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On approximation of solutions of multidimensional SDE's with reflecting boundary conditions

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Let D be either a convex domain in $\mathbb{R}^d$ or a domain satisfying the conditions (A) and (B) considered by Lions and Sznitman [7] and Saisho [11]. We estimate the rate of L^p convergence for Euler and Euler–Peano schemes for stochastic differential equations in D with normal reflection at the boundary of the form $X_t = X_0 + \int_0^t f(X_s)\,\mathrm{d}W_s + \int_0^t g(X_s)\,\mathrm{d}s + K_t$, $t \in \mathbb{R}^+$, where W is a d-dimensional Wiener process. As a consequence we give the rate of almost sure convergence for these schemes.

AMS Subject Classifications: 60H20, 60H99, 60F15.

stochastic differential equations * reflecting boundary * strong solutions * rate of L^p convergence * almost sure convergence

1. Introduction

In this paper we investigate L^p convergence as well as almost sure convergence of time-discretization schemes for d-dimensional stochastic differential equation (SDE) on a domain D with reflecting boundary condition. Given a function $f : \bar{D} = D \cup \partial D \to \mathbb{R}^d \otimes \mathbb{R}^d$, $f(x) = \{f_{ij}(x)\}_{i,j=1,\dots,d}$ we consider the following SDE:

$$X_t^i = X_0^i + \sum_{j=1}^{d} \int_0^t f_{ij}(X_s)\,\mathrm{d}W_s^j + \int_0^t g_i(X_s)\,\mathrm{d}s + K_t^i, \tag{1}$$

$i = 1,\dots, d$, $t \in \mathbb{R}^+$, where $W_t = (W_t^1,\dots, W_t^d)$ is a d-dimensional Wiener process $X_t = (X_t^1,\dots, X_t^d)$ is a reflecting process, on $\bar{D}$ and $K_t = (K_t^1,\dots, K_t^d)$ is a bounded variation process with variation $|K|_t$ increasing only when $X_t \in \partial D$ (the precise definition will be given in Section 2). This equation is called a Skorokhod SDE with the analogy to the one-

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dimensional case first discussed by Skorokhod [12] for $D=\mathbb{R}^+$. The case of reflecting processes in a domain more general than a half-line or half-space (i.e. $D=\mathbb{R}^+\times\mathbb{R}^{d-1}$) has been discussed firstly in the paper by Tanaka [14], where D is any convex subset of $\mathbb{R}^d$ and then by Lions and Sznitman [7] and Saisho [11], where D is a domain satisfying some mild conditions (A) and (B) given in Section 2. In particular in [14] and [11] it is proven, that if f, g are Lipschitz continuous and bounded on $\bar{D}$ i.e. there exists a constant $L>0$ such that for every $x, y\in\bar{D}$,

$$\|f(x)-f(y)\|+|g(x)-g(y)|\leqslant L|x-y|\,,\quad \|f(x)\|,\ |g(x)|\leqslant L\,, \tag{2}$$

where $\|\cdot\|$ denotes the usual norm in the space of linear operators from $\mathbb{R}^d$ into $\mathbb{R}^d$, then there exists a unique strong solution to the SDE (1).

Let us consider an array $\{\{t_{nk}\}\}$ of nonnegative numbers such that in each nth row the sequence $\{t_{nk}\}$ forms a partition on $\mathbb{R}^+$ with the property $0=t_{n0}<t_{n1}<\cdots$, $\lim_{k\to\infty}t_{nk}=+\infty$ and

$$\max(t_{nk}-t_{n,k-1})\leqslant 1/n\,,\quad n\in\mathbb{N}\,. \tag{3}$$

For the array $\{\{t_{nk}\}\}$ we define the sequence of summation rules $\{\rho^n\}$, $\rho^n:\mathbb{R}^+\to\mathbb{R}^+$ by $\rho^n_t=\max\{t_{nk};\ t_{nk}\leqslant t\}$. For every $x\in\mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$ ($\mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$ is the space of all mappings $x:\mathbb{R}^+\to\mathbb{R}^d$, which are right continuous and admit left-hand limits) we define the sequence $\{x^{\rho^n}\}$ of discretizations of x, $x^{\rho^n}_t=x_{\rho^n_t}=x_{t_{nk}}$ for $t\in[t_{nk}, t_{n,k+1})$, $k\in\mathbb{N}\cup\{0\}$, $n\in\mathbb{N}$. In the present paper we assume that D is either a convex set or a general domain satisfying the conditions (A) and (B) and we consider Euler and Euler–Peano schemes for the SDE (1). More precisely, we investigate the approximations $\{\bar{X}^n\}$ and $\{\hat{X}^n\}$, which are the solutions to the appropriate SDE's with reflecting boundary conditions

$$\bar{X}^n_t=X_0+\int_0^t f(\bar{X}^n_{s-})\,\mathrm{d}W^{\rho^n}_s+\int_0^t g(\bar{X}^n_{s-})\,\mathrm{d}\rho^n_s+\bar{K}^n_t\,,\quad t\in\mathbb{R}^+\,, \tag{4}$$

and

$$\hat{X}^n_t=X_0+\int_0^t f(\hat{X}^{n,\rho^n}_{s-})\,\mathrm{d}W_s+\int_0^t g(\hat{X}^{n,\rho^n}_{s-})\,\mathrm{d}s+\hat{K}^n_t\,,\quad t\in\mathbb{R}^+\,. \tag{5}$$

It has been observed in [13, Corollary 10] and [11, pp. 473–474] that

$$\sup_{t\leqslant q}|\bar{X}^n_t-X_t|\xrightarrow[\mathscr{P}]{}0\,,\quad q\in\mathbb{R}^+\,,$$

and

$$\sup_{t\leqslant q}|\hat{X}^n_t-X_t|\xrightarrow[\mathscr{P}]{}0\,,\quad q\in\mathbb{R}^+\,,$$

respectively. Note that if $D=\mathbb{R}^d$ then $\bar{X}^n_t=\hat{X}^{n,\rho^n}_t$ and (4) is a classical Euler scheme considered firstly in Maruyama [8]. In the case $D\neq\mathbb{R}^d$ the equality $\bar{X}^n_t=\hat{X}^{n,\rho^n}_t$ need not be satisfied. In this case the rate of mean-square convergence in the above schemes was considered before only if $D=\mathbb{R}^+\times\mathbb{R}^{d-1}$ by Chitashvili and Lazrieva [3], Kinkladze [5] (the scheme (4)) and Lépingle [6] (the scheme (5)). Let us observe that in this case we can write down the explicit formulas for the solutions $\bar{X}^n$ and $\hat{X}^n$. Namely,

$$\bar{X}_t^{n,i} = \begin{cases} \bar{Y}_t^{n,1} - 0 \wedge \inf_{s \leq t} \bar{Y}_t^{n,1} & \text{if } i=1 , \\ \bar{Y}_t^{n,i} & \text{if } i=2,\dots,d , \end{cases}$$

where

$$\bar{Y}_t^{n,i} = \begin{cases} X_0^i & \text{if } t \in [0, t_{n1}[, \\ \bar{Y}_{t_{n,k-1}}^{n,i} + \sum_{j=1}^d f_{ij}(\bar{Y}_{t_{n,k-1}}^n)(W_{t_{nk}}^j - W_{t_{n,k-1}}^j) & \\ \quad + g_i(\bar{Y}_{t_{n,k-1}}^n)(t_{nk} - t_{n,k-1}) & \text{if } t \in [t_{nk}, t_{n,k+1}[. \end{cases}$$

Similarly

$$\hat{X}_t^{n,i} = \begin{cases} \hat{Y}_t^{n,1} - 0 \wedge \inf_{s \leq t} \hat{Y}_t^{n,1} & \text{if } i=1 , \\ \hat{Y}_t^{n,i} & \text{if } i=2,\dots,d , \end{cases}$$

where

$$\hat{Y}_t^{n,i} = \begin{cases} X_0^i & \text{if } t=0 , \\ \hat{Y}_{t_{nk}}^{n,i} + \sum_{j=1}^d f_{ij}(\hat{Y}_{t_{nk}}^n)(W_t^j - W_{t_{nk}}^j) & \\ \quad + g_i(\hat{Y}_{t_{nk}}^n)(t - t_{nk}) & \text{if } t \in]t_{nk}, t_{n,k+1}] . \end{cases}$$

In our paper we assume that D is a domain in $\mathbb{R}^d$ with nonempty interior and we give the rate of L^p as well as almost sure convergence for the schemes (4), (5) for domain more general than half-space.

