Contribution to Nonserial Dynamic Programming*

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Submitted by Richard Bellman

A new property of nonserial dynamic programming is presented in this paper. This property allows cutting down considerably the computational effort required for solving the secondary optimization problem.

1. INTRODUCTION

The application of the principle of optimality of dynamic programming to the solution of the problem of optimizing non serial systems has been widely discussed in the literature [1, 5-7].

In this context the principle of optimality can conveniently be regarded as a decomposition technique which, at the cost of some (and often too much) storage, allows breaking the optimization problem in many smaller subproblems.

Two recent works [3, 4], deal with the problem of finding a decomposition which is optimal from the point of view of minimizing the number of operations required with the constraint that the storage space does not exceed a prescribed level. This paper follows closely the approach of [3] and [4], and presents a new mathematical result, which has some important computational implications. The paper is organized in sections as follows.

(a) Section 2 contains a short survey of those parts of [3] and [4] which are relevant to this work.

(b) Section 3 presents a short example.

(c) Section 4 introduces the definition of absence graph.

(d) Section 5 contains the mathematical results of the paper.

(e) Section 6 discusses some computational implications of the results of the preceding section.

Some elementary graph and set theory is used throughout the paper. An adequate reference is, for instance, Berge [2].

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2. A SUMMARY OF THE PREVIOUS RELATED WORK

Consider the following optimization problem

$$\min_X F(X) = \min_X \sum_{i \in I} f_i(X^i)$$

where

$$X = \{x_1, x_2, \ldots, x_M\}$$

is a set of discrete variables

$$I = \{1, 2, \ldots, n\} \quad \text{and} \quad X^i \subseteq X.$$  

Each component $f_i(X^i)$ of the cost function $F(X)$ is specified by means of a stored table with $|X^i| + 1$ columns and $|X^i|$ rows. For simplicity it has been assumed that all variables have the same range, namely that each variable can assume $\sigma$ values. The solution to the optimization problem stated above is now discussed.

Let $x_i \in X$ and $x_j \in X$. The two variables $x_i$ and $x_j$ are said to interact if there exists a component $f_k(X^k)$ such that both $x_i$ and $x_j$ belong to $X^k$.

One order, among the $M!$ possible ones, of the variables of the set $X$, is selected. Let $y_1, y_2, \ldots, y_M$ be such order. For this order the problem may be solved by dynamic programming. More specifically first the variable $y_1$ is considered. For minimizing $F(X)$ with respect to $y_1$ it is sufficient to compute

$$\min_{y_1} \sum_{i \in I_1} f_i(X^i) = g_1(I(y_1))$$

where

$$I_1 = \{i : X^i \cap \{y_1\} \neq \emptyset\}$$

and to store the optimizing assignment $y_1^*(I(y_1))$. Here $I(y_1)$ is the set of variables which interact with $y_1$. The minimization of $F(X)$ with respect to $y_1$, for all possible assignments of $I(y_1)$, is called the elimination of the variable $y_1$.

The problem remaining after the elimination of $y_1$

$$\min_{X \setminus \{y_1\}} g_1(I(y_1)) + \sum_{i \in I_1 \setminus I_4} f_i(X^i)$$

is of the same form of the original one (the function $g_1(I(y_1))$ may be regarded as a component of the new cost function).

Hence, letting $I(y_i | y_1, y_2, \ldots, y_{i-1})$ be the variables interacting with $y_i$ in the problem obtained after the elimination of $y_1, y_2, \ldots, y_{i-1}$ in that order, it follows that an optimal assignment for $X$ can be found, first
(1) eliminating the variables in the given order $y_1, y_2, \ldots, y_M$ and storing the optimizing assignment $y_1^*(\Gamma(y_1))$, $y_2^*(\Gamma(y_2 \mid y_1)), \ldots, y_M^*$ (note that, clearly, the set $\Gamma(y_M \mid y_1, y_2, \ldots, y_{M-1})$ is empty) and secondly,

(2) operating "backwards" determining successively $y_M^*$, $y_{M-1}^*$, $\ldots$, $y_1^*$ (i.e. the optimizing assignment for $X$) from the stored tables.

It is clear now how another optimization problem (the secondary optimization problem) emerges. An optimal assignment for $X$ can be equally obtained by $M!$ orders of elimination of the variables in the set $X$. Which, then among those $M!$ orders is the best from the point of view of minimizing the number of operations (i.e. the computing time) or the storage requirements or both? The elimination of the variable $y_1$ implies the construction and storage of two tables, one for the optimizing assignment $y_1^*(\Gamma(y_1 \mid y_1, y_2, \ldots, y_i))$ and the other for the new component $g_i(\Gamma(y_i \mid y_1, y_2, \ldots, y_{i-1}))$, with $|\Gamma(y_i \mid y_1, y_2, \ldots, y_{i-1})| + 1$ columns and $\sigma|\Gamma(y_i \mid y_1, y_2, \ldots, y_{i-1})| + 1$ rows.

