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# Solving Laplace's Equation in a Rectangle by Alternating Direction Implicit Methods

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## 1. INTRODUCTION

In 1967 Guittet [1] solved Laplace's equation numerically by using generalized (extrapolated) alternating direction implicit (ADI) methods. The problem was completely solved in the $p$-dimensional supercube $R$ where a uniform mesh of the same size $h$ in each coordinate direction was imposed on $R$, and Laplace's equation was approximated by $\alpha 2p + 1$–point difference formula. Moreover, explicit forms of the optimum values of the different parameters involved were given in the cases $p = 2$, 3, and 4.

In this paper we generalize the basic idea by Guittet in the 2-dimensional case by considering (i) that the region under consideration $R$ is a rectangle, instead of being a square, where the mesh sizes in the two coordinate directions are different in general, and (ii) that two different types of difference formulas are used to approximate Laplace's equation. It is effectively shown that the optimum results which are obtained in the general case considered here are quite different from those obtained by Guittet.

## 2. OPTIMUM EXTRAPOLATED ADI SCHEMES

We start with Laplace's equation (1), which is considered over the rectangle

$$R \equiv \{(x, y) \mid 0 < x < 1_1, 0 < y < 1_2\}, \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \tag{1}$$

where $u \equiv u(x, y)$ is prescribed on the boundary $\partial R$ of $R$ and impose a uniform mesh of mesh sizes $h_1$ and $h_2$ in the $x$- and $y$-directions, respectively,

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on $R \cup \partial R$. Thus, the number of mesh subdivisions in each coordinate direction is defined by

$$N_i = \frac{1_i}{h_i} \geqslant 3 \mid i = 1, 2.$$

At each mesh point

$$(i, j) \mid 1 \leqslant i \leqslant N_1 - 1, \qquad 1 \leqslant j \leqslant N_2 - 1,$$

Laplace's equation is approximated by the following difference formula

$$(\delta_x^2 + \sigma\delta_y^2 + \theta(1 + \sigma)\, \delta_x^2\delta_y^2)\, u_{ij} = 0, \tag{2}$$

where $\delta_x$ and $\delta_y$ are the central difference operators in the $x$- and $y$-directions, respectively, $\sigma$ is the ratio $h_1^2/h_2^2$, $u_{ij}$ is the approximate solution of (1) at the node $(i, j)$ and $\theta$ takes the values 0 and 1/12. $\theta = 0$ gives the well-known 5-point difference formula, while $\theta = 1/12$ gives the more accurate 9-point formula. Difference equation (2) yields the following extrapolated ADI scheme.

$$\begin{aligned} (1 - r\delta_x^2)\, u_{ij}^{(m+\frac{1}{2})} &= [(1 - r\delta_x^2) + \omega r(\delta_x^2 + \sigma\delta_y^2 + \theta(1 + \sigma)\, \delta_x^2\delta_y^2)]\, u_{ij}^{(m)}, \\ (1 - r\sigma\delta_y^2)\, u_{ij}^{(m+1)} &= u_{ij}^{(m+\frac{1}{2})} - r\sigma\delta_y^2 u_{ij}^{(m)}, \end{aligned} \tag{3}$$

where $r$ is a positive acceleration parameter, $\omega$ the extrapolation parameter, $u_{ij}^{(m)}$ the $m$th iteration approximation to $u_{ij}$ ($u_{ij}^{(0)}$ arbitrary), and $u_{ij}^{([m+1]/2)}$ can be regarded as an intermediate approximation to $u_{ij}^{(m+1)}$. Eliminating $u_{ij}^{([m+1]/2)}$ from (3), the following iterative scheme is produced.

$$\begin{aligned} &(1 - r\delta_x^2)\,(1 - r\sigma\delta_y^2)\, u_{ij}^{(m+1)} \\ &\quad = [(1 - r\delta_x^2)\,(1 - r\sigma\delta_y^2) + \omega r(\delta_x^2 + \sigma\delta_y^2 + \theta(1 + \sigma)\, \delta_x^2\delta_y^2)]\, u_{ij}^{(m)}, \end{aligned} \tag{4}$$

where, as can readily be seen, its amplification factor is given by

$$\rho(r, \omega) = 1 - \omega f, \tag{5}$$

with

$$f \equiv f(r, a_1, a_2) = \frac{r(a_1 + \sigma a_2 - \theta(1 + \sigma)\, a_1 a_2)}{(1 + ra_1)\,(1 + r\sigma a_2)},$$

and

$$a_i = 4 \sin^2 \frac{K_i \pi}{2N_i}, \qquad K_i = 1, 2, \ldots, N_i - 1, \quad \text{and} \quad i = 1, 2.$$

Starting with the relationships

$$0 < (1 + \sigma)\, a_1 a_2 = a_1 a_2 + \sigma a_1 a_2 < 4a_1 + 4\sigma a_2 = 4(a_1 + \sigma a_2),$$

which are always valid, we can easily obtain that

$$K\phi \leqslant f \leqslant \phi, \tag{6}$$

where

$$\phi \equiv \phi(r, a_1, a_2) = \frac{r(a_1 + \sigma a_2)}{(1 + ra_1)(1 + r\sigma a_2)}, \tag{7}$$

and $K = 1$ or $2/3$ corresponding to $\theta = 0$ and $1/12$. Then let $f_m$ and $f_M$ denote the minimum and the maximum values of the function $f$ defined previously in terms of $r$ when $a_1$ and $a_2$ vary so that

$$4 \sin^2 \frac{\pi}{2N_i} \leqslant a_i \leqslant 4 \cos^2 \frac{\pi}{2N_i} \mid i = 1, 2.$$