Let D be a convex domain in $\mathbb{R}^d$. For the Euler scheme we show that there exists a sequence of stopping times $\{\tau_n\}$, $\tau_n \to_{\mathscr{P}} +\infty$ such that

$$E \sup_{t \leq q} |\bar{X}_t^{n,\tau_n -} - X_t^{\tau_n}|^{2p} = O(1/n^{p/2-\varepsilon}) , \quad \varepsilon > 0, q \in \mathbb{R}^+, p \in \mathbb{N} , \tag{6}$$

and

$$n^{1/4-\varepsilon} \sup_{t \leq q} |\bar{X}_t^{n,\tau_n -} - X_t^{\tau_n}| \to 0 , \quad \mathscr{P}\text{-a.s.}, \varepsilon > 0, q \in \mathbb{R}^+ . \tag{7}$$

In this case for the Euler–Peano scheme we obtain

$$E \sup_{t \leq q} |\hat{X}_t^n - X_t|^{2p} = O(1/n^p) , \quad q \in \mathbb{R}^+, p \in \mathbb{N} , \tag{8}$$

and

$$n^{1/2-\varepsilon} \sup_{t \leq q} |\hat{X}_t^n - X_t| \to 0 , \quad \mathscr{P}\text{-a.s.}, \varepsilon > 0, q \in \mathbb{R}^+ . \tag{9}$$

For a large class of convex domains we are able to strengthen (6) and (7). If D is a convex subset of $\mathbb{R}^d$ satisfying the condition (β), which is automatically fulfilled if D is bounded or $d < 3$ (the precise definition is given in Section 2), then we prove that

$$E \sup_{t \leq q} |\bar{X}_t^n - X_t|^{2p} = O(1/n^{p/2-\varepsilon}) , \quad \varepsilon > 0, q \in \mathbb{R}^+, p \in \mathbb{N} ,$$

and

$$n^{1/4-\varepsilon}\sup_{t\leqslant q}|\bar{X}_t^n - X_t| \to 0, \quad \mathscr{P}\text{-a.s., } \varepsilon>0,\, q\in\mathbb{R}^+ .$$

If D is a convex polyhedron, i.e. $D=\bigcap_{i=1}^N D_i$, where D_i is a closed half-space, we prove even more, namely that

$$E\sup_{t\leqslant q}|\bar{X}_t^n - X_t|^{2p} = \mathrm{O}(1/n^{p-\varepsilon}), \quad \varepsilon>0,\, q\in\mathbb{R}^+,\, p\in\mathbb{N}, \tag{10}$$

and

$$n^{1/2-\varepsilon}\sup_{t\leqslant q}|\bar{X}_t^n - X_t| \to 0, \quad \mathscr{P}\text{-a.s., } \varepsilon>0,\, q\in\mathbb{R}^+ . \tag{11}$$

Let us mention that in the case of half-space some results of type (10) were obtained in [3,5], while (8) was announced in [6].

Now let D be a general domain satisfying the conditions (A) and (B). For the Euler scheme we prove that (6) and (7) are true, too. In the case of the Euler–Peano scheme we prove that there exists an array of stopping times $\{\{\tau_n^k\}\}$ such that $\lim_{k\to+\infty}\limsup_{n\to+\infty}\mathscr{P}(\tau_n^k\leqslant q)=0$ and

$$E\sup_{t\leqslant q}|\hat{X}_t^{n,\tau_n^k} - X_t^{\tau_n^k}|^{2p} = \mathrm{O}(1/n^{p}), \quad k\in\mathbb{N},\, q\in\mathbb{R}^+,\, p\in\mathbb{N}.$$

We can also find a sequence of stopping times $\{\tau_n\}$, $\tau_n\to_{\mathscr{P}}+\infty$ such that

$$n^{1/2-\varepsilon}\sup_{t\leqslant q}|\hat{X}_t^{n,\tau_n} - X_t^{\tau_n}| \to 0, \quad \mathscr{P}\text{-a.s., } \varepsilon>0,\, q\in\mathbb{R}^+ .$$

The paper is organised as follows. In Section 2 we give some basic definitions and basic facts about SDE's with reflecting boundary conditions. In particular, we give L^p versions of general inequalities for reflected processes proved earlier in [13], which are our main tool in the proofs. In Section 3 we consider the case of convex domain and in Section 4 the general case of domains satisfying the conditions (A) and (B). Finally, the Appendix contains some versions of Gronwall's lemma and other technical lemmas used in the proofs.

Let us introduce now some definitions and notations used further on. For $x\in\mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$, $A\subset\mathbb{R}^+$ we denote $\omega_x A=\sup_{s,t\in A}|x_t-x_s|$,

$$\omega_x(h,q)=\sup_{s,t\in[0,q],\,|s-t|\leqslant h}|x_t-x_s| ,$$

and

$$\omega_x'(h,q)=\inf_{(s_k)}\max_{1<k\leqslant r}\omega_x[s_{k-1},s_k[,$$

where $0=s_0<s_1<\cdots<s_r=q$, $s_k-s_{k-1}>h$, $k=1,2,\ldots,r$. $(\Omega,\mathscr{F},\mathscr{P})$ is a probability space and $(\mathscr{F}_t)$ a filtration on $(\Omega,\mathscr{F},\mathscr{P})$ satisfying the usual conditions. Let X be an $(\mathscr{F}_t)$ adapted process and τ be an $(\mathscr{F}_t)$ stopping time. We write X^τ and $X^{\tau-}$ to denote the stopped processes $X_{\cdot\wedge\tau}$ and $X_{\cdot\wedge\tau-}$, respectively. If $X=(X^1,\ldots,X^d)$ is a local martingale then $[X]_t$ stands for $\sum_{i=1}^d [X^i]_t$, where for $i=1,\ldots,d$, $[X^i]$ is a quadratic variation process of X^i. If $K=(K^1,\ldots,K^d)$ is a process with locally finite variation, then $|K|_t=\sum_{i=1}^d |K^i|_t$, where $|K^i|_t$ is a total variation of K^i on $[0,t]$.

2. Preliminaries

Let D be a domain in $\mathbb{R}^d$ with nonempty interior. Define the set $\mathcal{N}_x$ of inward normal unit vectors at $x\in\partial D$ by $\mathcal{N}_x=\bigcup_{r>0}\mathcal{N}_{x,r}$,

$$\mathcal{N}_{x,r}=\{\boldsymbol{n}\in\mathbb{R}^d;\ |\boldsymbol{n}|=1,\ B(x-r\boldsymbol{n},r)\cap D=\emptyset\}\,,$$

where $B(z,r)=\{y\in\mathbb{R}^d;\ |y-z|<r\}$, $z\in\mathbb{R}^d$, $r>0$.

Following Tanaka [14], Lions and Sznitman [7] and Saisho [11] we introduce three assumptions.

(β) There exist constants $\varepsilon>0$ and $\delta>0$ such that for every $x\in\partial D$ we can find $x_0\in D$ such that $B(x_0,\varepsilon)\subset D$ and $|x-x_0|\leqslant\delta$.

(A) There exists a constant $r_0>0$ such that

$$\mathcal{N}_x=\mathcal{N}_{x,r_0}\neq\emptyset \quad \text{for every } x\in\partial D\,.$$

(B) There exist constants $\delta>0$, $\beta\geqslant 1$ such that for every $x\in\partial D$ there is a unit vector $\boldsymbol{l}_x$ with the following property:

$$\langle \boldsymbol{l}_x,\boldsymbol{n}\rangle\geqslant\frac{1}{\beta}\quad\text{for every } \boldsymbol{n}\in\bigcup_{y\in B(x,\delta)\cap\partial D}\mathcal{N}_y\,,$$

where $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $\mathbb{R}^d$.

Remark 1 [14,7]. (i) If the condition (A) is satisfied and $\operatorname{dist}(x,\bar D)<r_0$, $x\notin\bar D$ then there exists a unique $[x]_\partial\in\bar D$ such that $|x-[x]_\partial|=\operatorname{dist}(x,\bar D)$ and, moreover, $([x]_\partial-x)/|[x]_\partial-x|\in\mathcal{N}_{\bar x}$.

(ii) If D is a convex domain in $\mathbb{R}^d$ with nonempty interior then $r_0=+\infty$ and the assumptions (β), (A) and (B) are satisfied for $d=1,2$. For $d>2$ there exists a sequence of bounded convex sets $\{D_k\}$ satisfying the conditions (β), (A) and (B) such that $D_k\uparrow D$. In this case (β) ⇒ (B) and we can put $D_k=D\cap\{x\in\mathbb{R}^d;\ |x|<k\}$, $k\in\mathbb{N}$.

The Skorokhod deterministic problem is stated in the following manner.

Definition 1. Let $y\in\mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$ and $y_0\in\bar D$. We will say that a pair $(x,k)\in\mathbb{D}(\mathbb{R}^+,\mathbb{R}^{2d})$ is a solution of the Skorokhod problem associated with y if

(i) $x_t=y_t+k_t$, $t\in\mathbb{R}^+$,

(ii) $x_t\in\bar D$, $t\in\mathbb{R}^+$,

(iii) k is a function with bounded variation on each finite interval such that $k_0=0$ and

$$k_t = \int_0^t \boldsymbol{n}_s \, \mathrm{d}|k|_s, \quad |k|_t = \int_0^t \mathbf{1}_{\{x_s \in \partial D\}} \, \mathrm{d}|k|_s,$$

where $\boldsymbol{n}_s \in \mathcal{N}_{x_s}$ if $x_s \in \partial D$ and $|k|_t$ denotes the total variation of k on $[0, t]$, $t \in \mathbb{R}^+$.

Remark 2 [14,11,13]. If either D is convex or the condition (A) is satisfied and y is continuous then there exists at most one solution to the Skorokhod problem associated with y. However if we assume only the conditions (A) and (B) then the solution of the Skorokhod problem associated with fixed discontinuous y is in general not unique (see Example 1 in [13]).

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space and let $(\mathcal{F}_t)$ be a filtration on $(\Omega, \mathcal{F}, \mathcal{P})$ satisfying the usual conditions.

Definition 2. Let Y be an $(\mathcal{F}_t)$ adapted process and $Y_0 \in \bar{D}$. We will say that a pair (X, K) of $(\mathcal{F}_t)$ adapted processes solves the Skorokhod problem associated with Y if and only if for every $\omega \in \Omega$, $(X(\omega), K(\omega))$ is a solution of the Skorokhod problem corresponding to $Y(\omega)$.

In this section we will give L^p versions of some estimates for reflected processes proved earlier in [13]. Let (X, K) be a solution to the Skorokhod problem associated with a semimartingale Y of the form

$$Y_t = Y_0 + M_t + V_t, \quad t \in \mathbb{R}^+, \tag{12}$$

where M is an $(\mathcal{F}_t)$ adapted local martingale, V is an $(\mathcal{F}_t)$ process with bounded variation, $M_0 = V_0 = 0$. Assume also, we have given another $(\mathcal{F}_t)$ adapted process $\hat{Y}$ and $\hat{Y}$ admits the decomposition

$$\hat{Y}_t = Y_0 + \hat{M}_t + \hat{V}_t, \quad t \in \mathbb{R}^+, \tag{13}$$

where $\hat{M}$ is an $(\mathcal{F}_t)$ adapted local martingale $\hat{M}_0 = 0$ and $\hat{V}$ is an $(\mathcal{F}_t)$ adapted process with bounded variation, $\hat{V}_0 = 0$. Let $(\hat{X}, \hat{K})$ be a solution of the Skorokhod problem corresponding to $\hat{Y}$.