The number of operations or table look-ups required is $\sigma|\Gamma(y_i \mid y_1, y_2, \ldots, y_{i-1})| + 1$ times the number of components in which $y_i$ appears.

Since the exponential factor is, usually, the most decisive, it is clear that the integer $|\Gamma(y_i \mid y_1, y_2, \ldots, y_{i-1})|$ is a reasonable index of the computational effort (number of operations) and of the storage requirements for the elimination of that variable.

Solving the secondary optimization problem consists in finding one order of elimination $y_1, y_2, \ldots, y_M$ for which the largest integer $|\Gamma(y_i \mid y_1, y_2, \ldots, y_{i-1})|$ is minimal.

It is now shown that this problem becomes a problem in graph theory. The interaction graph of the problem, $G(X, \Gamma)$ is an undirected graph defined by:

(1) The vertex set of the graph is the set of the variables of the problem.

(2) Two vertices are connected with an edge if an only if the corresponding variables interact.

The elimination of a variable $y_1$ from the original problem implies a new one in which all the tables containing $y_1$ are replaced by a new table containing all the variables interacting with $y_1$. Hence the interaction graph of the new problem is obtained from the original one deleting the variable $y_1$ and all the edges emanating from it, and connecting all the previously unconnected vertices corresponding to the variables interacting with $y_1$ (i.e. the vertices of $\Gamma(y_1)$).

Clearly the degree of the eliminated vertex $y_i$ in the graph resulting from the elimination of $y_1, y_2, \ldots, y_{i-1}$, equals $|\Gamma(y_i \mid y_1, y_2, \ldots, y_{i-1})|$. Hence the secondary optimization problem consists in finding an order of elimination of the vertices of $G(X, \Gamma)$ such that the largest degree of the eliminated vertices is minimal.
The degree of an eliminated vertex \( y_i \) in a given order of elimination \( y_1, y_2, ..., y_M \) is called the **dimension (of the stored table)** associated with the elimination of the vertex \( y_i \) and denoted by \( D(y_i) \).

The largest degree of the eliminated vertices for an order of elimination \( y_1, y_2, ..., y_M \) is called the **dimension of the order** and denoted by \( D(y_1, y_2, ..., y_M) \).

The minimal dimension for all possible orders is called the **dimension of the problem** or of the graph \( G \) or of the set \( X \) and denoted by \( D(G) \) or \( D(X) \).

Consider a subset \( X' \subset X \) with \( |X'| = m \leq M \) and an order of elimination \( y_1, y_2, ..., y_m \) of the variable of \( X' \).

The largest degree of the eliminated vertices for the order of elimination \( y_1, y_2, ..., y_m \) is called the **partial dimension** of that order and denoted by \( D^*(y_1, y_2, ..., y_m) \).

Among the results of Ref. [3] and [4] one is used in this work and is therefore reported here (clearly without proof).

**Theorem.** Let \( X' \subset X \). The graph which results from the elimination of the variables of the set \( X' \), provided that such variables are eliminated one by one, does not depend upon the order of elimination.

The optimization technique described in this paper has been given the name of **nonserial dynamic programming**.

In fact the method employs dynamic programming for solving the given optimization problem but, prior to that, determines one order of elimination of the variables, so that the dynamic programming procedure is most efficiently used.

Clearly when the system is serial namely when \( X^i = \{x_i, x_{i+1}\} \) and \( I = \{1, 2, ..., M - 1\} \) one optimal order is \( x_1, x_2, ..., x_M \) and no secondary optimization problem needs to be solved.

Finally it must be noted that the present statement of the secondary optimization problem is not the most general one. For more general decompositions, consisting in eliminating more than one variable at a time and a corresponding more general statement of the secondary optimization problem, see [4].

### 3. An Example

An example of [4] is for convenience reported here.