It is known (see [1, 2, and 3]) that optimum results, indicated from now on with a subscript opt, are obtained for that $r = r_{\text{opt}}$ for which the ratio $f_m/f_M$ or equivalently because of (6) the ratio $\phi_m/\phi_M$ is a maximum where $\phi_m$ and $\phi_M$ have obvious meanings. Therefore, if $r_{\text{opt}}$ can be determined, optimum values for the other parameters involved can be obtained by means of the relationships

$$\omega_{\text{opt}} = \frac{2}{\phi_{M\text{opt}} + K\phi_{m\text{opt}}}, \qquad \rho_{\text{opt}} = \frac{\phi_{M\text{opt}} - K\phi_{m\text{opt}}}{\phi_{M\text{opt}} + K\phi_{m\text{opt}}}, \tag{8}$$

where $\phi_{m\text{opt}}$, $\phi_{M\text{opt}}$, $\omega_{\text{opt}}$, and $\rho_{\text{opt}}$ stand for the optimum values of $\phi_m$, $\phi_M$, $\omega$, and $\rho(r, \omega)$, respectively, and $K = 1$ or $2/3$ depending on whether a 5-point or a 9-point difference formula is used. By putting

$$\begin{aligned} \mu_1 &\equiv 4 \sin^2 \frac{\pi}{2N_1} \leqslant a \equiv a_1 \leqslant 4 \cos^2 \frac{\pi}{2N_1} \equiv v_1, \\ \mu_2 &\equiv 4\sigma \sin^2 \frac{\pi}{2N_2} = b \equiv \sigma a_2 \leqslant 4\sigma \cos^2 \frac{\pi}{2N_2} \equiv v_2, \end{aligned} \tag{9}$$

we can define a new function $\psi$ as follows

$$\psi \equiv \psi(r, a, b) = \phi(r, a_1, a_2), \tag{10}$$

and therefore, the problem of determining $r_{\text{opt}}$ reduces to that of finding the extreme values $\psi_m(\equiv \phi_m)$ and $\psi_M(\equiv \phi_M)$ of the function $\psi$ in terms of $r$ and then maximizing the ratio $\Psi(r) \equiv \psi_m/\psi_M$ with respect to $r$. By forming the expressions for $\partial\psi/\partial a$ and $\partial\psi/\partial b$ it can be readily seen that neither of them changes sign as $a$ varies in $[\mu_1, v_1]$ and $b$ varies in $[\mu_2, v_2]$, respectively. This simply means that the extreme values of $\psi$ take place at

$$(a, b) \in \{(\mu_1, \mu_2), (\mu_1, v_2), (v_1, \mu_2), (v_1, v_2)\}.$$

Therefore, if we consider the expressions

$$A = \frac{r(\mu_1 + \mu_2)}{(1 + r\mu_1)(1 + r\mu_2)}, \qquad B = \frac{r(\mu_1 + v_2)}{(1 + r\mu_1)(1 + rv_2)},$$
$$C = \frac{r(v_1 + \mu_2)}{(1 + rv_1)(1 + r\mu_2)}, \quad \text{and} \quad D = \frac{r(v_1 + v_2)}{(1 + rv_1)(1 + rv_2)}, \tag{11}$$

then for a fixed $r$ we shall have that

$$\psi_m = \min(A, B, C, D) \quad \text{and} \quad \psi_M = \max(A, B, C, D). \tag{12}$$

Using (11) and the function sign $(x)$ which is defined as follows

$$\text{sign}(x) = \begin{cases} = +1 & \text{if} \quad x > 0 \\ = \ \ 0 & \text{if} \quad x = 0 \\ = -1 & \text{if} \quad x < 0 \end{cases}$$

it can easily be obtained that

$$\begin{aligned} \text{sign}(A - B) &= \text{sign}(r\mu_1 - 1), \\ \text{sign}(A - C) &= \text{sign}(r\mu_2 - 1), \\ \text{sign}(B - D) &= \text{sign}(rv_2 - 1), \\ \text{sign}(C - D) &= \text{sign}(rv_1 - 1). \end{aligned} \tag{13}$$

The order of $A$, $B$, $C$, and $D$, which will allow us to determine $\psi_m$ and $\psi_M$ through (12), mainly depends on the order of $r$, $1/\mu_1$, $1/v_1$, $1/\mu_2$, and $1/v_2$. We then distinguish the following three basic cases which will be examined separately.

*Case* I. $\mu_1 < \mu_2 < v_2 < v_1$,

*Case* II. $\mu_1 < \mu_2 < v_1 < v_2$,

*Case* III. $\mu_1 < v_1 < \mu_2 < v_2$.

Here it should be pointed out that besides the three basic cases just mentioned there exist three more cases which can be obtained by interchanging the indices 1 and 2 in the relationships above. The additional cases, however, can be treated in exactly the same way, so in the subsequent analysis we are not dealing with them at all. Moreover, it should be stressed that each of the above basic cases can be generalized in such a way that some strict inequalities can be replaced by nonstrict ones (e.g. a generalization of Case I may well be $\mu_1 \leqslant \mu_2 < v_2 \leqslant v_1$). To avoid unnecessary complications, we do not consider generalized cases in this sense. We simply note that such a generalized case can be regarded as a limiting case of one of the basic cases considered and therefore the corresponding results can be obtained as the limits of the results of the basic case.

