Necessary and Sufficient Conditions for Interval Polynomials to Have Only Real Distinct Roots

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ABSTRACT

This paper presents new necessary and sufficient conditions for a family of interval polynomials to have only real distinct roots. These new conditions require only a finite number of computations.

1. INTRODUCTION

Consider a family of real nth degree interval polynomials

$$I(s) = t_n s^n + t_{n-1} s^{n-1} + \cdots + t_0, \tag{1}$$

where $\alpha_i \leqslant t_i \leqslant \beta_i$, $(i = 0, \dots, n)$ and $0 < \alpha_n$. Recently, Soh and Berger [3] have shown that $I(s)$ has only real, negative, and distinct roots in the range $[\xi, \tau]$ if and only if two specified polynomials in $I(s)$ have only real, negative, and distinct roots in the range $[\xi, \tau]$. A similar result for $I(s)$ to have only real, positive and distinct roots in the range $[\xi, \tau]$ exists [3]. However, that the two specified polynomials in $I(s)$ given by Soh and Berger [3] have only real and distinct roots does not necessarily imply that $I(s)$ has only real and distinct roots.

For example, let

$$I(s) = s^2 + t_1 s + 2,$$

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where

$$-3 \leqslant t_1 \leqslant 3.$$

The two specified polynomials given by Soh and Berger [3] are

$$P(s) = s^2 - 3s + 2 = (s-1)(s-2)$$

and

$$P(s) = s^2 + 3s + 2 = (s+1)(s+2),$$

which have only real and distinct roots. However,

$$P(s) = s^2 + 2$$

in $I(s)$ clearly has imaginary roots.

This paper derives new necessary and sufficient conditions for the family of interval polynomials $I(s)$ to have only real, distinct roots. These new conditions also require only a finite number of computations.

2. NOTATION

Let

$$a(s) = \sum_{i=0}^{m} a_i s^i, \qquad a_m \neq 0,$$

and

$$b(s) = \sum_{i=0}^{m} b_i s^i.$$

The resultant matrix $R(a, b)$ associated with the polynomials $a(s)$ and $b(s)$

is defined as the $2m \times 2m$ matrix

$$R(a,b) = \begin{bmatrix} a_0 & a_1 & \cdot & \cdot & \cdot & a_{m-1} & a_m & & \\ & \ddots & \ddots & & & & \ddots & \ddots & \\ & & a_0 & a_1 & \cdot & \cdot & \cdot & a_{m-1} & a_m \\ b_0 & b_1 & \cdot & \cdot & \cdot & b_{m-1} & b_m & & \\ & \ddots & \ddots & & & & \ddots & \ddots & \\ & & b_0 & b_1 & \cdot & \cdot & \cdot & b_{m-1} & b_m \end{bmatrix}.$$

Let $P_1(s)$ and $P_2(s)$ be the two polynomials in $I(s)$ such that every polynomial $P(s) \in I(s)$ satisfies

$$P_1(x) \geqslant P(x) \geqslant P_2(x) \tag{2}$$

for all real nonpositive x. Similarly, let $P_3(s)$ and $P_4(s)$ be the two polynomials in $I(s)$ such that every polynomial $P(s) \in I(s)$ satisfies

$$P_3(x) \geqslant P(x) \geqslant P_4(x) \tag{3}$$

for all real nonnegative x. For the polynomials $P_i(s)$ $(i = 1,2,3,4)$ in $I(s)$, $P_i(s)$ $(i = 5,6)$, let

$$R_i^1 = R(a,b), \quad a(s) = P_i(s) \qquad b(s) = \frac{dp_i(s)}{ds};$$

$$R_i^2 = R(a,b), \quad a(s) = P_i(s) \qquad b(s) = (s-\tau)\frac{dp_i(s)}{ds};$$

$$R_i^3 = R(a,b), \quad a(s) = (s-\tau)(s-\xi)\frac{dp_i(s)}{ds}, \quad b(s) = P_i(s).$$

Similarly, let the above resultant matrices without subscript i (that is, R^1, R^2, and R^3) be associated with a polynomial $P(s)$ in $I(s)$.