Let

\[
F = \sum_{i=1}^{3} f_i(X^i)
\]
where \( \sigma = 2 \); \( X = \{x_1, x_2, x_3, x_4, x_5\} \), \( X^1 = \{x_1, x_3, x_4\} \), \( X^2 = \{x_1, x_2\} \), \( X^3 = \{x_2, x_4, x_5\} \) and the \( f_i \) - s are given in the following tables:

<table>
<thead>
<tr>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( f_1 )</th>
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It is now shown how an optimal assignment for \( X \), out of all the \( 2^5 = 32 \) possible assignments, is found by eliminating \( x_1, x_2, x_3, x_4, x_5 \) in that order. When eliminating \( x_1 \), only \( f_1 + f_2 \) is considered, since \( x_1 \) does not affect \( f_3 \). Next, a table with \( x_2, x_3, \) and \( x_5 \) as variables is constructed since they are interacting with \( x_1 \) through \( f_1 \) and \( f_2 \). For every assignment of \( x_2, x_3, \) and \( x_5 \) a best value for \( x_1 \) is chosen called \( x_1^* \), so that \( f_1 + f_2 \) is minimal. This value of \( f_1 + f_2 \), as a function of \( x_2, x_3, \) and \( x_5 \) is called \( g_1 \).

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<tr>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_5 )</th>
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<td>9</td>
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</table>

Next \( x_2 \) is eliminated in the optimization problem of \( g_1 + f_3 \). Let the new function of \( x_3, x_4, \) and \( x_5 \) be called \( g_2 \) and the optimizing value of \( x_2 \) be called \( x_2^* \).

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<tr>
<th>( x_2 )</th>
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<th>( x_5 )</th>
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<th>( x_2^* )</th>
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Next $x_3$ is eliminated in the optimization problem of $g_2$. The new function of $x_4$ and $x_5$ is called $g_3$ and the optimizing value of $x_3$ is called $x_3^*$. 

$$
\begin{array}{c|cc}
    & x_3^* & x_4^* \\
    \hline
    0 & 2 & 1 \\
    0 & 9 & 1 \\
    1 & 8 & 1 \\
    1 & 7 & 1 \\
\end{array}
$$

Next $x_4$ is eliminated in the optimization problem of $g_3$. The new function of $x_5$ is called $g_4$ and the optimizing value of $x_4$ is called $x_4^*$. 

$$
\begin{array}{c|cc}
    x_5 & x_4^* \\
    \hline
    0 & 2 \\
    1 & 7 \\
\end{array}
$$

Finally $x_5$ is eliminated, namely $g_4$ is optimized with respect to $x_5$. Thus, in this case, $\min F = \min g_4 = 2$ and $x_5^* = 0$, $x_4^* = 0$, $x_3^* = 1$, $x_2^* = 0$, $x_1^* = 1$ is one optimizing assignment.

The number of table look-ups in this solution is 

$$2.2^4 + 2.2^4 + 1.2^3 + 1.2^2 + 1.2 = 78.$$ 

If the order of elimination is $x_3$, $x_4$, $x_5$, $x_2$, $x_3$, as the reader can verify for himself, the number of table look-ups is 

$$1.2^3 + 1.2^3 + 2.2^3 + 2.2^2 + 1.2 = 42.$$ 

The interaction graph of the original problem is shown in Fig. 1. The graphs which result from the elimination of the variables in the order $x_1$, $x_2$, 

![Fig. 1. An example of interaction graph.](image-url)
$x_3, x_4, x_5$ are shown in Fig. 2 and those resulting from the elimination of the variables in the order $x_3, x_4, x_5, x_2, x_1$ are shown in Fig. 3.

The dimension of the order $x_1, x_2, x_3, x_4, x_5$ is $\max(3, 3, 2, 1, 0) = 3$, and the dimension of the order $x_3, x_4, x_1, x_2, x_5$ is $\max(2, 2, 2, 1, 0) = 2$.

Thus the example shows how the dimension of the order is an index of the number of the operations required.

However it also shows that, if the number of operations is counted exactly, the decomposition procedure which requires the least number of operations does not necessarily belong to the subclass of decompositions in which the variables are eliminated one at a time. For instance, when the problem consists of a single table (as $g_2$ in the example), it is not efficient to go on eliminating variables one by one; instead it is better to select directly a row which minimizes the function, i.e. to eliminate all the remaining variables simultaneously.
4. The Absence Graph

Definition 1. The complement of the interaction graph \( G(X, \Gamma) \) is called the absence graph and denoted by \( G(X, \Gamma') \).

Clearly it is possible to determine the absence graph which derives from the elimination of one variable directly i.e. without resorting to the interaction graph. Specifically the absence graph which derives from the elimination of \( x_i \) from the original absence graph is as follows:

1. The vertex set is \( X - \{x_i\} \).
2. The edges \((x_j, x_k)\) are those among the edges of \( G \) for which at least one, between \( x_j \) and \( x_k \), belongs to \( \Gamma(x_i) \). Clearly also the edges emanating from \( x_i \) are canceled.

An example of elimination of a variable from an absence graph is given in Fig. 4.

