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ON THE RELATIONS $P(X \times Y) = PX \times PY$

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Let P and Q be epireflective full subcategories of the category Haus of Hausdorff spaces and continuous functions, and also denote the corresponding reflectors by P: Haus $\to P$ and Q : Haus $\to Q$ respectively. Denote the class of P-regular spaces, i.e., of subspaces of P-spaces, by $\mathbf{R}P$. Embracing certain special cases which have been treated in the literature, we show that if $P \subset Q \subset \mathbf{R}P$ then for $X, Y \in \mathbf{R}P$ the relation $P(X \times Y) = PX \times PY$ implies $Q(X \times Y) = QX \times QY$. Applications to particular classes P, Q are given.

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realcompact space
topologically complete space

1. Introduction

Let **Haus** be the category of Hausdorff spaces and continuous functions. For a topological property P of Hausdorff spaces we do not distinguish between (1) this property, (2) the class of all P-spaces, i.e. of all spaces having property P, and (3) the full subcategory of **Haus** whose objects are the P-spaces. P is called *epireflective* in **Haus** if the following equivalent conditions hold:

(i) P is productive and closed-hereditary;

(ii) for every $X \in$ **Haus** there is $PX \in P$ and a continuous function $p_X : X \to PX$ such that $p_X[X]$ is dense in PX and for every continuous $f: X \to Y \in P$ there is a continuous $\bar{f}: PX \to Y$ such that $f = \bar{f} \circ p_X$.

If these conditions are satisfied, then the pair (p_X, PX), called the

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P-reflection of X, is determined uniquely up to homeomorphism. This gives rise to a functor P : **Haus** $\to P$ which commutes with colimits but usually not with limits. nor even with finite products. The aim of this paper is to show that for epireflective subcategories P and Q of **Haus** under suitable conditions the equality $P(X\times Y) = PX\times PY$ implies $Q(X\times Y) = QX\times QY$.

A space $X\in$ **Haus** is called *P-regular* if there is $P\in P$ such that $X\subset P$; the class of P-regular spaces is denoted **R***P*.

We refer the reader to [5,6,10,11,14] for additional background material, and to [10,12,18] for proofs of the following fact (which is crucial to the proof of our theorem below). For a productive, closed-hereditary class P, the P-reflection $p_X : X\to PX$ is a topological embedding if and only if $X\in \mathbf{R}P$; and if $X\in \mathbf{R}P$, then (p_X, PX) is characterized as follows:

(1) $PX\in P$;

(2) p_X is a P-embedding, i.e., an embedding of X onto a dense subspace of PX such that for every continuous $f: X\to Y\in P$ there is a continuous $\bar{f} : PX\to Y$ such that $f=\bar{f}\circ p_X$.

2. A two-class theorem

We here prove the result (Theorem 2.2) cited in the Abstract.

2.1. Lemma. *Let P and Q be epireflective subcategories of* **Haus** *such that $P\subset Q\subset \mathbf{R}P$. If $X, Y\in \mathbf{R}P$ and $X\times Y$ is P-embedded in $X\times QY$, then $X\times Y$ is Q-embedded in $X\times QY$.*

Proof. Let continuous $f: X\times Y\to Z\in Q$. Since $Z\in \mathbf{R}P$, the P-reflection $p_Z : Z\to PZ$ is a topological embedding. By assumption there is a continuous function $\bar{f} : X\times QY\to PZ$ such that $f\subset \bar{f}$, and it is enough to show that $\bar{f}[X\times QY]\subset Z$. For $x\in X$ we set $f_x = f|\{x\}\times Y$, we denote by $\bar{f}_x$ the continuous extension of f_x mapping $\{x\}\times QY$ to Z, and we note that for every $x\in X$ the functions $\bar{f}\mid\{x\}\times Y$ and $\bar{f}_x|\{x\}\times Y$ (which is f_x) are equal. Since $PZ\in$ **Haus**, we have $\bar{f}|\{x\}\times QY=\bar{f}_x$ for all $x\in X$, and hence

$$\bar{f}[X\times QY] = \bigcup_{x\in X} \bar{f}[\{x\}\times QY] = \bigcup_{x\in X} \bar{f}_x[\{x\}\times QY]\subset Z,$$

as required. □

2.2. Theorem. *Let P and Q be epireflective subcategories of* **Haus** *such that $P\subset Q\subset$ RP. If X, $Y\in$ RP and $P(X\times Y) = PX\times PY$, then $Q(X\times Y) = QX\times QY$.*

Proof. We note from the uniqueness result cited above that it follows that $X\subset QX\subset PX$ and $Y\subset QY\subset PY$ and that it is sufficient to prove that $X\times Y$ is Q-embedded in $QX\times QY$. Now $X\times Y$ is P-embedded in $PX\times PY$, and hence in $X\times QY$; hence (by Lemma 2.1) $X\times Y$ is Q-embedded in $X\times QY$. A second appeal to Lemma 2.1 shows that $X\times QY$ is Q-embedded in $QX\times QY$. The proof is complete. □

We note in 2.4 below certain instances of 2.1 and 2.2 (of which some have appeared already in the literature). We denote by $[0, 1]$ the closed unit interval and by D_2 the two-element discrete space. The following definitions, equivalent to the standard formulations, are convenient for our purposes.