Theorem 1. *Assume D satisfies* (A) *and* (B). *Let Y_0, $\hat{Y}_0 \in \bar{D}$ and let processes Y, $\hat{Y}$ fulfil* (12) *and* (13), *respectively, where M, $\bar{M}$ are square integrable martingales and V, $\hat{V}$ are processes with square integrable variation. If $r_0 < +\infty$ we assume additionally $|\Delta K|$, $|\Delta \hat{K}| \leqslant \frac{1}{4} r_0$ and that there exists a constant a such that $|K|_\infty$, $|\hat{K}|_\infty \leqslant a$. Then for every $p \in \mathbb{N}$ there exists a constant C_p depending on a and r_0 such that for every $(\mathcal{F}_t)$ stopping time σ:*

(i) $\quad E \sup_{t \leqslant \sigma} |X_s - \hat{X}_s|^{2p} \leqslant C_p E\{[M - \hat{M}]_\sigma^p + |V - \hat{V}|_\sigma^{2p}\}$.

(ii) $\quad E \sup_{t \leqslant \sigma} |K_s - \hat{K}_s|^{2p} \leqslant C_p E\{[M - \hat{M}]_\sigma^p + |V - \hat{V}|_\sigma^{2p}\}$.

(iii) $\quad E\sup_{t<\sigma}|K_s-\hat{K}_s|^{2p}\leqslant C_pE\{\langle M-\hat{M}\rangle^p_{\sigma-}+[M-\hat{M}]^p_{\sigma-}+|V-\hat{V}|^{2p}_{\sigma-}\}$.

Proof. (i) By [11, Lemma 2] for every $t\in\mathbb{R}^+$,

$$|X_t-\hat{X}_t|^2\leqslant\left\{|Y_t-\hat{Y}_t|^2+\frac{1}{r_0}\int_0^t|X_s-\hat{X}_s|^2\,\mathrm{d}(|K|_s+|\hat{K}|_s)\right.$$
$$\left.+2\int_0^t\langle Y_t-Y_s-\hat{Y}_t+\hat{Y}_s,\,\mathrm{d}(K_s-\hat{K}_s)\rangle\right\}. \tag{14}$$

Since $|\Delta K|,\ |\Delta\hat{K}|\leqslant\frac{1}{4}r_0$,

$$|X_t-\hat{X}_t|^2\leqslant2\left\{|Y_t-\hat{Y}_t|^2+\frac{1}{r_0}\int_0^{t-}|X_s-\hat{X}_s|^2\,\mathrm{d}(|K|_s+|\hat{K}|_s)\right.$$
$$\left.+2\int_0^t\langle Y_t-Y_s-\hat{Y}_t+\hat{Y}_s,\,\mathrm{d}(K_s-\hat{K}_s)\rangle\right\}.$$

On the other hand by the integration by parts formula

$$2\int_0^t\langle Y_t-Y_s-\hat{Y}_t+\hat{Y}_s,\,\mathrm{d}(K_s-\hat{K}_s)\rangle$$
$$=2\int_0^t\langle X_{s-}-\hat{X}_{s-},\,\mathrm{d}(M_s-\hat{M}_s)\rangle$$
$$+2\int_0^t\langle X_{s-}-\hat{X}_{s-},\,\mathrm{d}(V_s-\hat{V}_s)\rangle+[Y-\hat{Y}]_t-|Y_t-\hat{Y}_t|^2.$$

Hence for every stopping time τ and $p\in\mathbb{N}$,

$$E\sup_{t\leqslant\tau}|X_t-\hat{X}_t|^{2p}$$
$$\leqslant2^p4^{p-1}\left\{E\left(\frac{1}{r_0}\int_0^{\tau-}|X_s-\hat{X}_s|^2\,\mathrm{d}(|K|_s+|\hat{K}|_s)\right)^p\right.$$
$$+2^pE\sup_{t\leqslant\tau}\left|\int_0^t\langle X_{s-}-\hat{X}_{s-},\,\mathrm{d}(M_s-\hat{M}_s)\rangle\right|^p$$
$$\left.+2^pE\sup_{t\leqslant\tau}\left|\int_0^t\langle X_{s-}-\hat{X}_{s-},\,\mathrm{d}(V_s-\hat{V}_s)\rangle\right|^p+E[Y-\hat{Y}]^p_\tau\right\}.$$

Next by the Burkholder–Davis–Gundy and Schwarz's inequalities

$$E\sup_{t\leqslant\tau}\left|\int_0^t\langle(X_{s-}-\hat{X}_{s-}),\,\mathrm{d}(M_s-\hat{M}_s)\rangle\right|^p$$

$$\leqslant c(p)E\left(\int_0^\tau |X_{s-}-\hat{X}_{s-}|^2, \mathrm{d}[M-\hat{M}]_s\right)^{p/2}$$

$$\leqslant c(p)\left(E\sup_{s\leqslant\tau}|X_{s-}-\hat{X}_{s-}|^{2p}\right)^{1/2}(E[M-\hat{M}]_\tau^p)^{1/2},$$

and

$$E\sup_{t\leqslant\tau}\left|\int_0^t \langle (X_{s-}-\hat{X}_{s-}), \mathrm{d}(V_s-\hat{V}_s)\rangle\right|^p$$

$$\leqslant E\sup_{s\leqslant\tau}|X_{s-}-\hat{X}_{s-}|^p|V-\hat{V}|_\tau^p$$

$$\leqslant\left(E\sup_{s\leqslant\tau}|X_{s-}-\hat{X}_{s-}|^{2p}\right)^{1/2}(E|V-\hat{V}|_\tau^{2p})^{1/2}.$$

By simple calculations

$$E[Y-\hat{Y}]_\tau^p\leqslant 2^pE[M-\hat{M}]_\tau^p+2^pE[V-\hat{V}]_\tau^p$$
$$\leqslant 2^pE[M-\hat{M}]_\tau^p+2^pE|V-\hat{V}|_\tau^{2p}.$$

If $r_0<+\infty$ then there exists a constant $c(p, a, r_0)$ such that

$$\left(\frac{1}{r_0}\int_0^{\tau-}|X_s-\hat{X}_s|^2\,\mathrm{d}(|K|_s+|\hat{K}|_s)\right)^p$$
$$\leqslant c(p, a, r_0)\int_0^{\tau-}|X_s-\hat{X}_s|^{2p}\,\mathrm{d}(|K|_s+|\hat{K}|_s).$$

If we denote

$$x=\left(E\sup_{t\leqslant\tau}|X_t-\hat{X}|^{2p}\right)^{1/2},$$

$$b_1=(E[M-\hat{M}]_\tau^p+E|V-\hat{V}|_\tau^{2p})^{1/2},$$

$$b_2=\frac{1}{r_0}E\int_0^{\tau-}|X_s-\hat{X}_s|^{2p}\,\mathrm{d}(|K|_s+|\hat{K}|_s),$$

then by the above calculations we deduce that there exists a constant C_1 such that

$$x^2\leqslant C_1(b_2+2b_1x+b_1^2).$$

Since $0\leqslant b_1$, $x<+\infty$ it is clear that $x^2\leqslant C_2(b_2+b_1^2)$ for some constant C_2. To finish the proof it is sufficient to use Lemma 1 of the Appendix. If we set in Lemma 1 $Y_t^1=\sup_{s\leqslant t\wedge\sigma}|X_s-\hat{X}_s|^{2p}$, $Y_t^2=C_2(|K|_{t\wedge\sigma}+|\hat{K}|_{t\wedge\sigma})<C_2 2a+1$ then the proof of (i) is complete.

(ii) Is an obvious consequence of (i) and of the Burkholder–Davis–Gundy inequality.

(iii) We can deduce from (ii) by using the version of Métivier–Pellaumail inequality proved in Pratelli [10] and the arguments from the paper of Chaleyat-Maurel, El Karoui and Marchal [2]. □

Corollary 1. *Assume D is a convex subset in $\mathbb{R}^d$. Let $Y_0, \hat{Y}_0 \in \bar{D}$ and let processes, $Y, \hat{Y}$ fulfil* (12) *and* (13), *respectively, where $M, \hat{M}$ are square integrable martingales and $V, \hat{V}$ are processes with square integrable variation. Then the estimations* (i), (ii) *and* (iii) *are true, too.*

Proof. It is clear that in this case the condition (A) is satisfied with $r_0 = +\infty$. Let $\tau_k = \inf\{t \in \mathbb{R}^+;\ |X_t| \text{ or } |\hat{X}_t| \geqslant k\}$, $k \in \mathbb{N}$. Then of course $X_t^{\tau_k -}$, $\hat{X}_t^{\tau_k} \in \bar{D}_k$, where $D_k = D \cap \{x \in \mathbb{R}^d;\ |x| < k\}$. Next due to Remark 1(ii) and Theorem 2(iii) for every $k \in \mathbb{N}$,

$$E \sup_{t < \tau_k \wedge \sigma} |K_s - \hat{K}_s|^{2p} \leqslant C_p E\{\langle M - \hat{M}\rangle^p_{(\tau_k \wedge \sigma)-} + [M - \hat{M}]^p_{(\tau_k \wedge \sigma)-} + |V - \hat{V}|^{2p}_{(\tau_k \wedge \sigma)-}\}.$$