In each basic case, the analysis follows the same general pattern. First, the whole interval $(0, \infty)$ is split into five subintervals which are defined by using the order of $\mu_1$, $v_1$, $\mu_2$, and $v_2$. Then by using (13), $\psi_m$ and $\psi_M$ are determined for $r$ taking values in each of the five subintervals. Second, the ratio $\Psi \equiv \Psi(r) = \psi_m/\psi_M$ is formed, the function sign $\partial\Psi/\partial r$ is studied in each of the subintervals and a table indicating the increasing or decreasing character of $\Psi$ in specified subintervals is constructed. Third, by studying the behavior of the function $\Psi$ in the whole interval $(0, \infty)$, $r_{\text{opt}}$ can be determined.

*Case* I. $\mu_1 < \mu_2 < v_2 < v_1$

The interval $(0, \infty)$ is split into the following subintervals: $(0, 1/v_1]$, $[1/v_1, 1/v_2]$, $[1/v_2, 1/\mu_2]$, $[1/\mu_2, 1/\mu_1]$, and $[1/\mu_1, \infty)$. Five subcases are then considered.

*Subcase* $\text{I}_\text{a}$. $0 < r \leqslant 1/v_1$. Using (13), it can be found that

$$A < \min(B, C) < \max(B, C) \leqslant D.$$

Therefore, $\psi_m = A$ and $\psi_M = D$ so that $\Psi = A/D$. This gives that $\partial\Psi/\partial r > 0$ which implies that $\Psi$ is increasing in $(0, 1/v_1]$.

*Subcase* $\text{I}_\text{b}$. $1/v_1 \leqslant r \leqslant 1/v_2$. This time we have $A < B \leqslant D \leqslant C$. Therefore, $\psi_m = A$, $\psi_M = C$, and $\Psi = A/C$. Again $\partial\Psi/\partial r > 0$, implying that $\Psi$ is increasing in $[1/v_1, 1/v_2]$.

*Subcase* $\text{I}_\text{c}$. $1/v_2 \leqslant r \leqslant 1/\mu_2$. It is readily obtained from (13) that $A, D < B, C$. Therefore, to decide which one of $A$ and $D$ is the smallest, as well as which one of $B$ and $C$ is the largest, we form and study the following two functions.

$$\begin{aligned} &\operatorname{sign}(A - D) \\ &\quad = \operatorname{sign}\{[(v_1 - \mu_1)\,\mu_2 v_2 + (v_2 - \mu_2)\,\mu_1 v_1]\,r^2 - [(v_1 - \mu_1) + (v_2 - \mu_2)]\}, \end{aligned} \tag{14}$$

$$\begin{aligned} &\operatorname{sign}(B - C) \\ &\quad = \operatorname{sign}\{[(v_1 - \mu_1)\,\mu_2 v_2 - (v_2 - \mu_2)\,\mu_1 v_1]\,r^2 - [(v_1 - \mu_1) - (v_2 - \mu_2)]\}. \end{aligned} \tag{15}$$

Because of the inequalities

$$(v_1 - \mu_1)\,\mu_2 v_2 + (v_2 - \mu_2)\,\mu_1 v_1 > 0 \qquad \text{and} \qquad (v_1 - \mu_1) + (v_2 - \mu_2) > 0$$

which are valid in this present case, the quadratic on the r.h.s. of (14) has two real roots $r_1$ and $r_2$ satisfying $r_1 < 0 < r_2$. It can be verified that $r_2 = r_{AD}$ where

$$r_{AD} = \left(\frac{(v_1 - \mu_1) + (v_2 - \mu_2)}{(v_1 - \mu_1)\,\mu_2 v_2 + (v_2 - \mu_2)\,\mu_1 v_1}\right)^{1/2} \tag{16}$$

belongs to the interval $[1/v_2, 1/\mu_2]$. Therefore, by using (14), it can be found that for $r \in [1/v_2, r_{AD}]$, $A \leqslant D$, implying that $\psi_m = A$, while for $r \in [r_{AD}, 1/\mu_2]$, $D \leqslant A$ implying that $\psi_m = D$. Now using the relationship $v_1\mu_2 > v_2\mu_1$ which holds in this case, we obtain

$$\frac{v_1 - \mu_1}{v_2 - \mu_2} > \frac{v_1}{v_2}\left(> \frac{\mu_1 v_1}{\mu_2 v_2}\right),$$

which in turn gives that $(v_1 - \mu_1)\,\mu_2 v_2 - (v_2 - \mu_2)\,\mu_1 v_1 > 0$. On the other hand we have that $(v_1 - \mu_1) - (v_2 - \mu_2) > 0$. Therefore, the quadratic on the r.h.s. of (15) has two real roots $r_1$ and $r_2$ such that $r_1 < 0 < r_2$. If we put $r_2 = r_{BC}$ where

$$r_{BC} = \left(\frac{(v_1 - \mu_1) - (v_2 - \mu_2)}{(v_1 - \mu_1)\,\mu_2 v_2 - (v_2 - \mu_2)\,\mu_1 v_1}\right)^{1/2}, \tag{17}$$

it can be verified that $r_{BC} \in [1/v_2, 1/\mu_2]$. Therefore, for $r \in [1/v_2, r_{BC}]$ $B \leqslant C$ giving that $\psi_M = C$, while for $r \in [r_{BC}, 1/\mu_2]$ $C \leqslant B$ giving that $\psi_M = B$.

It can be proved that

$$\operatorname{sign}(r_{AD} - r_{BC}) = \operatorname{sign}(\mu_2 v_2 - \mu_1 v_1).$$

Therefore, two cases must be distinguished according to whether $\mu_1 v_1 \leqslant \mu_2 v_2$ or not.