Let

$$A_r = (1-r)A_0 + rA_1, \qquad 0 \leqslant r \leqslant 1,$$

represent a convex combination of two $p \times p$ real matrices A_0 and A_1.

Similarly, let

$$A_k = A_0 + kA_1$$

denote a linear combination of the two real matrices. Finally, $\lambda_{\max}^{+}(B)$ will denote the maximum positive (real) eigenvalue of a matrix B. In case B has no positive eigenvalues, we adopt the convention $\lambda_{\max}^{+}(B) = 0^{+}$. Similarly, $\lambda_{\min}^{-}(B)$ will denote the minimum negative (real) eigenvalue of matrix B, and if B has no negative eigenvalues, we take $\lambda_{\min}^{-}(B) = 0^{-}$.

3. SUPPORTING RESULTS

In this section, we develop the tools necessary for attaining the ultimate objective.

LEMMA 1 (Fu and Barmish [4]). *Suppose A_0 is nonsingular. Then*

(i) *A_r is nonsingular for all $r \in [0,1]$ if and only if $A_0^{-1} A_1$ has no eigenvalues on $(-\infty, 0]$:*

(ii) *the maximal range $(k_{\min}, k_{\max})$ for all A_k to be nonsingular is given by*

$$k_{\min} = \frac{1}{\lambda_{\min}^{-}\left(-A_0^{-1}A_1\right)}$$

and

$$k_{\max} = \frac{1}{\lambda_{\max}^{+}\left(-A_0^{-1}A_1\right)}.$$

LEMMA 2. *Let M be a connected set of nth degree real polynomials. Then every $P(s) \in M$ has only real and distinct roots if and only if*

(i) *there exists at least one polynomial $P_0(s) \in M$ with all its roots real and distinct;*

(ii) *the associated resultant matrix R^1 for all $P(s) \in M$ is nonsingular.*

Proof. Necessity: The necessity of condition (i) is obvious. To prove the necessity of condition (ii), we recall that the resultant matrix R^1 is nonsingular if and only if $a(s) = P(s)$ and $b(s) = dP(s)/ds$ do not have common roots [2].

It has been established that if all the roots of $P(s)$ are real and distinct, then all the roots of $dP(s)/ds$ are also real, distinct, and different from the real roots of $P(s)$ [1]. Therefore $P(s)$ and $dP(s)/ds$ do not have common roots if all the roots of $P(s)$ are real and distinct. This implies that the associated resultant matrix R^1 is nonsingular.

Sufficiency: Proceeding by contradiction, suppose (i) and (ii) hold but there exists $P_x(s) \in M$ which does not have all its roots real and distinct. We need to show that there exists some $P(s) \in M$ such that the resultant matrix R^1 is singular. There are two possibilities for the root locations of $P_x(s)$:

(a) $P_x(s)$ has only real roots, with at least one multiple real root.
(b) $P_x(s)$ has at least one root which lies off the real axis.

For case (a), the associated resultant matrix R^1 for $P_x(s)$ is singular. This is because $P_x(s)$ and $dP_x(s)/ds$ have at least one common root if $P_x(s)$ has at least one multiple real root [1]. For case (b), by the connectedness of M, we can construct a continuous path Γ in M connecting $P_0(s)$ and $P_x(s)$. Then Γ induces at least one continuous path in the complex plane connecting a real root of $P_0(s)$ with the root lying outside the real axis. This guarantees the existence of some $P(s) \in \Gamma$ with $P(s)$ having a multiple real root. This is because roots can only leave the real axis in pairs. This implies that the associated resultant matrix R^1 for $P(s)$ having a multiple real root is singular. ∎

LEMMA 3. *Let M be a connected set of nth degree real polynomials. Then every $P(s) \in M$ has only real, distinct roots, with h roots lying in the real segment $(-\infty, \tau)$ and $n-h$ roots lying in the real segment (τ, ∞), if and only if*