Since C_p does not depend on δ and β and $\tau_k \uparrow +\infty$, letting $k \uparrow +\infty$ we have (iii) for every convex set in $\mathbb{R}^d$. Similarly

$$\begin{aligned} &E \sup_{t \leqslant \sigma} |K_s - \hat{K}_s|^{2p} \\ &= \lim_{k \to +\infty} E \sup_{t < \tau_k} |K_t^\sigma - \hat{K}_t^\sigma|^{2p} \\ &\leqslant C_p \lim_{k \to +\infty} E\{\langle M^\sigma - \hat{M}^\sigma\rangle^p_{\tau_k -} + [M^\sigma - \hat{M}^\sigma]^p_{\tau_k -} + |V^\sigma - \hat{V}^\sigma|^{2p}_{\tau_k -}\} \\ &= C_p E\{\langle M - \hat{M}\rangle^p_\sigma + [M - \hat{M}]^p_\sigma + |V - \hat{V}|^{2p}_\sigma\} \\ &\leqslant (C_p + p^p) E\{[M - \hat{M}]^p_\sigma + |V - \hat{V}|^{2p}_\sigma\} \end{aligned}$$

and we get (ii). Finally, the estimation (i) is an obvious consequence of (ii) and of the Burkholder–Davis–Gundy inequality. □

Now, let W be an $(\mathscr{F}_t)$ adapted Wiener process. We will say that the SDE (1) has a strong solution if there exists a pair (X, K) of $(\mathscr{F}_t)$ adapted processes such that (X, K) is the solution of the Skorokhod problem associated with

$$X_0 + \int_0^\cdot f(X_s)\, \mathrm{d}W_s + \int_0^\cdot g(X_s)\, \mathrm{d}s .$$

If for two $(\mathscr{F}_t)$ adapted solutions (X, K), (X', K') on $(\Omega, \mathscr{F}, \mathscr{P})$ of the SDE (1) satisfy $\mathscr{P}[(X, K) = (X', K');\ t \in \mathbb{R}^+] = 1$ then we say that strong uniqueness holds for the SDE (1). Similarly we will say that the SDE (5) has a strong solution if there exists a pair $(\hat{X}^n, \hat{K}^n)$ of $(\mathscr{F}_t)$ adapted processes such that $(\hat{X}^n, \hat{K}^n)$ is the solution of the Skorokhod problem associated with

$$X_0 + \int_0^\cdot f(\hat{X}^{n,\rho^n}_{s-})\, \mathrm{d}W_s + \int_0^\cdot g(\hat{X}^{n,\rho^n}_{s-})\, \mathrm{d}s .$$

Let $(\mathscr{F}_t^{\rho^n})$ be a discretization of $(\mathscr{F}_t)$ i.e. $\mathscr{F}_t^{\rho^n} = \mathscr{F}_{t_{nk}}$, $t \in [t_{nk}, t_{n,k+1}[$. We will say that the SDE (4) has a strong solution if there exists a pair $(\bar{X}^n, \bar{K}^n)$ of $(\mathscr{F}_t^{\rho^n})$ adapted processes such that $(\bar{X}^n, \bar{K}^n)$ is the solution of the Skorokhod problem associated with

$$X_0 + \int_0^{\cdot} f(\bar{X}_{s-}^n)\, \mathrm{d}W_s^{\rho^n} + \int_0^{\cdot} g(\bar{X}_{s-}^n)\, \mathrm{d}\rho_s^n .$$

It is easy to prove that the pair $(\bar{X}^n, \bar{K}^n)$ defined by the recurrent formula

$$\bar{X}_t^n = \begin{cases} X_0 & \text{if } t \in [0, t_{n1}[, \\ [\bar{X}_{t_{n,k-1}}^n + f(\bar{X}_{t_{n,k-1}}^n)(W_{t_{nk}} - W_{t_{n,k-1}}) & \\ \quad + g(\bar{X}_{t_{n,k-1}}^n)(t_{nk} - t_{n,k-1})]_{\partial} & \text{if } t \in [t_{nk}, t_{n,k+1}[, \end{cases}$$

and

$$\bar{K}_t^n = \begin{cases} 0 & \text{if } t \in [0, t_{n1}[, \\ \bar{K}_{t_{n,k-1}}^n + \bar{X}_t^n - \bar{X}_{t_{n,k-1}}^n & \\ \quad - \{f(\bar{X}_{t_{n,k-1}}^n)(W_{t_{nk}} - W_{t_{n,k-1}}) & \\ \qquad + g(\bar{X}_{t_{n,k-1}}^n)(t_{nk} - t_{n,k-1})\} & \text{if } t \in [t_{nk}, t_{n,k+1}[\end{cases}$$

is a strong solution to (4) on an interval $[0, \gamma_n[$, where $\gamma_n = \inf\{t;\ |\Delta W_t^{\rho^n}| + 1/n \geqslant r_0/L\}$. If D is a convex domain then due to [14] strong uniqueness for the SDE (4) holds, too.

The solution $(\hat{X}^n, \hat{K}^n)$ to the SDE (5) we construct also recurrently. We put $\hat{X}_0 = X_0$ and then for $t \in [0, t_{n1}]$, $(\hat{X}_t^n, \hat{K}_t^n)$ is a solution to the Skorokhod problem associated with

$$X_0 + f(X_0) W_t + g(X_0) t$$

(we know that under our assumptions on a domain D the solution really exists and is unique). If we have defined $(\hat{X}_t^n, \hat{K}_t^n)$ for $t \in [0, t_{nk}]$ then for $t \in [t_{nk}, t_{n,k+1}]$, $(\hat{X}_t^n, \hat{K}_t^n)$ is a solution to the Skorokhod problem associated with

$$\hat{X}_{t_{nk}}^n + f(\hat{X}_{t_{nk}}^n)(W_t - W_{t_{nk}}) + g(\hat{X}_{t_{nk}}^n)(t - t_{nk}) .$$

By construction, strong uniqueness for the SDE (5) holds for every D, which is either convex or verifies both (A) and (B).

3. Convex domains

In this section we assume that D is a convex domain in $\mathbb{R}^d$. We would like to stress that in this case, due to Corollary 1,

$$E \sup_{t \leqslant q} |X_t - X_0|^{2p} \leqslant C_p d^p L^{2p} (q^p + q^{2p}) .$$

Let (X^n, K^n) be a solution of the Skorokhod problem associated with

$$X_0 + \int_0^{\rho_\cdot^n} f(X_s)\, \mathrm{d}W_s + \int_0^{\rho_\cdot^n} g(X_s)\, \mathrm{d}s .$$

Using once more Corollary 1 and the Burkholder–Davis–Gundy inequality

$$E\sup_{t\leqslant q}|X_t^n-X_0|^{2p}\leqslant C_pE\left\{\left[\int_0^{\rho^n_\cdot} f(X_s)\,\mathrm{d}W_s\right]_q^p+\left|\int_0^{\rho^n_\cdot} g(X_s)\,\mathrm{d}s\right|_q^p\right\}$$

$$\leqslant C(p,d)E\left\{\sup_{t\leqslant\rho^n_q}\left|\int_0^t f(X_s)\,\mathrm{d}W_s\right|^{2p}+\left|\int_0^{\cdot} g(X_s)\,\mathrm{d}s\right|_{\rho^n_q}^p\right\}$$

$$=C(p,d,L,q)<+\infty\,.$$

Notice that Corollary 1 implies also integrability of $\sup_{t\leqslant q}|\bar{X}_t^n-X_0|^{2p}$, and $\sup_{t\leqslant q}|\hat{X}_t^n-X_0|^{2p}$.

Proposition 1. *Assume D is a convex domain in $\mathbb{R}^d$. Then for every sequence $\{\tau_n\}$ of stopping times*

$$E\sup_{t<q}|X_t^{n,\tau_n-}-\bar{X}_t^{n,\tau_n-}|^{2p}=O\left(E\sup_{t<q}|X_t^{\tau_n}-X_t^{n,\tau_n-}|^{2p}\right),\quad q\in\mathbb{R}^+,\ p\in\mathbb{N}\,.$$

Proof. For every $q\in\mathbb{R}^d$,

$$E\sup_{t<q}|X_t^{n,\tau_n-}-\bar{X}_t^{n,\tau_n-}|^{2p}$$

$$\leqslant 3^{p-1}\left\{E\sup_{t<q\wedge\tau_n}\left|\int_0^t f(X_{s-}^n)-f(\bar{X}_{s-}^n)\,\mathrm{d}W_s^{\rho^n}+\int_0^t g(X_{s-}^n)-g(\bar{X}_{s-}^n)\,\mathrm{d}\rho_s^n\right|^{2p}\right.$$

$$\left.+E\sup_{t<q\wedge\tau_n}|K_t^n-\bar{K}_t^n|^{2p}+\varepsilon_1^n\right\}$$

$$=3^{p-1}\{I_1^n+I_2^n+\varepsilon_1^n\}\,,$$

where

$$\varepsilon_1^n=E\sup_{t<q\wedge\tau_n}\left|\int_0^{\rho_t^n} f(X_s)-f(X_{s-}^n)\,\mathrm{d}W_s+\int_0^{\rho_t^n} g(X_s)-g(X_{s-}^n)\,\mathrm{d}s\right|^{2p}.$$