*Subcase* $\mathrm{I}_{\mathrm{c}_1}$. $\mu_1 v_1 \leqslant \mu_2 v_2$. The interval $[1/v_2, 1/\mu_2]$ is split into the subintervals $[1/v, r_{BC}]$, $[r_{BC}, r_{AD}]$, and $[r_{AD}, 1/\mu_2]$. In each subinterval $\psi_m$ and $\psi_M$ can readily be determined from the analysis above, and the corresponding results are obtained straightforward (see Table I).

*Subcase* $\mathrm{I}_{\mathrm{c}_2}$. $\mu_2 v_2 \leqslant \mu_1 v_1$. This time the three subintervals are $[1/v_2, r_{AD}]$, $[r_{AD}, r_{BC}]$, and $[r_{BC}, 1/\mu_2]$, and the analysis is exactly the same as in the previous subcase $\mathrm{I}_{\mathrm{c}_1}$. The corresponding results are given in Table I.

*Subcase* $\mathrm{I}_{\mathrm{d}}$. $1/\mu_2 \leqslant r \leqslant 1/\mu_1$. It can be seen from (13) that $D < C \leqslant A \leqslant B$. Therefore, $\psi_m = D$, $\psi_M = B$, which give that $\partial\Psi/\partial r < 0$. Thus, $\Psi$ is decreasing in $[1/\mu_2, 1/\mu_1]$.

TABLE I

Case I: $\mu_1 < \mu_2 < v_2 < v_1$

| Subcase | | $r$ | $\psi_m$ | $\psi_M$ | $\partial\Psi/\partial r$ | $\Psi$ |
|---|---|---|---|---|---|---|
| $I_a$ | | $0 < r \leqslant 1/v_1$ | $A$ | $D$ | $>0$ | increasing |
| $I_b$ | | $1/v_1 \leqslant r \leqslant 1/v_2$ | ,, | $C$ | ,, | ,, |
| $I_c$ | $I_{c_1}: \mu_1 v_1 \leqslant \mu_2 v_2$ | $1/v_2 \leqslant r \leqslant r_{BC}$ [a] | ,, | ,, | ,, | ,, |
| | | $r_{BC} \leqslant r \leqslant r_{AD}$ [b] | ,, | $B$ | ,, | ,, |
| | | $r_{AD} \leqslant r \leqslant 1/\mu_2$ | $D$ | ,, | $<0$ | decreasing |
| | $I_{c_2}: \mu_1 v_1 \geqslant \mu_2 v_2$ | $1/v_2 \leqslant r \leqslant r_{AD}$ | $A$ | $C$ | $>0$ | increasing |
| | | $r_{AD} \leqslant r \leqslant r_{BC}$ | $D$ | ,, | $<0$ | decreasing |
| | | $r_{BC} \leqslant r \leqslant 1/\mu_2$ | ,, | $B$ | ,, | ,, |
| $I_d$ | | $1/\mu_2 \leqslant r \leqslant 1/\mu_1$ | ,, | ,, | ,, | ,, |
| $I_e$ | | $1/\mu_1 \leqslant r < \infty$ | ,, | $A$ | ,, | ,, |

[a] $r_{BC}$ is given by (17).
[b] $r_{AD}$ is given by (16).

*Subcase* $I_e$. $1/\mu_1 \leqslant r < \infty$. This time we have that

$$D < \min(B, C) < \max(B, C) \leqslant A,$$

which imply that $\psi_m = D$ and $\psi_M = A$, and from these we can find out that $\Psi$ is decreasing in $[1/\mu_1, \infty)$.

Having studied the five basic subcases, we have constructed Table I which gives a summary of the results obtained so far.

*Case* II. $\mu_1 < \mu_2 < v_1 < v_2$

By splitting the interval $(0, \infty)$ into subintervals $(0, 1/v_2]$, $[1/v_2, 1/v_1]$, $[1/v_1, 1/\mu_2]$, $[1/\mu_2, 1/\mu_1]$, and $[1/\mu_1, \infty)$, five subcases are distinguished. The four subcases $II_a$, $II_b$, $II_d$, and $II_e$, which arise when $r$ takes values from the first two and the last two subintervals defined above, are easily studied as in the previous Case I; the results obtained are given in Table II. The analysis in subcase $II_c$, however, is different from the corresponding one made in subcase $I_c$, mainly because the inequalities $v_1\mu_2 > v_2\mu_1$ and $v_1 - \mu_1 > v_2 - \mu_2$ used there do not always hold. Therefore, in what follows only subcase $II_c$ is studied.

TABLE II

Case II: $\mu_1 < \mu_2 < v_1 < v_2$

| Subcase | $r$ | $\psi_m$ | $\psi_M$ | $\partial\Psi/\partial r$ | $\Psi$ |
|---|---|---|---|---|---|
| $\mathrm{II_a}$ | $0 < r \leqslant 1/v_2$ | $A$ | $D$ | $>0$ | increasing |
| $\mathrm{II_b}$ | $1/v_2 \leqslant r \leqslant 1/v_1$ | ,, | $B$ | ,, | ,, |
| $\mathrm{II_c}$ | $1/v_1 \leqslant r \leqslant r_{AD}$[a] | ,, | ,, | ,, | ,, |
| | $r_{AD} \leqslant r \leqslant 1/\mu_2$ | $D$ | ,, | $<0$ | decreasing |
| $\mathrm{II_d}$ | $1/\mu_2 \leqslant r \leqslant 1/\mu_1$ | ,, | ,, | ,, | ,, |
| $\mathrm{II_e}$ | $1/\mu_1 \leqslant r < \infty$ | ,, | $A$ | ,, | ,, |

[a] $r_{AD}$ is given by (16).