2.3. Definition. A topological space is:

(a) *zero-dimensional* if it is homeomorphic to a subspace of a power of D_2;

(b) *realcompact* if it is homeomorphic to a closed subspace of a product of real lines;

(c) *topologically complete* if it is homeomorphic to a closed subspace of a product of metric spaces.

We denote by Z the class of zero-dimensional spaces and by **Tych** the class of completely regular Hausdorff spaces. For $X\in$ **Tych** we denote by βX, υX and γX the Stone-Čech compactification, the Hewitt realcompactification and the topological completion of X, respectively.

Concerning Corollary 2.4(b), we note that (for P closed-hereditary in **Haus**) the hypothesis $D_2\in P$ is fulfilled whenever P is non-trivial (i.e., whenever there is $Z\in P$ such that $|Z| > 1$).

2.4. Corollary. *Let P and Q be epireflective subcategories of* **Haus** *and let X, $Y\in$* **Tych**.

(a) *If $[0, 1]\in P\subset Q\subset$* **Tych** *and $P(X\times Y) = PX\times PY$, then $Q(X\times Y) = QX\times QY$.*

(b) *If $D_2\in P\subset Q\subset Z$ and X, $Y\in Z$ and $P(X\times Y) = PX\times PY$, then $Q(X\times Y) = QX\times QY$.*

(c) *If $\beta(X\times Y) = \beta X\times\beta Y$, then $\upsilon(X\times Y) = \upsilon X\times\upsilon Y$.*

(d) *If $\upsilon(X\times Y) = \upsilon X\times\upsilon Y$, then $\gamma(X\times Y) = \gamma X\times\gamma Y$.*

We have shown above (for suitably restricted subcategories $\boldsymbol{P}$ and $\boldsymbol{Q}$ of **Haus** and for $X, Y \in \boldsymbol{RP}$) that if $X\times Y$ is $\boldsymbol{P}$-embedded in $QX\times QY$, then $X\times Y$ is $\boldsymbol{Q}$-embedded in $QX\times QY$. For $\boldsymbol{P}$ the class of compact spaces and $\boldsymbol{Q}$ the class of realcompact spaces, the result is given in [3, Theorem 5.2]. Corollary 2.3(c) has been noticed by several authors, using a theorem of Glicksberg [8] (cf. (2) of 3.4 below); and statement (d) is due to Isiwata [13, Theorem 2.1].

3. Spaces X such that $P(X\times Y) = PX\times PY$

3.1. Definition. If X is a space and $A, B \subset X$, then A and B are said to be *completely separated* (in X) if there is a continuous function $f: X\to [0,1]$ such that $f[A] \subset \{0\}$ and $f[B]\subset\{1\}$.

The following lemma is from [19, Theorem 3.7]. Tamano's proof and a number of applications are given in [4].

3.2. Lemma. *If $X \in$ **Tych**, then X is topologically complete if and only if for every $p\in \beta X\setminus X$ the sets $X\times\{p\}$ and $\{(x,x): x\in X\}$ are completely separated in $X\times\beta X$.*

3.3. Corollary. *Let $\boldsymbol{P}$ be an epireflective subcategory of **Tych** such that $[0,1]\in\boldsymbol{P}$, and let X be a topologically complete space. The following are equivalent.*

(a) $P(X\times Y) = PX\times PY = PX\times Y$ *for all* $Y\in\boldsymbol{P}$;

(b) $P(X\times PX) = PX\times PX$;

(c) $P(X\times\beta X) = PX\times\beta X$;

(d) $X = PX$.