By the Burkholder–Davis–Gundy inequality

$$\varepsilon_1^n\leqslant E\sup_{t<q\wedge\tau_n}\left|\int_0^t f(X_s)-f(X_{s-}^n)\,\mathrm{d}W_s+\int_0^t g(X_s)-g(X_{s-}^n)\,\mathrm{d}s\right|^{2p}$$

$$\leqslant E\sup_{t\leqslant q}\left|\int_0^t f(X_s^{\tau_n})-f(X_{s-}^{n,\tau_n-})\,\mathrm{d}W_s+\int_0^t g(X_s^{\tau_n})-g(X_{s-}^{n,\tau_n-})\,\mathrm{d}s\right|^{2p}$$

$$\leqslant C(p,d,L,q)E\sup_{t\leqslant q}|X_t^{\tau_n}-X_{t-}^{n,\tau_n-}|^{2p}$$

$$=C(p,d,L,q)E\sup_{t<q}|X_t^{\tau_n}-X_t^{n,\tau_n-}|^{2p}$$

and

$$I_1^n \leqslant E \sup_{t<q\wedge\tau_n} \left| \int_0^t f(X_{s-}^n) - f(\bar{X}_{s-}^n)\, \mathrm{d}W_s + \int_0^t g(X_{s-}^n) - g(\bar{X}_{s-}^n)\, \mathrm{d}s \right|^{2p}$$

$$\leqslant C(p, d, L, q) E\left\{ \left(\int_0^q \sup_{u\leqslant s} |X_{u-}^{n,\tau_n-} - \bar{X}_{u-}^{n,\tau_n-}|^2\, \mathrm{d}s \right)^p + \left(\int_0^q \sup_{u\leqslant s} |X_{u-}^{n,\tau_n-} - \bar{X}_{u-}^{n,\tau_n-}|\, \mathrm{d}s \right)^{2p} \right\}$$

$$\leqslant C(p, d, L, q) \int_0^q E \sup_{u<s} |X_u^{n,\tau_n-} - \bar{X}_u^{n,\tau_n-}|^{2p}\, \mathrm{d}s\,,$$

where the constant $C(p, d, L, q)$ changes from place to place in the preceding. For every τ_n define

$$\sigma_n = \min\{t_{nk};\ t_{nk} \geqslant \tau_n\}\,.$$

Then by simple calculations σ_n is an $\mathscr{F}^{\rho^n}$ stopping time such that for every $\mathscr{F}^{\rho^n}$ adapted, step process Y (i.e. of the form $Y_t = Y_{t_{nk}}$, $t \in [t_{nk}, t_{n,k+1}[$) we have $Y_t^{\tau_n-} = Y_t^{\sigma_n-}$ and $\sup_{t<q\wedge\tau_n} |Y_t| = \sup_{t<q\wedge\sigma_n} |Y_t|$. Therefore by using Corollary 1(iii) and the Burkholder–Davis–Gundy inequality

$$I_2^n = \sup_{t<q\wedge\sigma_n} |K_t^n - \bar{K}_t^n|^{2p}$$

$$\leqslant C(p) E\left\{ \left[\int_0^\cdot f(X_{s-}^n) - f(\bar{X}_{s-}^n)\, \mathrm{d}W_s^{\rho^n} \right]_{(q\wedge\sigma_n)-}^p + \left\langle \int_0^\cdot f(X_{s-}^n) - f(\bar{X}_{s-}^n)\, \mathrm{d}W_s^{\rho^n} \right\rangle_{(q\wedge\sigma_n)-}^p + \left| \int_0^\cdot g(X_{s-}^n) - g(\bar{X}_{s-}^n)\, \mathrm{d}\rho_s^n \right|_{(q\wedge\sigma_n)-}^{2p} + \varepsilon_2^n \right\}$$

$$\leqslant C(p, d, L, q) E\left\{ \left[\int_0^\cdot f(X_{s-}^{n,\tau_n-}) - f(\bar{X}_{s-}^{n,\tau_n-})\, \mathrm{d}W_s^{\rho^n} \right]_q^p + \int_0^q E \sup_{u\leqslant s} |X_{u-}^{n,\tau_n-} - \bar{X}_{u-}^{n,\tau_n-}|^{2p}\, \mathrm{d}\rho_s^n + \varepsilon_2^n \right\}$$

$$\leqslant C(p, d, L, q) E\left\{ \sup_{t\leqslant q} \left| \int_0^t f(X_{s-}^{n,\tau_n-}) - f(\bar{X}_{s-}^{n,\tau_n-})\, \mathrm{d}W_s \right|_q^{2p} + \int_0^q E \sup_{u\leqslant s} |X_{u-}^{n,\tau_n-} - \bar{X}_{u-}^{n,\tau_n-}|^{2p}\, \mathrm{d}s + \varepsilon_2^n \right\}$$

$$\leqslant C(p, d, L, q) \left\{ \int_0^q E \sup_{u<s} |X_u^{n,\tau_n-} - \bar{X}_u^{n,\tau_n-}|^{2p}\, \mathrm{d}s + \varepsilon_2^n \right\},$$

where

$$\varepsilon_2^n = E\left\{\left[\int_0^{\rho^n} f(X_s)\, \mathrm{d}W_s - \int_0^{\cdot} f(X_{s-}^n)\, \mathrm{d}W_s^{\rho^n}\right]^p_{(q\wedge\sigma_n)-} + \left\langle \int_0^{\rho^n} f(X_s)\, \mathrm{d}W_s - \int_0^{\cdot} f(X_{s-}^n)\, \mathrm{d}W_s^{\rho^n}\right\rangle^p_{(q\wedge\sigma_n)-} + E\left|\int_0^{\rho^n} g(X_s)\, \mathrm{d}s - \int_0^{\cdot} q(X_{s-}^n)\, \mathrm{d}\rho_s^n\right|^{2p}_{(q\wedge\sigma_n)-}\right\}$$

$$\leqslant C(p, d, L, q) E \sup_{t<q} |X_t^{\tau_n} - X_t^{n,\tau_n -}|^{2p}.$$

From the above estimations it is clear that there exist two constant C_1, C_2 such that for every $q \in \mathbb{R}^d$,

$$E \sup_{t<q} |X_t^{n,\tau_n -} - \bar{X}_t^{n,\tau_n -}|^{2p} \leqslant C_1 E \sup_{t<q} |X_t^{\tau_n} - X_t^{n,\tau_n -}|^{2p} + C_2 \int_0^q E \sup_{u<s} |X_u^{n,\tau_n -} - \bar{X}_u^{n,\tau_n -}|^{2p}\, \mathrm{d}s.$$

To finish the proof we use Lemma 2 of the Appendix. If we set in Lemma 2 $h_q = E \sup_{t<q} |X_t^{n,\tau_n -} - \bar{X}_t^{n,\tau_n -}|^{2p}$, $k_q = q$ we obtain

$$E \sup_{t<q} |X_t^{n,\tau_n -} - \bar{X}_t^{n,\tau_n -}|^{2p} \leqslant 2C_1 E \sup_{t<q} |X_t^{\tau_n} - X_t^{n,\tau_n -}|^{2p} \exp\{2C_2 q\}$$

and the proof is complete. □

Theorem 2. *Assume f, g satisfy* (2).

(i) *If D is a convex polyhedron then*

$$E \sup_{t\leqslant q} |\bar{X}_t^n - X_t|^{2p} = \mathrm{O}(1/n^{p-\varepsilon}), \quad \varepsilon > 0,\ q \in \mathbb{R}^+,\ p \in \mathbb{N}.$$

(ii) *If D is a convex domain satisfying the condition* (β) *then*

$$E \sup_{t\leqslant q} |\bar{X}_t^n - X_t|^{2p} = \mathrm{O}(1/n^{p/2-\varepsilon}), \quad \varepsilon > 0,\ q \in \mathbb{R}^+,\ p \in \mathbb{N}.$$

(iii) *If D is any convex domain then there exists a sequence of stopping times* $\{\tau_n\}$, $\tau_n \to_{\mathcal{P}} +\infty$ such that

$$E \sup_{t\leqslant q} |\bar{X}_t^{n,\tau_n -} - X_t^{\tau_n}|^{2p} = \mathrm{O}(1/n^{p/2-\varepsilon}), \quad \varepsilon > 0,\ q \in \mathbb{R}^+,\ p \in \mathbb{N}.$$

Proof. (i) Due to Proposition 1 (with $\tau_n = +\infty$, $n \in \mathbb{N}$) it is sufficient to prove that

$$E \sup_{t\leqslant q} |X_t^n - X_t|^{2p} = \mathrm{O}(1/n^{p-\varepsilon}), \quad \varepsilon > 0,\ q \in \mathbb{R}^+.$$

Next, by [4, Theorem 2.2] there exists a constant $c(p)>0$ such that

$$\sup_{t\leqslant q} |X_t^n - X_t|^{2p} \leqslant c(p) \sup_{t\leqslant q} \left| \int_{\rho_t^n}^t f(X_s)\, \mathrm{d}W_s + \int_{\rho_t^n}^t g(X_s)\, \mathrm{d}s \right|^{2p}, \tag{15}$$

and therefore in order to prove (i) it is enough to use Lemma 3 of the Appendix.

(ii) For a general convex domain we have the following inequality [14, Lemma 2.2]:

$$\begin{aligned}
&\sup_{t\leqslant q} |X_t^n - X_t|^{2p} \\
&\leqslant C(p)\left\{ \sup_{t\leqslant q} \left| \int_{\rho_t^n}^t f(X_s)\, \mathrm{d}W_s + \int_{\rho_t^n}^t g(X_s)\, \mathrm{d}s \right|^{2p} \right. \\
&\qquad \left. + \sup_{t\leqslant q} \left| \int_{\rho_t^n}^t f(X_s)\, \mathrm{d}W_s + \int_{\rho_t^n}^t g(X_s)\, \mathrm{d}s \right|^{p} (|K^n|_p^p + |K|_p^p) \right\}
\end{aligned}$$

instead of (15). Therefore by Schwarz's inequality we have

$$\begin{aligned}
&E \sup_{t\leqslant q} |X_t^n - X_t|^{2p} \\
&\leqslant C(p)\left\{ E \sup_{t\leqslant q} \left| \int_{\rho_t^n}^t f(X_s)\, \mathrm{d}W_s + \int_{\rho_t^n}^t g(X_s)\, \mathrm{d}s \right|^{2p} \right. \\
&\qquad + \left(E \sup_{t\leqslant q} \left| \int_{\rho_t^n}^t f(X_s)\, \mathrm{d}W_s + \int_{\rho_t^n}^t g(X_s)\, \mathrm{d}s \right|^{2p} \right)^{1/2} \\
&\qquad \left. \times (E|K^n|_q^{2p} + E|K|_q^{2p})^{1/2} \right\}.
\end{aligned}$$