*Subcase* $\mathrm{II_c}$. $1/v_1 \leqslant r \leqslant 1/\mu_2$. It can readily be obtained, from (13), that $A, D < B, C$. To find out which one of $A$ and $D$ is the smallest, we consider again the function sign$(A - D)$ given by (14). The results obtained are the same as in subcase $\mathrm{I_c}$, the only exception being that the lower bound $1/v_2$ considered there is replaced by $1/v_1$. To find the largest of $B$ and $C$, the function sign$(B - C)$ given by (15) is formed. We put

$$g(r) \equiv \alpha r^2 + j \tag{18}$$

where

$$\alpha = (v_1 - \mu_1)\,\mu_2 v_2 - (v_2 - \mu_2)\,\mu_1 v_1,$$

and

$$j = -[(v_1 - \mu_1) - (v_2 - \mu_2)] \tag{19}$$

and distinguish three subcases.

*Subcase* $\mathrm{II_{c_1}}$. $[(v_1 - \mu_1)/(v_2 - \mu_2)] \geqslant 1 > \mu_1 v_1/\mu_2 v_2$. In this case $\alpha > 0$ and $j \leqslant 0$, which implies that $g(r)$ has two real roots $r_1$ and $r_2$ such that $r_1 \leqslant 0 \leqslant r_2$. It can easily be found out that

$$\operatorname{sign}\left(g(0)\cdot g\left(\frac{1}{v_1}\right)\right) = \operatorname{sign}(j(v_2 - v_1)) = -1 \quad \text{or} \quad 0.$$

Therefore, $0 \leqslant r_2 < 1/v_2$. This implies that sign$(g(r)) = $ sign$(\alpha) = +1$ for all $r \in [1/v_1, 1/\mu_2]$ and by virtue of (15), (18), and (19) we obtain that $B > C$.

*Subcase* $\mathrm{II}_{c_2}$. $1 > [(v_1 - \mu_1)/(v_2 - \mu_2)] \geqslant \mu_1 v_1/\mu_2 v_2$. This time we have that $\alpha \geqslant 0$ and $j > 0$, implying that $g(r) > 0$ for all $r \in [1/v_1, 1/\mu_2]$ and therefore $B > C$.

*Subcase* $\mathrm{II}_{c_3}$. $1 > \mu_1 v_1/\mu_2 v_2 > [(v_1 - \mu_1)/(v_2 - \mu_2)]$. We have now that $\alpha < 0$ and $j > 0$, so the two roots $r_1$ and $r_2$ of $g(r)$ are such that $r_1 < 0 < r_2$. Because of

$$\operatorname{sign}\left(g\left(\frac{1}{\mu_2}\right) \cdot g(\infty)\right) = \operatorname{sign}(\alpha(\mu_2 - \mu_1)) = -1,$$

we obtain that $1/\mu_2 < r_2$, and consequently

$$\operatorname{sign}(B - C) = \operatorname{sign}(g(r)) = -\operatorname{sign}(\alpha) = +1$$

for all $r \in [1/v_1, 1/\mu_2]$ implying that $B > C$. As has been seen in all three subcases examined above, we always have $B > C$, namely $\psi_M = B$. Forming now the expression $\partial\Psi/\partial r$ for $r \in [1/v_1, r_{AD}]$ and $r \in [r_{AD}, 1/\mu_2]$, we can easily find out how the function $\Psi$ behaves. The corresponding results are given in Table II. Table II has been constructed in the same way as Table I and gives a summary of the results which hold in this present Case II.

*Case* III. $\mu_1 < v_1 < \mu_2 < v_2$

The five subintervals into which the interval $(0, \infty)$ is split are: $(0, 1/v_2]$, $[1/v_2, 1/\mu_2]$, $[1/\mu_2, 1/v_1]$, $[1/v_1, 1/\mu_1]$, and $[1/\mu_1, \infty)$. Subcases $\mathrm{III_a}$, $\mathrm{III_b}$, $\mathrm{III_d}$, and $\mathrm{III_e}$, arising when $r$ lies in the first two and the last two subintervals above, are easily studied as before. The corresponding results are presented in Table III. Subcase $\mathrm{III_c}$, however, presents a certain amount of difficulty, and this is therefore the case which is studied in detail in what follows.

*Subcase* $\mathrm{III_c}$. $1/\mu_2 \leqslant r \leqslant 1/v_1$. In view of (13) we obtain that

$$C \leqslant \min(A, D) < \max(A, D) < B.$$

These inequalities imply that $\psi_m = C$, $\psi_M = B$ and so $\Psi = C/B$. Therefore,

$$\operatorname{sign}\left(\frac{\partial\Psi}{\partial r}\right) = \operatorname{sign}\{g(r)\}, \tag{20}$$

where

$$g(r) \equiv \alpha r^2 + 2\beta r + j, \tag{21}$$

and

$$\alpha = (v_2 - \mu_2)\,\mu_1 v_1 - (v_1 - \mu_1)\,\mu_2 v_2, \qquad \beta = \mu_1 v_2 - \mu_2 v_1,$$