Let \( y_1, y_2, \ldots, y_i, \ldots, y_M \) be an order of elimination; let \( G = G_1 \) and \( G_i \) be the absence graph which results from the elimination of the vertices \( y_1, y_2, \ldots, y_{i-1} \).

The notation \( G(X, \Gamma) \) for the absence graph implies that \( \Gamma(y_i) \) is the set of variables connected to \( y_i \) in \( G_1 \), namely those variables which do not interact with \( y_i \) in the original interaction graph. Similarly the notation \( \Gamma(y, y_1, y_2, \ldots, y_{i-1}) \) refers to the set of variables connected with \( y_i \) in \( G_i \).

It is clear that the dimension associated with the elimination of \( y_i \) from \( G_i \) is given by

\[
D(y_i) = (M - i) - |\Gamma(y_i, y_1, y_2, \ldots, y_{i-1})|.
\]

5. The Mathematical Results

Lemma 1. Let \( G(X, \Gamma) \) be an interaction graph and \( G(X, \Gamma') \) be the corresponding absence graph. There exists at least one minimal dimension order \( y_1 \),
$y_2, \ldots, y_M$ with the following property: If the absence graph $G_i$ is not empty and has isolated vertices, then $y_i$ is not one of such isolated vertices.

**Proof.** Obvious.

It is worthwhile noting that there exist minimal dimension orders not satisfying Lemma 1. The order $x_1, x_2, x_3, x_4$ for the absence graph of Fig. 5 is an example of a minimal dimension order of this kind.

![Fig. 5](image)

**Fig. 5.** The order $x_1, x_2, x_3, x_4$ is a nonregular minimal dimension order.

**Definition 2.** A minimal dimension order is **regular** if it satisfies the property of Lemma 1.

**Lemma 2.** Let $G(X, \Gamma)$ be an interaction graph. Let $T_1(| T_1 | \geq 1)$ be a subset of $X$, fully connected in $G$.

Then there exists an order of elimination, beginning with a vertex of $X - T_1$, which has minimal dimension.

**Proof.** If $G$ is fully connected the lemma trivially holds. Otherwise consider a subset $T_1$ fully connected in $G$ and let $Z_1 = X - T_1$. Since $G$ is not fully connected $Z_1$ is not empty. The proof is given considering, for convenience, the absence graph $G(X, \bar{\Gamma})$. Clearly in $G$ the vertices of $T_1$ have connections only with those of $Z_1$ (see Fig. 6).

Consider a minimal dimension order beginning with a vertex of $T_1$, if such order exists. In such case, by Lemma 1, there exists also a regular minimal dimension order beginning with a vertex of $T_1$. Let $y_1, y_2, \ldots, y_M$ be such order.

It will now be shown that there exists another order beginning with a vertex of $Z_1$ which has the same dimension.

Since, by assumption, $y_1 \in T_1$ and the vertices of $T_1$ have connections in $G_1$ only with those in $Z_1$, it follows that $\bar{F}(y_1) \subset Z_1$. 
Let $Z_2 = \overline{I}(y_1)$ and $T_2 = (X - \{y_1\}) - Z_2$. Since, by the elimination of $y_1$ from $G_1$, all the edges connecting two vertices, both not belonging to $\overline{I}(y_1)$, disappear, in $G_2$ the vertices of $T_2$ have connections only with the vertices of $Z_2$. If $y_2 \in T_2$ the sets

$$Z_3 = \overline{I}(y_2, y_1) \quad \text{and} \quad T_3 = (X - \{y_1, y_2\}) - Z_3$$

are defined and, in the absence graph $G_3$, the vertices of $T_3$ have connections only with the vertices of $Z_3$. This procedure is repeated until either

1. a vertex $y_h$ is found such that $y_h \in Z_h (h \in \{2, 3, \ldots, M\})$, or
2. an integer $k$ is found such that $Z_k = \emptyset$ (and $y_i \in T_i$, $i \in \{1, 2, \ldots, k - 1\}$).

Case (2) is now examined. Since the minimal dimension order $y_1, y_2, \ldots, y_M$ is regular, $G_{k-1}$ is empty. Consider a vertex $z \in Z_{k-1}$. It is clear that the order $y_1, y_2, \ldots, y_{k-2}, z$ (with all the other vertices following in any order) is also optimal.

Thus it has been shown that, among the regular minimal dimension orders beginning with a vertex of $T_1$, if such orders exist, there is at least one $z_1, z_2, \ldots, z_M$ for which there exists an integer $l$ ($l \in \{2, 3, \ldots, M\}$) with $z_i \in Z_l$.