To finish the proof in view of Proposition 1 (as before with $\tau_n = +\infty$, $n\in\mathbb{N}$) and Lemma 3 it is enough to show

$$E|K|_q^{2p} < +\infty, \tag{16}$$

and

$$\sup_n E|K^n|_q^{2p} < +\infty. \tag{17}$$

Now, let Y be a process with continuous trajectories and let (X, K) be a solution of the Skorokhod problem associated with Y. Due to [14, 2.11(b)] there exist constants $c, h>0$ such that if n is so large that $\omega_{Y(\omega)}(1/n, q) < h$ then

$$|K(\omega)|_q \leqslant (nq+1) c \sup_{t\leqslant q} |Y_t(\omega)|, \quad \omega\in\Omega. \tag{18}$$

Define

$$N = \inf\{n\in\mathbb{N};\ \omega_Y(1/n, q) < \tfrac{1}{2}h\} - 1 .$$

Then if $N\geqslant 1$ we have

$$\tfrac{1}{2}h \leqslant \omega_Y(1/N, q) < h \tag{19}$$

and

$$|K|_q^{2p} \leqslant c(p)(Nq+1)^{2p} \sup_{t \leqslant q} |Y_t|^{2p} . \tag{20}$$

Set $Y_t = \int_0^t f(X_s)\, dW_s + \int_0^t g(X_s)\, ds$, $t \in \mathbb{R}^+$. By (19) and Lemma 3,

$$\begin{aligned} E\, N^{2p} &\leqslant (\tfrac{1}{2}h)^{-8p} E\, N^{2p} \omega_Y(1/N, q)^{8p} \\ &= (\tfrac{1}{2}h)^{-8p} \sum_{n=1}^{\infty} E\, n^{2p} \omega_Y(1/n, q)^{8p} 1_{\{N=n\}} \\ &\leqslant (\tfrac{1}{2}h)^{-8p} \sum_{n=1}^{\infty} n^{-2p+\varepsilon} n^{4p-\varepsilon} E \omega_Y(1/n, q)^{8p} \\ &< +\infty \end{aligned}$$

for every $p \in \mathbb{N}$. In view of Schwarz's inequality (16) is immediate from (20). On the other hand it is known that (18) is true also if $\omega'_{Y(\omega)}(1/n, q) < h$ and Y is any process with trajectories in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ (see e.g. [1]). Since by simple calculations

$$\omega'_{Y^{\rho^n}}(h, q) \leqslant \omega_Y(h, q) ,$$

it is clear that

$$|K^n|_q^{2p} \leqslant c(p)(Nq+1)^{2p} \sup_{t \leqslant q} |Y_t^{\rho^n}|^{2p} , \quad n \in \mathbb{N} ,$$

and by the arguments used previously the property (17) easily follows.

(iii) Denote $\tau_n^k = \inf\{t \in \mathbb{R}^+;\ |X_t| + |X_t^n| + |\bar{X}_t^n| > k\}$, $n, k \in \mathbb{N}$. Then of course

$$\lim_{k \to +\infty} \limsup_{n \to +\infty} \mathscr{P}(\tau_n^k \leqslant q) = 0 \tag{21}$$

and for fixed $k \in \mathbb{N}$, $X_t^{\tau_n^k}$, $X_t^{n, \tau_n^k -}$, $\bar{X}_t^{n, \tau_n^k -} \in \bar{D}_k$. Since D_k satisfies the condition (β) (see Remark 1(ii)), from the proof of (ii) we deduce

$$E \sup_{t \leqslant q} |X_t^{n, \tau_n^k -} - X_t^{\tau_n^k}|^{2p} = O(1/n^{p/2-\varepsilon}) , \quad \varepsilon > 0,\ q \in \mathbb{R}^+,\ k \in \mathbb{N},\ p \in \mathbb{N} .$$

Hence, due to Proposition 1 also

$$E \sup_{t \leqslant q} |\bar{X}_t^{n, \tau_n^k} - X_t^{\tau_n^k}|^{2p} = O(1/n^{p/2-\varepsilon}) , \quad \varepsilon > 0,\ q \in \mathbb{R}^+,\ k \in \mathbb{N},\ p \in \mathbb{N} .$$

Since $\varepsilon > 0$ can be chosen as small as desired, we have in fact the convergence

$$\sup_{p' \leqslant p} n^{p'/2-\varepsilon} E \sup_{t \leqslant q} |\bar{X}_t^{n, \tau_n^k -} - X_t^{\tau_n^k}|^{2p'} \to 0 \tag{22}$$

for every $\varepsilon > 0$, $k \in \mathbb{N}$, $q \in \mathbb{R}^+$, $p \in \mathbb{N}$. Set

$$a_{nk}(\varepsilon^{-1}, q, p) = \sup_{p' \leqslant p} n^{p'/2-\varepsilon} E \sup_{t \leqslant q} |\bar{X}_t^{n, \tau_n^k -} - X_t^{\tau_n^k}|^{2p'} + \mathscr{P}(\tau_n^k \leqslant q)$$

for every $\varepsilon > 0, k \in \mathbb{N}, q \in \mathbb{R}^+, p \in \mathbb{N}$. By (21), (22) and Lemma 4 we can choose sequence $\{k_n\}$, $k_n \uparrow +\infty$ sufficiently slowly such that

$$a_{nk_n}(\varepsilon^{-1}, q, p) \to 0$$

and

$$\tau_n^{k_n} \xrightarrow[\mathscr{P}]{} +\infty .$$

Thus, setting $\tau_n = \tau_n^{k_n}$, $n \in \mathbb{N}$ we complete the proof. □

Remark 3. As observed in [4, Proposition 4.1], the estimation (15) is true only for a convex polyhedron with nonempty interior. Therefore it seems not to be possible to obtain the rate of convergence $O(1/n^{p-\varepsilon})$ for other convex domains in $\mathbb{R}^d$.

Corollary 2. *Assume f, g satisfy* (2).

(i) *If D is a convex polyhedron then*

$$n^{1/2-\varepsilon} \sup_{t \leqslant q} |\bar{X}_t^n - X_t| \to 0 , \quad \mathscr{P}\text{-a.s., } \varepsilon > 0, q \in \mathbb{R}^+ .$$

(ii) *If D is a convex domain satisfying the condition* (β) then

$$n^{1/2-\varepsilon} \sup_{t \leqslant q} |\bar{X}_t^n - X_t| \to 0 , \quad \mathscr{P}\text{-a.s., } \varepsilon > 0, q \in \mathbb{R}^+ .$$

(iii) *If D is any convex domain then there exists a sequence* $\{\tau_n\}$ *of stopping times such that* $\tau_n \to_{\mathscr{P}} +\infty$ *and*

$$n^{1/4-\varepsilon} \sup_{t \leqslant q} |\bar{X}_t^{n,\tau_n -} - X_t^{\tau_n}| \to 0 , \quad \mathscr{P}\text{-a.s., } \varepsilon > 0, q \in \mathbb{R}^+ .$$

Proof. (i) Due to the Borel–Cantelli lemma it is sufficient to find a sequence $\{\alpha_n\}$, $\alpha_n \downarrow 0$ such that

$$\sum_{n=1}^{\infty} \mathscr{P}\left(n^{1/2-\varepsilon} \sup_{t \leqslant q} |\bar{X}_t^n - X_t| > \alpha_n\right) < +\infty .$$

Let p be such large that $2p\varepsilon > 1$. Set $\varepsilon' = \frac{1}{4}(2p\varepsilon - 1)$, $\alpha_n = n^{-\varepsilon'/(2p)}$. Then by Chebyshev's inequality and Theorem 2(i) used for ε' instead of ε,

$$\sum_{n=1}^{\infty} \mathscr{P}\left(n^{1/2-\varepsilon} \sup_{t \leqslant q} |\bar{X}_t^n - X_t| > \alpha_n\right)$$

$$\leqslant \sum_{n=1}^{\infty} \frac{1}{n^{-\varepsilon'}} E n^{p-2p\varepsilon} \sup_{t \leqslant q} |\bar{X}_t^n - X_t|^{2p}$$

$$\leqslant C(p, d, L, q) \sum_{n=1}^{\infty} \frac{1}{n^{-\varepsilon'}} \frac{n^{p-2p\varepsilon}}{n^{p-\varepsilon'}}$$

$$= C(p, d, L, q) \sum_{n=1}^{\infty} n^{-2p\varepsilon + 2\varepsilon'} .$$

Since $-2p\varepsilon + 2\varepsilon' < -1$, the proof is finished.