TABLE III

Case III: $\mu_1 < v_1 < \mu_2 < v_2$

| Subcase | | | | $r$ | $\psi_m$ | $\psi_M$ | $\partial\Psi/\partial r$ | $\Psi$ |
|---|---|---|---|---|---|---|---|---|
| III$_a$ | | | | $0 < r \leqslant 1/v_2$ | $A$ | $D$ | $>0$ | increasing |
| III$_b$ | | | | $1/v_2 \leqslant r \leqslant 1/\mu_2$ | ” | $B$ | ” | ” |
| III$_c$ | III$_{c_1}$ : $\mu_1 v_2 = \mu_2 v_1$ | | | $1/\mu_2 \leqslant r \leqslant r_2$ [b] | $C$ | ” | ” | ” |
| | | | | $r_2 \leqslant r \leqslant 1/v_1$ | ” | ” | $<0$ | decreasing |
| | III$_{c_2}$ : $\mu_1 v_2 < \mu_2 v_1$ | $(v_2 - \mu_2)/(v_1 - \mu_1) > 1$ | $g(1/\mu_2) \geqslant 0$ [a] | $1/\mu_2 \leqslant r \leqslant r_2$ | ” | ” | $>0$ | increasing |
| | | | | $r_2 \leqslant r \leqslant 1/v_1$ | ” | ” | $<0$ | decreasing |
| | | | $g(1/\mu_2) < 0$ | $1/\mu_2 \leqslant r \leqslant 1/v_1$ | ” | ” | ” | ” |
| | | $(v_2 - \mu_2)/(v_1 - \mu_1) \leqslant 1$ | | $1/\mu_2 \leqslant r \leqslant 1/v_1$ | ” | ” | ” | ” |
| | III$_{c_3}$ : $\mu_1 v_2 > \mu_2 v_1$ | $(v_2 - \mu_2)/(v_1 - \mu_1) \geqslant \mu_2 v_2/\mu_1 v_1$ | | $1/\mu_2 \leqslant r \leqslant 1/v_1$ | ” | ” | $>0$ | increasing |
| | | $(v_2 - \mu_2)/(v_1 - \mu_1) < \mu_2 v_2/\mu_1 v_1$ | $g(1/v_1) \leqslant 0$ | $1/\mu_2 \leqslant r \leqslant r_2$ | ” | ” | ” | ” |
| | | | | $r_2 \leqslant r \leqslant 1/v_1$ | ” | ” | $<0$ | decreasing |
| | | | $g(1/v_1) > 0$ | $1/\mu_2 \leqslant r \leqslant 1/v_1$ | ” | ” | $>0$ | increasing |
| III$_d$ | | | | $1/v_1 \leqslant r \leqslant 1/\mu_1$ | $D$ | ” | $<0$ | decreasing |
| III$_e$ | | | | $1/\mu_1 \leqslant r < \infty$ | ” | $A$ | ” | ” |

[a] $g(r)$ is given by (21).
[b] $r_2$ is the positive root of $g(r)$.

and

$$j = (v_2 - \mu_2) - (v_1 - \mu_1). \tag{22}$$

If we observe that

$$\operatorname{sign}(\Psi(1/\mu_2) - \Psi(1/v_1)) = -\operatorname{sign}(\beta), \tag{23}$$

then the subsequent analysis can be simplified by distinguishing three basic cases.

*Subcase* $\mathrm{III}_{c_1}$. $\beta = 0$. $\beta = 0$ implies that $\mu_1 v_2 = \mu_2 v_1$, which in turn gives that $\mu_2 v_2/\mu_1 v_1 > (v_2 - \mu_2)/(v_1 - \mu_1) = \mu_2/\mu_1 > 1$. Because of these relationships and in view of (22), we have that $\alpha < 0$ and $j > 0$. Hence, the two roots $r_1$ and $r_2$ of $g(r)$ are such that $r_1 < 0 < r_2$. By virtue of $\Psi(1/\mu_2) = \Psi(1/v_1)$, which comes from (23), it can be proved that

$$\frac{1}{\mu_2} < r_2 < \frac{1}{v_1}.$$

Therefore $\partial\Psi/\partial r > 0$ for $r \in [1/\mu_2, r_2]$ and $\Psi$ is increasing in $[1/\mu_2, r_2]$, while for $r \in [r_2, 1/v_1]$, $\partial\Psi/\partial r < 0$ and $\Psi$ is decreasing in $[r_2, 1/v_1]$.

*Subcase* $\mathrm{III}_{c_2}$. $\beta < 0$. In this case, we have that $\mu_1 v_2 < \mu_2 v_1$, which implies that $(v_2 - \mu_2)/(v_1 - \mu_1) < \mu_2/\mu_1 < \mu_2 v_2/\mu_1 v_1$, therefore $\alpha < 0$. If $(v_2 - \mu_2)/(v_1 - \mu_1) > 1$, then $j > 0$ which together with $\alpha < 0$ implies that the two roots $r_1$ and $r_2$ of $g(r)$ are such that $r_1 < 0 < r_2$. The root $r_2$ can not be greater than $1/v_1$, for if it were, then for all $r \in [1/\mu_2, 1/v_1]$, we would have $\operatorname{sign}(g(r)) = -\operatorname{sign}(\alpha) = +1$, which would imply that $\Psi$ is increasing in $[1/\mu_2, 1/v_1]$. Therefore, $\Psi(1/\mu_2) < \Psi(1/v_1)$. The latter contradicts (23). Consequently, we have that either $1/\mu_2 \leqslant r_2 \leqslant 1/v_1$, and therefore $\Psi$ is increasing in $[1/\mu_2, r_2]$ and decreasing in $[r_2, 1/v_1]$ or $r_2 \leqslant 1/\mu_2$, which implies that $\Psi$ is decreasing in $[1/\mu_2, 1/v_1]$. If, on the other hand $(v_2 - \mu_2)/(v_1 - \mu_1) \leqslant 1$, then $j \leqslant 0$ implying that $g(r) < 0$ for all $r \in [1/\mu_2, 1/v_1]$. Therefore, $\Psi$ is decreasing in this interval.