(i) *there exists at least one polynomial $P_0(s) \in M$ having only real, distinct roots with h roots lying in the real segment $(-\infty, \tau)$ and $n-h$ roots lying in the segment (τ, ∞);*
(ii) *the associated resultant matrix R^2 for all $P(s) \in M$ is nonsingular.*

Proof. First note that the resultant matrix R^2 is nonsingular if and only if $a(s) = P(s)$ and $b(s) = (s-\tau)\, dP(s)/ds$ do not have common roots [2]. This implies that the resultant matrix R^2 is nonsingular if and only if the resultant R^1 is nonsingular and $P(s)$ does not have a real root equals to τ. The proof then follows similar arguments to those used in the proof of Lemma 2. ∎

LEMMA 4. *Let M be a connected set of nth degree real polynomials. Then every $P(s) \in M$ has only real, distinct roots, with h roots lying in the real segment (ξ, τ), p roots lying in the real segment $(-\infty, \xi)$, and q roots lying in the real segment (τ, ∞), where $h + p + q = n$, if and only if*

(i) *there exists at least one polynomial $P_0(s) \in M$ having only real, distinct roots with h roots lying in the real segment (ξ, τ), p roots lying in the real segment $(-\infty, \xi)$, and q roots lying in the segment (τ, ∞), where $h + p + q = n$;*

(ii) *the associated resultant matrix R^3 for all $P(s) \in M$ is nonsingular.*

Proof. First note that the resultant matrix R^3 is nonsingular if and only if $a(s) = (s - \tau)(s - \xi)\, dp(s)/ds$ and $b(s) = P(s)$ do not have common roots [2]. This implies that the resultant matrix R^3 is nonsingular if and only if the resultant matrix R^2 is nonsingular and $P(s)$ does not have a real root equal to ξ. The proof then follows similar arguments to those used in the proof of Lemma 2 and Lemma 3. ■

LEMMA 5. *The family of interval polynomials $I(s)$ has only real and distinct roots if and only if*

(i) *$(1 - r)P_1(s) + rP_2(s)$ has only real and distinct roots in n disjoint real segments for all $r \in [0, 1]$;*

(ii) *$(1 - r)P_3(s) + rP_4(s)$ has only real and distinct roots in n disjoint real segments for all $r \in [0, 1]$.*

Proof. Sufficiency: First note that every polynomial $P(s) \in I(s)$ satisfies

$$P_1(x) \geqslant P(x) \geqslant P_2(x) \tag{4}$$

for all real nonpositive x, and

$$P_3(x) \geqslant P(x) \geqslant P_4(x) \tag{5}$$

for all real nonnegative x. It follows that the graph of $P(x)$ lies within the envelope given by

$$[\min\{P_2(x), P_4(x)\}, \max\{P_1(x), P_3(x)\}].$$

Therefore conditions (i) and (ii) guarantee that the envelope intersects the real axis in the n disjoint real segments. This implies that every $P(s) \in I(s)$ has only real and distinct roots.

Necessity: Every polynomial $P(s) \in I(s)$ can be rewritten as

$$P(s) = H(s^2) + SG(s^2),$$

where

$$H_1(\lambda^2) \geqslant H(\lambda^2) \geqslant H_2(\lambda^2), \qquad \lambda \text{ real}, \tag{6}$$

and

$$G_1(\lambda^2) \geqslant G(\lambda^2) \geqslant G_2(\lambda^2), \qquad \lambda \text{ real}. \tag{7}$$