Proof. Since $PX\in\boldsymbol{P}$ and $\beta X\in\boldsymbol{P}$, we have (a) ⇒ (b) and (a) ⇒ (c). We show next that (b) ⇒ (d) and (c) ⇒ (d). Indeed, if (d) fails there is

$$p\in PX\setminus X\subset \beta X\setminus X,$$

and since X is topologically complete there is (by Lemma 3.2) a continuous function $f: X\times\beta X\to[0,1]$ such that

$$f[X\times\{p\}] = \{0\},$$
$$f(x,x) = 1 \quad \text{for all } x\in X.$$

It is clear that f has no continuous extension to the point $\langle p, p\rangle \in PX\times\beta X$ (so that (c) fails), and that $f|X\times PX$ has no continuous extension to the point $\langle p, p\rangle \in PX\times PX$ (so that (b) fails).

That (d) ⇒ (a) follows from the uniqueness of $P(X\times Y)$ as described in the introduction. □

3.4. Remarks. The requirement in Corollary 3.3 that X be topologically complete is not artificial and cannot be omitted. The failure of the equivalence of (a), (b), (c) and (d) for spaces X that are not topologically complete for suitable choices of the class $\boldsymbol{P}$ is given in Table 1, where $\boldsymbol{T}$ denotes the class of topologically complete spaces and $\boldsymbol{C}$ the class of compact Hausdorff spaces (so that $CY = \beta Y$ for all $Y\in$ **Tych**). The entry Y in this table means "Yes, the statement in question (a), (b), (c) or (d) of Corollary 3.3) does hold"; the entry N means "No, it does not"; the single entry (*) is discussed below.

The reader will easily verify the Y-N entries by using the following facts.

(1) If $X\notin \boldsymbol{T}$, there is $Y\in \boldsymbol{T}$ such that $T(X\times Y)\neq TX\times TY$. This is shown by Isiwata [13, Theorem 3.2], following McArthur's [15] use of a construction of Hager and Mrowka (see [9, Theorem 3.2]).

(2) $\beta(X\times Y) = \beta X\times\beta Y$ if and only if X or Y is finite or $X\times Y$ is pseudocompact. This result is due to Glicksberg [8]; see also [7].

(3) If $Y\in \boldsymbol{T}$ and Y is locally compact then $T(X\times Y) = TX\times TY$ for all $X\in$ **Tych**. This is shown by Pupier [17, Théorème 4.3] and Morita [16, Theorem 5.1], see [2, Corollary 2.2] for the appropriate analogous result concerning realcompact spaces.

(4) If X is pseudocompact and Y is compact, then $X\times Y$ is pseudocompact. This familiar result is proved (for example) in [8].

(5) Every pseudocompact, topologically complete space is compact (and hence $TX = \beta X$ for every pseudocompact space X). This familiar result is stated (for example) in [1, Theorem 8].

As to the entry (*) we show that the relation $T(X\times TX) = TX\times TX$ holds for certain spaces X that are not topologically complete and not pseudocompact, and fails for certain other such spaces X. If X_0 is locally compact and topologically complete and X_1 is pseudocompact but not compact, then $T(X_1\times X_0) = (\beta X_1)\times X_0$ by (4) and (5), and the "disjoint union" $X = X_0 + X_1$ satisfies $TX = X_0 + \beta X_1$ and $T(X\times TX) = TX\times TX$. (Essentially this is remarked by McArthur [15, Example 5.8].) On the other hand, if X_0 is any space that is not topologically complete and $X_1\in \boldsymbol{T}$ is chosen so that $T(X_0\times X_1)\neq TX_1$, then

Table 1
Behavior of spaces X such that $X \notin T$

		(a)	(b)	(c)	(d)
X pseudocompact	$P = C$	Y	Y	Y	N
X pseudocompact	$P = T$	N	Y	Y	N
X not pseudocompact	$P = C$	N	N	N	N
X not pseudocompact	$P = T$	N	(*)	Y	N

with $X = X_0 + X_1$ we have $TX = TX_0 + X_1$ and hence $T(X \times TX) \neq TX \times TX$.

We conclude with a corollary to 2.2 and 3.2. Again, we denote by T the class of topologically complete spaces.

3.5. Corollary. *Let* P *be an epireflective subcategory of* **Tych** *such that* $[0, 1] \in P$ *and let* $X \in$ **Tych**. *Then*:

(a) *If* $P \subset T$ *and* $P(X \times Y) = PX \times PY$ *for all* $Y \in$ **Tych**, *then* $X \in P$.

(b) *If* $T \subset P$, X *is locally compact and* $X \in T$, *then* $P(X \times Y) = PX \times PY$ *for all* $Y \in$ **Tych**.

Proof. (a) It follows from Theorem 2.2 that $T(X \times Y) = TX \times TY$ for all $Y \in$ **Tych**, so that $X \in T$ by 3.4(1); hence $X \in P$ by Corollary 3.3(a).

(b) This follows from (3) above and Theorem 2.2.□

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