complex case. A precise statement of the result we have in mind is given in [H9] in the case of hyperbolic 3-space.

Meanwhile, we have recently received a preprint by Krötz et al. [KOS] that establishes an isometry formula for general functions (not necessarily radial) on general symmetric spaces of the noncompact type (not necessarily of the complex type). However, this isometry formula does not, at least on the surface, seem parallel to the compact case. In particular, in the complex case, this isometry formula does *not* reduce to the one we have in mind, at least not without some substantial manipulation of the formula in [KOS, Theorem 3.3]. Nevertheless, the result of [KOS] is a big step toward understanding the situation for general symmetric spaces of the noncompact type. There may well be a connection, in the complex case, between the results of [KOS] and the isometry formula we have in mind, but this remains to be worked out. If the isometry formula can be understood better for general noncompact symmetric spaces, this understanding may pave the way for progress on the inversion formula as well.

Note that in the case of compact symmetric spaces, the results take on a particularly simple and explicit form in the compact group case. (Compare Theorem 10 to Theorem 11.) Our results in this paper are for the noncompact symmetric spaces of the complex type; this case is just the dual of the compact group case. Thus, one cannot expect the same level of explicitness for noncompact symmetric spaces that are not of the complex type. Instead, we may hope for results that involve some suitably “unwrapped” version of the heat kernel measure on the dual compact symmetric space, where in general there will not be an explicit formula for this unwrapped heat kernel.

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