From (4) and (5),

$$P_1(s) = H_1(s^2) + sG_2(s^2), \tag{8}$$

$$P_2(s) = H_2(s^2) + sG_1(s^2), \tag{9}$$

$$P_3(s) = H_1(s^2) + sG_1(s^2), \tag{10}$$

$$P_4(s) = H_2(s^2) + sG_2(s^2). \tag{11}$$

Let

$$P_0(s) = \frac{P_1(s) + P_2(s)}{2}. \tag{12}$$

From (8)–(12), $P_0(s)$ can also be written as

$$P_0(s) = \frac{P_3(s) + P_4(s)}{2}.$$

Suppose $P_0(s)$ has only real and distinct roots. Then the graph of $P_0(x)$ intersects the real axis at n distinct locations. Consider the graphs of

$$G(x,r) = (1-r)P_1(x) + rP_2(x)$$

and

$$H(x,r) = (1-r)P_3(x) + rP_4(x).$$

Since

$$G(x,\tfrac{1}{2}) = H(x,\tfrac{1}{2}) = P_0(x), \tag{13}$$

there exists a number $\xi^* \leqslant \frac{1}{2}$ such that the graphs of $G(x,r)$ and $H(x,r)$ intersect the real axis in n disjoint real segments for all $r \in [\frac{1}{2} - \xi, \frac{1}{2} + \xi]$, $0 \leqslant \xi < \xi^*$.

Suppose the graphs of $G(x,r)$ and $H(x,r)$ do not intersect the real axis in n disjoint segments for all $r \in [\frac{1}{2} - \xi^*, \frac{1}{2} + \xi^*]$. That is, two of the n disjoint real segments become one at $\xi = \xi^*$. From (4), (5), and (13),

$$G(x,\tfrac{1}{2}-\xi) \geqslant G(x,r) \geqslant G(x,\tfrac{1}{2}+\xi) \qquad x \leqslant 0,$$

$$G(x,\tfrac{1}{2}-\xi) \geqslant H(x,r) \geqslant G(x,\tfrac{1}{2}+\xi) \qquad x \leqslant 0,$$

$$H(x,\tfrac{1}{2}-\xi) \geqslant H(x,r) \geqslant H(x,\tfrac{1}{2}+\xi) \qquad x \geqslant 0,$$

$$H(x,\tfrac{1}{2}-\xi) \geqslant G(x,r) \geqslant H(x,\tfrac{1}{2}+\xi), \qquad x \geqslant 0,$$

for all $r \in [\frac{1}{2} - \xi, \frac{1}{2} + \xi]$, $0 \leqslant \xi \leqslant \frac{1}{2}$. It follows that at least one of the graphs of

$$G(x,\tfrac{1}{2}-\xi^*), \qquad G(x,\tfrac{1}{2}+\xi^*)$$

$$H(x,\tfrac{1}{2}-\xi^*), \qquad H(x,\tfrac{1}{2}+\xi^*)$$

does not intersect the real axis n times. Instead, the graph has a turning point touching the real axis. This implies that the polynomial corresponding to this graph has a multiple real root. Hence, conditions (i) and (ii) are necessary for the family of interval polynomials $I(s)$ to have only real and distinct roots. ■

Remark. From the proof of Lemma 5, it also follows that the family of interval polynomials $I(s)$ has only real and distinct roots if and only if the convex combinations of polynomials in conditions (i) and (ii) of Lemma 5 have only real and distinct roots. Similarly, the family of interval polynomials $I(s)$ has only real and distinct roots distributed on the real axis in a specified manner if and only if the convex combinations of polynomials in conditions

(i) and (ii) of Lemma 5 have only real and distinct roots distributed in the specified manner.

4. MAIN RESULTS

We now derive the necessary and sufficient conditions for a family of interval polynomials $I(s)$ to have only real and distinct roots. These conditions require only a finite number of computations.

THEOREM 1. *The family of interval polynomials $I(s)$ has only real and distinct roots if and only if*

(i) *$P_1(s)$ and $P_3(s)$ have only real and distinct roots;*
(ii) *$(R_1^1)^{-1}R_2^1$ and $(R_3^1)^{-1}R_4^1$ have no eigenvalues on $(-\infty, 0]$.*

Proof. From Lemma 5 and the remark on it, the family of interval polynomials $I(s)$ has only real and distinct roots if and only if

$$(1-r)P_1(s) + rP_2(s), \qquad r \in [0,1],$$

and

$$(1-r)P_3(s) + rP_4(s), \qquad r \in [0,1],$$

have only real and distinct roots. From Lemma 2,

$$(1-r)P_1(s) + rP_2(s), \qquad r \in [0,1],$$

have only real and distinct roots if and only if

(a) $P_1(s)$ has only real and distinct roots,
(b) $(1-r)R_1^1 + rR_2^1$ is nonsingular for all $r \in [0,1]$.