Next it is shown that the partial dimensions of the two orders $O_1 = z_1, z_2, \ldots, z_l$ and $O_2 = z_1, z_1, z_2, \ldots, z_{l-1}$ are equal.
Since the graph resulting from the elimination of the variables of the set \(x_1, x_2, \ldots, x_l\) does not depend upon the order of elimination (see the theorem of Section 2) this is sufficient for proving the lemma.

Let \(D_1(x_j)\) and \(D_2(x_j)\) be the dimensions associated with the elimination of \(x_j\) \((j \in \{1, 2, \ldots, l\})\) in the two orders \(O_1\) and \(O_2\), respectively.

It results
\[
D_1(x_j) = (M - j) - |\bar{\Gamma}(x_j | x_1, x_2, \ldots, x_{j-1})|,
\]
j \(\in\) \(\{1, 2, \ldots, l\}\)
and
\[
D_2(x_j) = (M - (j + 1)) - |\bar{\Gamma}(x_j | x_1, x_2, \ldots, x_{j-1})|,
\]
j \(\in\) \(\{1, 2, \ldots, l - 1\}\).

Since \(x_i \in Z_i\) and hence \(x_i \in Z_i\) for \(i \in \{1, 2, \ldots, l - 1\}\), it is clear that \(x_i\) is connected to \(x_1, x_2, \ldots, x_{i-1}\) in \(G\). Then it follows
\[
|\bar{\Gamma}(x_i | x_1, x_2, \ldots, x_{i-1})| = |\bar{\Gamma}(x_i)| - (l - 1)
\]
and, for \(j \in \{1, 2, \ldots, l - 1\}\)
\[
|\bar{\Gamma}(x_j | x_1, x_2, \ldots, x_{j-1})| = |\bar{\Gamma}(x_j | x_1, x_2, \ldots, x_{j-1})| - 1.
\]

Then it follows that for \(j \in \{1, 2, \ldots, l\}\)
\[
D_1(x_j) = D_2(x_j).
\]

This completes the derivation. Q.E.D.

**Theorem 1.** Let \(G(X, \Gamma)\) be an interaction graph and let \(T\) be a fully connected subset of \(X\). There exists at least one minimal dimension order of elimination in which the variables of \(T\) have the last \(|T|\) places.

**Proof.** If \(T = X\) the theorem trivially holds.

Otherwise the proof is by induction on the number of vertices. Clearly the theorem holds for \(M = 2\). Consider an interaction graph \(G_1(X_1, \Gamma_1)\) with \(|X_1| = M\) and let \(T \subseteq X_1\). According to the inductive hypothesis, it is assumed that there exists a minimal dimension order in which the variables of \(T\) have the last \(|T|\) places. Consider a graph \(G_2(X_2, \Gamma_2)\) with \(|X_2| = M + 1\) and let \(T \subseteq X_2\). By Lemma 2 there exists an order of elimination beginning with a vertex of \(X_2 - T\) and, in the graph resulting from the elimination of such vertex, the set \(T\) is still fully connected.

Q.E.D.
6. SOME COMPUTATIONAL IMPLICATIONS

The computational importance of the theorem for the solution of the secondary optimization problem is clear. Letting $M^*$ be the number of vertices of the maximal fully connected subset of $X$, the number of orders, which must be checked in the exhaustive or straightforward approach, is reduced to $(M - M^*)!$. It is easy to see that also the computational complexity of the algorithm of [3] and [4] is similarly reduced.

Since each vertex is a fully connected subset of $X$, the theorem also implies that at least $M$ minimal dimension orders, each terminating with a different node, exist.

Another interesting implication of Theorem 1 is the following.

Consider a problem

$$\min_X \sum_{i \in I} f_i(X_i)$$

which must be solved repeatedly in correspondence of the changes of one component. Let this component be $f_1(X^1)$. The set $X^1$ is not supposed to change.

It is easy to conceive many cases for which this can happen. It is clear that each different $f_1$ may yield a totally different solution to the optimization problem, namely all the $x_i$ - $s$ (not only those belonging to $X^1$) may assume a different value.

It is also clear that, if an order of elimination is chosen, in which the variables belonging to $X^1$ have the last $|X^1|$ places, a part of the "forward" dynamic programming procedure, namely that part consisting in the elimination of the variables of $X - X^1$, must not be repeated each time.

The subset $X^1$ is a fully connected subset of $G$; hence there exists a minimal dimension order in which the variables of $X^1$ have the last $|X^1|$ places.

This means that there exists an order which has minimal dimension and is the best from the point of view of the repetitive solution of the problem in correspondence of changes in $f_1$.

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