(ii) and (iii) It is enough to use Theorem 2(ii) and (iii), respectively and to follow proof of part (i). □

Corollary 3. *Assume f·g satisfy* (2). *If D is a convex domain then*

(i) $E \sup_{t \leqslant q} |\bar{X}_t^n - X_t|^{2p} \to 0, \quad q \in \mathbb{R}^+, p \in \mathbb{N},$

(ii) $n^{1/4-\varepsilon} \sup_{t \leqslant q} |\bar{X}_t^n - X_t| \xrightarrow[\mathscr{P}]{} 0, \quad q \in \mathbb{R}^+, \varepsilon > 0.$

Proof. In view of Corollary 2(iii),

$$n^{p/2-\varepsilon} \sup_{t \leqslant q} |\bar{X}_t^n - X_t|^{2p} \xrightarrow[\mathscr{P}]{} 0, \quad q \in \mathbb{R}^+, p \in \mathbb{N}, \varepsilon > 0,$$

and we get (ii). Then to prove (i) it is enough to observe that by Theorem 1(i), $\sup_n E \sup_{t \leqslant q} |\bar{X}_t^n - X_t|^{2p} < +\infty$ for every $p \in \mathbb{N}$. □

Theorem 3. *Assume f, g satisfy* (2). *If D is a convex domain then*

$$E \sup_{t \leqslant q} |\hat{X}_t^n - X_t|^{2p} = \mathrm{O}(1/n^p), \quad q \in \mathbb{R}^+ .$$

Proof. For every $t \in \mathbb{R}^+$,

$$\begin{aligned} &E|\hat{X}_t^n - X_t|^{2p} \\ &\leqslant 3^{p-1}\left\{ E \left| \int_0^t f(\hat{X}^n)_s - f(X_s)\, \mathrm{d}W_s + \int_0^t g(\hat{X}_s^n) - g(X_s)\, \mathrm{d}s \right|^{2p} \right. \\ &\qquad + E|\hat{K}_t^n - K_t|^{2p} \\ &\qquad \left. + E \left| \int_0^t f(\hat{X}_{s-}^{n,\rho^n}) - f(\hat{X}_s^n)\, \mathrm{d}W_s + \int_0^t g(\hat{X}_{s-}^{n,\rho^n}) - g(\hat{X}_s^n)\, \mathrm{d}s \right|^{2p} \right\} \\ &= 3^{p-1}\{I_1^n + I_2^n + \varepsilon_n\} . \end{aligned}$$

It is clear that

$$I_1^n \leqslant C(p, d, L, q) \int_0^t E|\hat{X}_s^n - X_s|^{2p}\, \mathrm{d}s$$

and

$$\varepsilon_n \leqslant C(p, d, L, q) \int_0^t E|\hat{X}_{s-}^{n,\rho^n} - \hat{X}_s^n|^{2p}\, \mathrm{d}s .$$

Let $s \in]t_{nk}, t_{n,k+1}]$, $s \leqslant q$. Then of course $\hat{X}^{n,\rho^n}_{s-} = \hat{X}^n_{t_{nk}}$. By using Theorem 1(i) for $\mathscr{F}_{t_{nk}+}$-adapted processes we have

$$E|\hat{X}^n_{t_{nk}} - \hat{X}^n_s|^{2p} \leqslant C_p E\left\{\left[\int_{t_{nk}}^{\cdot} f(\hat{X}^{n,\rho^n}_{u-})\, \mathrm{d}W_u\right]_s^p + \left|\int_{t_{nk}}^{\cdot} g(\hat{X}^{n,\rho^n}_{u-})\, \mathrm{d}s\right|_s^{2p}\right\}$$

$$\leqslant C(p, d, L, q)/n^p .$$

Hence for every $s \leqslant q$,

$$E|\hat{X}^{n,\rho^n}_{s-} - \hat{X}^n_s|^{2p} \leqslant C(p, d, L, q)/n^p . \tag{23}$$

On the other hand, due to Theorem 1(ii),

$$I^n_2 \leqslant E \sup_{t \leqslant q} |\hat{K}^n_t - K_t|^{2p}$$

$$\leqslant C_p\left\{E\left[\int_0^{\cdot} f(\hat{X}^{n,\rho^n}_{s-}) - f(X_s)\, \mathrm{d}W_s\right]_q^p + E\left|\int_0^{\cdot} g(\hat{X}^{n,\rho^n}_{s-}) - g(X_s)\, \mathrm{d}s\right|_q^{2p}\right\}$$

$$\leqslant C(p, d, L, q)\left\{\int_0^q E|\hat{X}^n_s - X_s|^{2p}\, \mathrm{d}s + \int_0^q E|\hat{X}^{n,\rho^n}_{s-} - \hat{X}^n_s|^{2p}\, \mathrm{d}s\right\}.$$

Putting the above estimates together we see that for every $t \leqslant q$ there exist constants C_1, $C_2 > 0$ such that

$$E|\hat{X}^n_t - X_t|^{2p} \leqslant \frac{C_1}{n^p} + C_2 \int_0^t E|\hat{X}^n_s - X_s|^{2p}\, \mathrm{d}s ,$$

hence by Lemma 2 of the Appendix,

$$\sup_{t \leqslant q} E|\hat{X}^n_t - X_t|^{2p} \leqslant \frac{C_1}{n^p} \exp\{C_2 q\} , \tag{24}$$

which, when combined with (23), yields

$$\sup_{t \leqslant q} E|\hat{X}^{n,\rho^n}_{t-} - X_t|^{2p} = \mathrm{O}(1/n^p) , \quad q \in \mathbb{R}^+ . \tag{25}$$

Finally, by the Burkholder–Davis–Gundy inequality and (24),

$$E \sup_{t \leqslant q} |\hat{X}^n_t - X_t|^{2p}$$

$$\leqslant 3^{p-1}\left\{E \sup_{t \leqslant q}\left|\int_0^t f(\hat{X}^{n,\rho^n}_{s-}) - f(X_s)\, \mathrm{d}W_s\right|^{2p}\right.$$

$$\left. + E \sup_{t \leqslant q}\left|\int_0^t g(\hat{X}^{n,\rho^n}_{s-}) - g(X_s)\, \mathrm{d}s\right|^{2p} + E \sup_{t \leqslant q} |\hat{K}^n_t - K_t|^{2p}\right\}$$

$$\leqslant C(p, d, L, q)\left\{\int_0^q E|\hat{X}^{n,\rho^n}_{s-} - X_s|^{2p}\, \mathrm{d}s + E \sup_{t \leqslant q} |\hat{K}^n_t - K_t|^{2p}\right\}$$

$$\leqslant C(p,d,L,q)\left\{\int_0^q E|\hat{X}_{s-}^{n,\rho^n}-X_s|^{2p}\,\mathrm{d}s + \int_0^q E|\hat{X}_s^n-X_s|^{2p}\,\mathrm{d}s\right\}+\mathrm{O}(1/n^p)\,,$$

and thus, in view of (24) and (25) the result follows. □

By using the arguments from the proof of Corollary 2 we can easily deduce from Theorem 3 the following rate of almost sure convergence.

Corollary 4. *Assume f, g satisfy* (2). *If D is a convex domain in* $\mathbb{R}^d$ *then*

$$n^{1/2-\varepsilon}\sup_{t\leqslant q}|\hat{X}_t^n-X_t|\to 0\,,\quad \mathscr{P}\text{-a.s.},\ \varepsilon>0,\ q\in\mathbb{R}^+\,.\qquad \square$$

4. General domains

In this section we assume that D satisfies the conditions (A) and (B). In this case, if $r_0<+\infty$, the processes $|X_t-X_0|^{2p}$, $|X_t^n-X_0|^{2p}$, $|\bar{X}_t^n-X_0|^{2p}$, and $|\hat{X}_t^n-X_0|^{2p}$ are not necessarily integrable. Therefore the results are weaker than their analogues in Section 3.

Theorem 4. *Assume f, g satisfy* (2). *If the conditions* (A) *and* (B) *are satisfied then there exists a sequence of stopping times* $\{\tau_n\}$, $\tau_n\to_{\mathscr{P}}+\infty$ such that

$$E\sup_{t\leqslant q}|\bar{X}_t^{n,\tau_n-}-X_t^{\tau_n}|^{2p}=\mathrm{O}(1/n^{p/2-\varepsilon})\,,\quad \varepsilon>0,\ q\in\mathbb{R}^+,\ p\in\mathbb{N}\,.$$

Proof. Denote

$$\tau_n^k=\inf\left\{t;\ |K|_t+|K^n|_t+|\bar{K}^n|_t>k \text{ or } |\Delta W_t^{\rho^n}|+\frac{1}{n}\geqslant\frac{r_0}{4L}\right\},\quad k,n\in\mathbb{N}\,.$$

Due to [11, Proposition 3.1],

$$\lim_{k\to\infty}\limsup_{n\to\infty}\mathscr{P}(\tau_n^k\leqslant q)=0\,,\quad q\in\mathbb{R}^+\,. \tag{26}$$

By using the arguments from the proof of Proposition 1 we show that

$$E\sup_{t<q}|X_t^{n,\tau_n^k-}-\bar{X}_t^{n,\tau_n^k-}|^{2p}=\mathrm{O}\left(E\sup_{t<q}|X_t^{\tau_n^k}-X_t^{n,\tau_n^k}|^{2p}\right). \tag{27}$$

On the other hand since $|\Delta K^{n,\tau_n^k-}|\leqslant\frac{1}{4}r_0$ it follows by [11, Lemma 2.3(i)] that

$$\sup_{t\leqslant q}|X_t^{n,\tau_n^k-}-X_t^{\tau_n^k}|^{2p}$$

$$\leqslant C(p)\Big\{\sup_{t\leqslant q}\Big|\int_{\rho_t^n}^t f(X_s)\,\mathrm{d}W_s+\int_{\rho_t^n}^t g(X_s)\,\mathrm{d}s\Big|^{2p}$$

$$+\sup_{t\leqslant q}\Big|\int_{\rho_t^n}^t f(X_s)\,\mathrm{d}W_s+\int_{\rho_t^n}^t g(X_s)\,\mathrm{d}s\Big|^p(|K^{n,\tau_n^k-}|_q^p+|K^{\tau_n^k}|_q^p)$$

$$+\Big(\frac{1}{r_0}\int_0^{q-}|X_s^{n,\tau_n^k-}-X_s^{\tau_n^k}|^2\,(\mathrm{d}|K^{n,\tau_n^k-}|_s+|K^{\tau_n^k}|_s)\Big)^p\Big\}.$$

By the definition of τ_n^k, $|K^{n,\tau_n^k-}|\leqslant k$ and $|K^{\tau_n^k}\leqslant k$. Therefore by using Lemma 2 we have

$$\sup_{t\leqslant q}|X_t^{n,\tau_n^k-}-X_t^{\tau_n^k}|^{2p}$$

$$\leqslant C(p,k)\Big\{\sup_{t\leqslant q}\Big|\int_{\rho_t^n}^t f(X_s)\,\mathrm{d}W_s+\int_{\rho_t^n}^t g(X_s)\,\mathrm{d}s\Big|^{2p}$$

$$+\sup_{t\leqslant q}\Big|\int_{\rho_t^n}^t f(X_s)\,\mathrm{d}W_s+\int_{\rho_t^n}^t g(X_s)\,\mathrm{d}s\Big|^p\Big\}. \tag{28}$$