Using Lemma 1,

$$(1-r)R_1^1 + rR_2^1$$

is nonsingular for all $r \in [0,1]$ if and only if $(R_1^1)^{-1}R_2^1$ has no eigenvalues on

$(-\infty, 0]$ and R_1^1 is nonsingular. Since $P_1(s)$ having only real and distinct roots implies that R_1^1 is nonsingular (see Lemma 2),

$$(1-r)P_1(s) + rP_2(s), \qquad r \in [0,1],$$

have only real and distinct roots if and only if $P_1(s)$ has only real and distinct roots and $(R_1^1)^{-1}R_2^1$ has no eigenvalues on $(-\infty, 0]$. Similarly,

$$(1-r)P_3(s) + rP_4(s), \qquad r \in [0,1],$$

have only real and distinct roots if and only if $P_3(s)$ has only real and distinct roots and $(R_3^1)^{-1}R_4^1$ has no eigenvalues on $(-\infty, 0]$. It follows that the family of polynomials $I(s)$ has only real and distinct roots if and only if conditions (i) and (ii) of Theorem 1 are satisfied. ■

REMARK. For the special case where the real and distinct roots are restricted to be negative, Soh and Berger [3] have shown that $P_1(s)$ and $P_2(s)$ having only real, negative, and distinct roots in the range $[\xi, \tau]$ are necessary and sufficient conditions for the family of interval polynomials $I(s)$ to have only real, negative, and distinct roots in the range $[\xi, \tau]$. Similarly, $P_3(s)$ and $P_4(s)$ having only real, positive, and distinct roots in the range $[\xi, \tau]$ are necessary and sufficient conditions for the family of interval polynomials $I(s)$ to have only real, positive, and distinct roots in the range $[\xi, \tau]$. However, it has been shown in the introduction that $P_1(s)$, $P_2(s)$, $P_3(s)$, and $P_4(s)$ having only real distinct roots does not necessarily imply that the family of interval polynomials $I(s)$ has only real distinct roots.

THEOREM 2. *Every polynomial in the family of interval polynomials $I(s)$ has only real, distinct roots, with h roots lying in the real segment $(-\infty, \tau)$ and $n-h$ roots lying in the real segment (τ, ∞), if and only if*

(i) *$P_1(s)$ and $P_2(s)$ have only real and distinct roots distributed in the same manner;*
(ii) *$(R_1^2)^{-1}R_2^2$ and $(R_3^2)^{-1}R_4^2$ have no eigenvalues on $(-\infty, 0]$.*

Proof. The proof is similar to the proof of Theorem 1, using Lemmas 1, 3, 5 and the remark on Lemma 5. ■

THEOREM 3. *Every polynomial in the family of interval polynomials $I(s)$ has only real, distinct roots, with h roots lying in the real segment (ξ, τ), p* *roots lying in the real segment* $(-\infty, \xi)$, *and* q *roots lying in the real segment* (τ, ∞), *where* $h + p + q = n$, *if and only if*

(i) $P_1(s)$ *and* $P_3(s)$ *have only real and distinct roots distributed in the same manner*;

(ii) $(R_1^3)^{-1}R_2^3$ *and* $(R_3^3)^{-1}R_4^3$ *have no eigenvalues on* $(-\infty, 0]$.

Proof. The proof is similar to the proof of Theorem 1, using Lemmas 1, 4, 5 and the remark on Lemma 5. ■

We now consider the linear combination of an nth degree real polynomial $P_6(s)$ and an mth degree real polynomial $P_5(s)$, where $n \geqslant m$. Let k^+ be the positive value of k such that

$$P_k(s) = P_6(s) + kP_5(s)$$

is of order less than n. In case there is no such positive value $k^+ = \infty$. Similarly, let k^- be the negative value such that $P_k(s)$ is of order less than n. In case there is no such negative value, $k^- = -\infty$.