*Subcase* $\mathrm{III}_{c_3}$. $\beta > 0$. This time $\mu_1 v_2 > \mu_2 v_1$, giving that

$$\frac{v_2 - \mu_2}{v_1 - \mu_1} > \frac{\mu_2}{\mu_1} > 1,$$

which implies that $j > 0$. If

$$\frac{v_2 - \mu_2}{v_1 - \mu_1} \geqslant \frac{\mu_2 v_2}{\mu_1 v_1} \left(> \frac{\mu_2}{\mu_1}\right),$$

then $\alpha \geqslant 0$ and $g(r) > 0$ for all $r \in [1/\mu_2, 1/v_1]$. Therefore $\Psi$ is increasing in this interval. However, if $\mu_2 v_2/\mu_1 v_1 > (v_2 - \mu_2)/(v_1 - \mu_1)$, then $\alpha < 0$, and because of $j > 0$, $g(r)$ has two real roots satisfying $r_1 < 0 < r_2$. A reasoning similar to the one made previously in subcase $\text{III}_{c_2}$ leads to the conclusion that if $1/v_1 \leqslant r_2$, $\Psi$ is increasing in $[1/\mu_2, 1/v_1]$ while if $r_2 \in [1/\mu_2, 1/v_1]$, then $\Psi$ is increasing in $[1/\mu_2, r_2]$ and decreasing in $[r_2, 1/v_1]$. A summary of the results obtained in this present Case III is given in Table III.

Having constructed Tables I, II, and III, it is easy to follow the behavior of the function $\Psi$ as $r$ increases from zero to infinity, and therefore the value of $r = r_{\text{opt}}$ at which $\Psi$ attains its maximum value can readily be determined. To summarize the optimum results a further table, Table IV, has also been constructed. Table IV gives the values for $r_{\text{opt}}$ as well as $\psi_{m\text{opt}}$ and $\psi_{M\text{opt}}$ in each case. We note that in order to obtain the optimum values of the other two parameters, $\omega_{\text{opt}}$ and $\rho_{\text{opt}}$, in any of the three basic cases studied in this paper Table IV together with Eqs. (8) must be used.

TABLE IV

Optimum Parameters

| Case | | | $r_{\text{opt}}$ | $\psi_{m\text{opt}}$ | $\psi_{M\text{opt}}$ |
|---|---|---|---|---|---|
| I | $\text{I}_1 : \mu_1 v_1 \leqslant \mu_2 v_2$ | | $r_{AD}$ [b] | $A_{\text{opt}} = D_{\text{opt}}$ | $B_{\text{opt}}$ |
| | $\text{I}_2 : \mu_1 v_1 \geqslant \mu_2 v_2$ | | ” | ” | $C_{\text{opt}}$ |
| II | | | ” | ” | $B_{\text{opt}}$ |
| III | $\text{III}_1 : \mu_1 v_2 = \mu_2 v_1$ | | $r_2$ [c] | $C_{\text{opt}}$ | ” |
| | $\text{III}_2 : \mu_1 v_2 < \mu_2 v_1$ | $g(1/\mu_2) > 0$ [a] | $r_2$ | ” | ” |
| | | $g(1/\mu_2) \leqslant 0$ | $1/\mu_2$ | ” | ” |
| | $\text{III}_3 : \mu_1 v_2 > \mu_2 v_1$ | $g(1/v_1) \geqslant 0$ | $1/v_1$ | ” | ” |
| | | $g(1/v_1) < 0$ | $r_2$ | ” | ” |

[a] $g(r)$ is given by (21).
[b] $r_{AD}$ is given by (16).
[c] $r_2$ is the positive root of $g(r)$.

## 3. Final Remarks

The analysis made so far shows that the values for $r_{\text{opt}}$ and $\omega_{\text{opt}}$ $(=2)$ obtained by Guittet [1] do not hold in the general case. We note that if

$h_1 \neq h_2$ (i.e., $\sigma \neq 1$) the values of $r_{\text{opt}}$ and consequently of $\omega_{\text{opt}}$ depend mainly on the order of $\mu_1$, $v_1$, $\mu_2$, and $v_2$ (see Table IV). In the special case $h_1 = h_2$ (i.e., $\sigma = 1$), the formulas giving the optimum values of the various parameters involved can be simplified. For example if $N_1 = N_2$, we have that $\mu_1 = \mu_2$ and $v_1 = v_2$, so this case can be regarded as a limiting case of either Case I or II, and therefore for the 5-point formula we have

$$r_{\text{opt}} = \frac{1}{(\mu_1 v_1)^{1/2}}, \qquad \omega_{\text{opt}} = 2, \qquad \text{and} \qquad \rho_{\text{opt}} = \frac{B_{\text{opt}} - A_{\text{opt}}}{B_{\text{opt}} + A_{\text{opt}}}, \tag{24}$$

(these results were obtained by Guittet [1]) while for the 9-point formula we have

$$r_{\text{opt}} = \frac{1}{(\mu_1 v_1)^{1/2}}, \qquad \omega_{\text{opt}} = \frac{2}{B_{\text{opt}} + \frac{2}{3}A_{\text{opt}}},$$

and

$$\rho_{\text{opt}} = \frac{B_{\text{opt}} - \frac{2}{3}A_{\text{opt}}}{B_{\text{opt}} + \frac{2}{3}A_{\text{opt}}}, \tag{25}$$

(these results were obtained by Hadjidimos [3]). On the other hand, if $N_1 \neq N_2$ ($N_1 > N_2$), we have that $\mu_1 < \mu_2 < v_2 < v_1$, so this case is Case I. Therefore, $r_{\text{opt}} = r_{AD}$ and the optimum results here are quite different from those given by (24) and (25) above.