Because of Lemma 3 and (28),

$$E\sup_{t\leqslant q}|X_t^{n,\tau_n^k-}-X_t^{\tau_n^k-}|^{2p}=\mathrm{O}(1/n^{p/2-\varepsilon}),\quad \varepsilon>0,\ q\in\mathbb{R}^+,\ k\in\mathbb{N},\ p\in\mathbb{N}.$$

Next, in view of (27),

$$E\sup_{t\leqslant q}|\bar{X}_t^{n,\tau_n^k-}-X_t^{\tau_n^k}|^{2p}=\mathrm{O}(1/n^{p/2-\varepsilon}),\quad \varepsilon>0,\ q\in\mathbb{R}^+,\ k\in\mathbb{N},\ p\in\mathbb{N}.$$

Hence

$$\sup_{p'\leqslant p} n^{p'/2-\varepsilon}E\sup_{t\leqslant q}|\bar{X}_t^{n,\tau_n^k-}-X_t^{\tau_n^k}|^{2p'}\to 0 \tag{29}$$

for every $\varepsilon>0$, $k\in\mathbb{N}$, $q\in\mathbb{R}^+$, $p\in\mathbb{N}$. Finally, the result follows from (26), (29) and Lemma 4. □

From the above theorem we can easily deduce:

Corollary 5. *Assume f, g satisfy* (2). *If the conditions* (A) *and* (B) *are satisfied then*:

(i) *there exists a sequence of stopping times* $\{\tau_n\}$, $\tau_n\to_{\mathscr{P}}+\infty$ such that

$$n^{1/4-\varepsilon}\sup_{t\leqslant q}|\bar{X}_t^{n,\tau_n-}-X_t^{\tau_n}|\to 0,\quad \mathscr{P}\text{-a.s.},\ k\in\mathbb{N},\ \varepsilon>0,\ q\in\mathbb{R}^+;$$

(ii) $$n^{1/4-\varepsilon}\sup_{t\leqslant q}|\bar{X}_t^n-X_t|\xrightarrow[\mathscr{P}]{}0,\quad k\in\mathbb{N},\ \varepsilon>0,\ q\in\mathbb{R}^+.\qquad\square$$

Theorem 5. *Assume f, g satisfy* (2). *If the conditions* (A) *and* (B) *are satisfied then there exists an array of stopping times* $\{\{\tau_n^k\}\}$ *such that* $\lim_{k\to\infty} \limsup_{n\to\infty} \mathscr{P}(\tau_n^k \leqslant q) = 0$, $q \in \mathbb{R}^+$ *and*

$$E \sup_{t \leqslant q} |\hat{X}_t^{n,\tau_n^k -} - X_t^{\tau_n^k}|^{2p} = O(1/n^p), \quad q \in \mathbb{R}^+, k \in \mathbb{N}.$$

Proof. It is sufficient to put $\tau_n^k = \inf\{t;\ |\hat{K}^n|_t + |K|_t > k\}$ and use the arguments from the proof of Theorem 4. □

Theorem 5 and Lemma 4 immediately lead to the following:

Corollary 6. *Assume f, g satisfy* (2). *If the conditions* (A) *and* (B) *are satisfied then*:

(i) *there exists a sequence of stopping times* $\{\tau_n\}$, $\tau_n \rightarrow_{\mathscr{P}} +\infty$ *such that*

$$n^{1/2-\varepsilon} \sup_{t \leqslant q} |\hat{X}_t^{n,\tau_n -} - X_t^{\tau_n}| \rightarrow 0, \quad \mathscr{P}\text{-a.s., } k \in \mathbb{N},\ \varepsilon > 0,\ q \in \mathbb{R}^+;$$

(ii) $$n^{1/2-\varepsilon} \sup_{t \leqslant q} |\hat{X}_t^n - X_t| \xrightarrow[\mathscr{P}]{} 0, \quad k \in \mathbb{N},\ \varepsilon > 0,\ q \in \mathbb{R}^+. \qquad \square$$

5. Appendix

Lemma 1 [13, Lemma 2]. *Let* Y^1, Y^2 *be two increasing processes*, $Y_0^1 = Y_0^2 = 0$ *such that* $EY_\infty^1 < +\infty$, $Y_\infty^2 < C_2$ *for some constant* C_2. *If for every stopping time* τ,

$$EY_\tau^1 \leqslant C_1 + E \int_0^{\tau -} Y_s^1 \, \mathrm{d}Y_s^2,$$

then $EY_\infty^1 \leqslant C_1 \exp\{C_2\}$. □

Lemma 2 [11, Lemma 2.2]. *Let* $k \in \mathbb{D}(\mathbb{R}^+, \mathbb{R})$ *be a non-decreasing function with* $k_0 = 0$ *and let h be a nonnegative Borel measurable function on* $\mathbb{R}^+$. *If*

$$h_q \leqslant C_1 + C_2 \int_0^{q-} h_s \, \mathrm{d}k_s, \quad q \in \mathbb{R}^+,$$

for some $C_1, C_2 > 0$ *then* $h_q \leqslant C_1 \exp\{C_2 k_q\}$. □

Lemma 3. *Let H, G be two predictable processes with values in* $\mathbb{R}^d \otimes \mathbb{R}^d$ *and* $\mathbb{R}^d$, *respectively such that* $\sup_{t \leqslant q} \|H_t\|$, $\sup_{t \leqslant q} |G_t| \leqslant L < +\infty$ *for some constant* $L > 0$ *and let Y be a process with continuous trajectories of the form* $Y_t = \int_0^t H_s \, \mathrm{d}W_s + \int_0^t G_s \, \mathrm{d}s$, $t \in \mathbb{R}^+$. *Then*

$$E \sup_{t \leqslant q} |Y_t - Y_t^{\rho^n}|^{2p} \leqslant E\omega_Y(1/n, q)^{2p} = O(1/n^{p-\varepsilon}), \quad q \in \mathbb{R}^+, p \in \mathbb{N}, \varepsilon > 0.$$

Proof. Without loss of generality we may and will assume that $d=1$, $g=0$. Then $\int_0^t H_s \, \mathrm{d}W_s = B \circ A_t$, for some standard (possibly stopped) Wiener process B and A defined by $A_t = \int_0^t H_s^2 \, \mathrm{d}s$. Hence $A_t - A_s \leqslant L^2(t-s)$, $A_q \leqslant L^2 q$ and $\omega_{B \circ A}(\delta, q) \leqslant \omega_B(\delta L^2, L^2 q)$. Since

$$\mathscr{P}(\omega_B(\delta L^2, L^2 q)^{2p} \geqslant y) \leqslant \frac{16L\sqrt{\delta}((q+2\delta)/\delta)}{\sqrt{2\pi y^{1/p}}} \exp\left(-\frac{y^{1/p}}{8\delta L^2}\right)$$

(see e.g. [15]) we can write

$$\begin{aligned} & n^{p-\varepsilon} E\omega_B(L^2/n, L^2 q)^{2p} \\ & \quad \leqslant 1 + n^{p-\varepsilon} \int_{1/n^{p-\varepsilon}}^{\infty} \mathscr{P}(\omega_B(L^2/n, L^2 q)^{2p} > y) \, \mathrm{d}y \\ & \quad \leqslant 1 + n^{-\varepsilon} \int_{n^{\varepsilon/p}}^{\infty} \frac{16Lp(qn+2)}{\sqrt{2\pi y}} \exp\left(-\frac{y}{8L^2}\right) y^{p-1} \, \mathrm{d}y \\ & \quad \leqslant 1 + C(p, L, q, \pi) P(n) \exp(-n^{\varepsilon/p}) \\ & \quad = \mathrm{O}(1), \end{aligned}$$

where $P(n)$ is some polynomial of variable n, $n \in \mathbb{N}$. □

Lemma 4. *Let $S = \prod_{j=1}^{l} S_j$ where S_j are equal $\mathbb{N}$ or $\mathbb{R}^+$. Assume that for every $\alpha \in S \{\{a_{nk}(\alpha)\}\}$ is an array of nonnegative numbers such that*

$$\lim_{k \to \infty} \limsup_{n \to \infty} a_{nk}(\alpha) = 0, \quad \alpha \in S, \tag{30}$$

$$\alpha \leqslant \alpha' \Rightarrow a_{nk}(\alpha) \leqslant a_{nk}(\alpha'). \tag{31}$$

Then there exists a sequence $\{k_n\}$, $k_n \uparrow +\infty$ sufficiently slowly for which

$$a_{nk_n}(\alpha) \to 0, \quad \alpha \in S.$$

Proof. For every $m \in \mathbb{N}$ there exists a number $c(m) > c(m-1)$ $(c(0)=1)$ such that for all $k \geqslant c(m)$,

$$\limsup_{n \to +\infty} a_{nk}((m, m, \ldots, m)) \leqslant \frac{1}{m},$$

where $(m, m, \ldots, m) \in S$. Set $\alpha_k = (m, m, \ldots, m)$ if $k \in [c(m), c(m+1)[$, $m \in \mathbb{N}$. Then

$$\lim_{k \to +\infty} \limsup_{n \to +\infty} a_{nk}(\alpha_k) = 0$$

and

$$a_k \uparrow +\infty.$$

Similarly for every $m \in \mathbb{N}$ there exists a number $b(m) > b(m-1)$ $(b(0)=1)$ such that for $n \geqslant b(m)$,

$$a_{nm}(\alpha_m) \leqslant 2 \limsup_{n\to+\infty} a_{nm}(\alpha_m) .$$

Finally, if we set $k_n = m$ for $n \in [b(m), b(m+1)[$ then

$$a_{n,k_n}(\alpha_{k_n}) \to 0$$

and the desired result follows by (31). □

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