THEOREM 4. *Suppose* $P_6(s)$ *has only real and distinct roots. Then the maximal range* $(k_{\min}, k_{\max})$ *for the family of nth degree polynomials* $P_k(s)$ *to have only real and distinct roots is given by*

$$k_{\min} = \max\left\{k^-, \frac{1}{\lambda_{\min}^-\left[-\left(R_6^1\right)^{-1}R_5^1\right]}\right\}$$

and

$$k_{\max} = \min\left\{k^+, \frac{1}{\lambda_{\max}^+\left[-\left(R_6^1\right)^{-1}R_5^1\right]}\right\}.$$

The resultant matrix R_5^1 *is obtained by treating* $P_5(s)$ *an an nth degree polynomial for the purpose of conformability of matrix multiplication.*

Proof. First, note that it is necessary to have $k_{\min} \geqslant k^-$ and $k_{\max} \leqslant k^+$ to guarantee that all $P_k(s)$ will be nth degree. Therefore, we assume $k_{\min} \geqslant k^-$ and $k_{\max} \leqslant k^+$ in the remainder of the proof. Since $P_6(s)$ has only real and distinct roots, in accordance with Lemma 2, $P_k(s)$ has only real and distinct roots for all $k \in (k_{\min}, k_{\max})$ if and only if

$$R_6^1 + kR_5^1$$

is nonsingular for all $k \in (k_{\min}, k_{\max})$. The maximal range $(k_{\min}, k_{\max})$ is then obtained using Lemma 1, part (ii). ∎

THEOREM 5. *Suppose $P_6(s)$ has only real and distinct roots, with h roots lying in the real segment $(-\infty, \tau)$ and $n-h$ roots lying in the real segment (τ, ∞). Then the maximal range $(k_{\min}, k_{\max})$ for every polynomial in the family of nth degree polynomials $P_k(s)$ to have only real and distinct roots distributed in the same manner is given by*

$$k_{\min} = \max\left\{k^-, \frac{1}{\lambda_{\min}^-\left[-\left(R_6^2\right)^{-1}R_5^2\right]}\right\}$$

and

$$k_{\max} = \min\left\{k^+, \frac{1}{\lambda_{\max}^+\left[-\left(R_6^2\right)^{-1}R_5^2\right]}\right\}.$$

The resultant matrix R_5^2 is obtained by treating $P_5(s)$ as an nth degree polynomial for the purpose of conformability of matrix multiplication.

Proof. The proof is similar to the proof of Theorem 4, using Lemma 1 and Lemma 3. ∎

THEOREM 6. *Suppose $P_6(s)$ has only real and distinct roots, with h roots lying in the segment (ξ, τ), p roots lying in the real segment $(-\infty, \xi)$, and q roots lying in the real segment (τ, ∞), where $h + p + q = n$. Then the maximal range $(k_{\min}, k_{\max})$ for every polynomial in the family of nth degree polynomials $P_k(s)$ to have only real and distinct roots distributed in the same manner is given by*

$$k_{\min} = \max\left\{k^-, \frac{1}{\lambda_{\min}^-\left[-\left(R_6^3\right)^{-1}R_5^3\right]}\right\}$$

and

$$k_{\max} = \min\left\{ k^{+}, \frac{1}{\lambda_{\max}^{+}\left[-\left(R_6^3\right)^{-1} R_5^3 \right]} \right\}.$$

The resultant matrix R_5^3 is obtained by treating $P_5(s)$ as an nth degree polynomial for the purpose of conformability of matrix multiplication.

Proof. The proof is similar to the proof of Theorem 4, using Lemma 1 and Lemma 4. ■

5. CONCLUSION

We have obtained necessary and sufficient conditions (requiring only a finite number of computations) for a family of interval polynomials to have only real and distinct roots. These results complement the necessary and sufficient conditions obtained by Soh and Berger [3].

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