Perhaps the most interesting remark is the following. In the case of a 5-point formula the optimum extrapolation parameter $\omega_{\text{opt}}$ is such that $1 < \omega_{\text{opt}} \leqslant 2$ (equality holds if and only if $\mu_1 v_1 = \mu_2 v_2$) while in the case of a 9-point formula $\omega_{\text{opt}}(>1)$ can be either less than or greater than 2, as many numerical examples we have run on a computer have shown it.

Before we close this paper we state and prove a theorem concerning the values of $\omega_{\text{opt}}$.

THEOREM. *In the case of a 5-point difference formula,* $\omega_{\text{opt}}$ *satisfies* $1 < \omega_{\text{opt}} \leqslant 2$ *with equality on the right holding if and only if* $\mu_1 v_1 = \mu_2 v_2$.

*Proof.* We know that

$$\phi_{m\text{opt}} = \min\{A_{\text{opt}}, B_{\text{opt}}, C_{\text{opt}}, D_{\text{opt}}\},$$

$$\phi_{M\text{opt}} = \max\{A_{\text{opt}}, B_{\text{opt}}, C_{\text{opt}}, D_{\text{opt}}\}.$$

On the other hand, from (11) we have that $A, B, C, D < 1$ for any $r > 0$. Therefore, $\phi_{m\text{opt}} + \phi_{M\text{opt}} < 2$ and $\omega_{\text{opt}} = 2/(\phi_{m\text{opt}} + \phi_{M\text{opt}}) > 1$. To prove that $\omega_{\text{opt}} \leqslant 2$, we distinguish three cases.

*Case* 1. (*It consists of Cases* $\mathrm{I}_1$ *and* II)

Starting with $\mu_1 v_1 \leqslant \mu_2 v_2$, we obtain

$$\frac{1}{\mu_2 v_2} \leqslant \frac{(v_1 - \mu_1) + (v_2 - \mu_2)}{(v_1 - \mu_1)\,\mu_2 v_2 + (v_2 - \mu_2)\,\mu_1 v_1} = r_{\text{opt}}^2 .$$

This gives

$$\frac{(1 - r_{\text{opt}}\mu_2)}{(1 + r_{\text{opt}}\mu_2)} + \frac{(1 - r_{\text{opt}} v_2)}{(1 + r_{\text{opt}} v_2)} \leqslant 0$$

or

$$\frac{(1 - r_{\text{opt}}\mu_1)(1 - r_{\text{opt}}\mu_2)}{(1 + r_{\text{opt}}\mu_1)(1 + r_{\text{opt}}\mu_2)} + \frac{(1 - r_{\text{opt}}\mu_1)(1 - r_{\text{opt}} v_2)}{(1 + r_{\text{opt}}\mu_1)(1 + r_{\text{opt}} v_2)} \leqslant 0$$

or

$$(\tfrac{1}{2} - A_{\text{opt}}) + (\tfrac{1}{2} - B_{\text{opt}}) \leqslant 0 \qquad \text{or} \qquad \omega_{\text{opt}} = \frac{2}{A_{\text{opt}} + B_{\text{opt}}} \leqslant 2,$$

where the equality holds if and only if $\mu_1 v_1 = \mu_2 v_2$.

*Case* 2. (*It is identical with Case* $\mathrm{I}_2$)

The proof follows the same steps as in the previous Case 1. The only exceptions are that we start with $\mu_2 v_2 \leqslant \mu_1 v_1$ and that $C_{\text{opt}}$ is used instead of $B_{\text{opt}}$.

*Case* 3. (*It is identical with Case* III)

Because of $r_{\text{opt}} \in [1/\mu_2, 1/v_1]$, we have that

$$\frac{(1 - r_{\text{opt}}\mu_1)(1 - r_{\text{opt}} v_2)}{(1 + r_{\text{opt}}\mu_1)(1 + r_{\text{opt}} v_2)} + \frac{(1 - r_{\text{opt}}\mu_2)(1 - r_{\text{opt}} v_1)}{(1 + r_{\text{opt}}\mu_2)(1 + r_{\text{opt}} v_1)} < 0$$

or

$$(\tfrac{1}{2} - B_{\text{opt}}) + (\tfrac{1}{2} - C_{\text{opt}}) < 0 \qquad \text{or} \qquad \omega_{\text{opt}} = \frac{2}{B_{\text{opt}} + C_{\text{opt}}} < 2.$$

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## References

1. J. Guittet, Une nouvelle méthode de directions alternées à $q$ variable, *J. Math. Anal. Appl.* **17** (1967), 199–213.

2. A. Hadjidimos, Extrapolated alternating direction implicit methods for the numerical solution of elliptic partial differential equations, Ph.D. Dissertation, Liverpool University, Liverpool, England, 1968.
3. A. Hadjidimos, Extrapolated alternating direction implicit methods, *BIT*, **10** (1970), 465–475.