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Growth of heat trace and heat content asymptotic coefficients

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Abstract

We show in the smooth category that the heat trace asymptotics and the heat content asymptotics can be made to grow arbitrarily rapidly. In the real analytic context, however, this is not true and we establish universal bounds on their growth.

Keywords: Heat trace; Heat content; Asymptotics; Compact Riemannian manifold

1. Introduction

1.1. Heat trace asymptotics

Let (M, g) be a compact Riemannian manifold of dimension m with smooth (possibly empty) boundary ∂M. Let dvol_m and dvol_{m-1} be the Riemannian volume elements on M and on ∂M, respectively. Let Δ_g be the scalar Laplacian. Let ν be the inward unit normal on the boundary; we extend ν by parallel translation to a vector field defined on a collared neighborhood of the bound-

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ary so $\nabla_\nu \nu = 0$; this means that the integral curves of ν are unit speed geodesics perpendicular to ∂M. Let

$$\mathcal{B}^-\phi := \phi|_{\partial M} \quad \text{and} \quad \mathcal{B}^+\phi := \nu\phi|_{\partial M}$$

be the Dirichlet and Neumann boundary operators, respectively. Impose boundary conditions $\mathcal{B} = \mathcal{B}^-$ or $\mathcal{B} = \mathcal{B}^+$. Let $u : M \times (0, \infty) \to \mathbb{R}$ be the unique solution of

$$\begin{aligned}
&(\partial_t + \Delta_g)u(x,t) = 0 && \text{(heat equation)},\\
&\lim_{t\to 0} u(\cdot,t) = \phi_1(\cdot) && \text{in } L^2 \text{ (initial condition)},\\
&\mathcal{B}u(\cdot,t) = 0 && \text{for } t > 0 \text{ (boundary condition)},
\end{aligned}$$

where ϕ_1 is real-valued and smooth on M. Then $u(x,t)$ represents the temperature at $x \in M$ at time $t > 0$ if M has initial temperature distribution ϕ_1 where the boundary condition $\mathcal{B}$ is imposed on u for $t > 0$. The solution is formally given by

$$u(x,t) := e^{-t\Delta_{g,\mathcal{B}}}\phi_1(x),$$

where $\Delta_{g,\mathcal{B}}$ is the associated realization of the Laplacian. The operator $e^{-t\Delta_{g,\mathcal{B}}}$ is a smoothing operator of trace class and, as $t \downarrow 0$, there is a complete asymptotic series of the form [29,30,43–47]

$$\mathrm{Tr}_{L^2}\left\{e^{-t\Delta_{g,\mathcal{B}}}\right\} \sim (4\pi t)^{-m/2} \sum_{n=0}^{\infty} a_n(M,g,\mathcal{B})t^{n/2}.$$

If M is a closed manifold, the boundary condition plays no role and we shall denote these coefficients by $a_n(M,g)$. They vanish if n is odd in this instance.

The asymptotic coefficients $\{a_1, a_2, \ldots\}$ are locally computable invariants of M and of ∂M as we shall see presently in Section 2. In mathematical physics, they occur for example in the calculation of Casimir forces [5,18,33] or in the study of the partition function of quantum mechanical systems [7,6,33]. They are known in the category of manifolds with boundary for $n \leqslant 5$ [19,32], and in the category of closed manifolds for $n \leqslant 8$ [1,4]. These coefficients play a crucial role in the study of isospectral questions. Related invariants for more general operators of Laplace type also play a crucial role in the local index theorem. See, for example, the discussion and references in [2,3,20,21,23,26–28,37–39]. They have also been studied with nonlocal boundary conditions [34]. We also refer to [24] where the heat trace itself is studied and not just the asymptotic coefficients. For the study of the asymptotic behaviour of the eigenvalues of $\Delta_{g,\mathcal{B}}$ we refer to [41] and the references therein. The field is vast and it is only possible to cite a few references.

1.2. Planar domains

In the case of a planar domain Ω, the heat trace asymptotic coefficients (with Dirichlet boundary conditions) have been computed for $n \leqslant 13$ by Berry and Howls [17]. Berry and Howls computed a_n for $n \leqslant 31$ in the case of a disc [17], and were led to conjecture that for planar domains Ω and for $n \to \infty$,

$$a_n(\Omega) = \alpha \Gamma(n-\beta+1)\Gamma(n/2)^{-1}\ell(\Omega)^{2-n}\big(1+o(1)\big), \tag{1.a}$$

where α and β are dimensionless quantities and where $\ell(\Omega)$ is the length of the shortest accessible periodic geodesic in Ω. In particular, for a disk of radius R and shortest accessible periodic geodesic $4R$, they further conjectured that Eq. (1.a) holds with $\alpha = (8\sqrt{2\pi})^{-1}$ and $\beta = \frac{3}{2}$. While the latter conjecture remains open to date, it is instructive to see that Eq. (1.a) cannot hold in general. The following counter examples were given in [8].

Example 1.1. Let $0 < \varepsilon < \frac{1}{5}$, and let

$$\tilde{P}_\varepsilon = \big\{(x_1, x_2) \in \mathbb{R}^2\colon\ |x| \leqslant 1,\ |x_2| \leqslant 1-\varepsilon\big\},$$
$$\tilde{Q}_\varepsilon = \big\{(x_1, x_2) \in \mathbb{R}^2\colon\ |x| \leqslant 1,\ x_1 \leqslant 1-\varepsilon,\ x_2 \leqslant 1-\varepsilon\big\}.$$

We smooth out the corners of $\partial\tilde{P}_\varepsilon$ at $x_2 = \pm(1-\varepsilon)$ and of $\partial\tilde{Q}_\varepsilon$ at $x_1 = 1-\varepsilon$, $x_2 = 1-\varepsilon$ isometrically to obtain two convex domains P_ε and Q_ε with smooth boundary and with $a_n(P_\varepsilon) = a_n(Q_\varepsilon)$ and $\ell(P_\varepsilon) = 4(1-\varepsilon)$, $\ell(Q_\varepsilon) = 2(2-\varepsilon)$. This then contradicts Eq. (1.a).

Example 1.2. Let $0 < \varepsilon < 1$, $0 < \rho < 1-\varepsilon$, and let

$$\Omega_\varepsilon := \big\{(x_1, x_2) \in \mathbb{R}^2\colon\ \varepsilon \leqslant |x| \leqslant 1\big\},$$
$$\Omega_\varepsilon^\rho := \big\{(x_1, x_2) \in \mathbb{R}^2\colon\ |x| \leqslant 1,\ |(x_1-\rho, x_2)| \geqslant \varepsilon\big\}.$$

We then have that $a_n(\Omega_\varepsilon) = a_n(\Omega_\varepsilon^\rho)$ and $\ell(\Omega_\varepsilon) = 2(1-\varepsilon)$, $\ell(\Omega_\varepsilon^\rho) = 2(1-\varepsilon-\rho)$ which once again contradicts Eq. (1.a).

It remains an open problem to construct a pair of iso – a_n real analytic simply connected planar domains which have different shortest periodic geodesics. It has been conjectured that Eq. (1.a) also holds for balls in $\mathbb{R}^m$ where β depends on m only [31].

1.3. The heat trace asymptotics in the real analytic category

The calculus of Seeley [43–47] and Greiner [29,30] shows that a_n is given by a local formula; the following result will then follow from the analysis of Section 2:

Theorem 1.1. *Let $\mathcal{B}$ be either Dirichlet or Neumann boundary conditions. There exist universal constants $\kappa_{n,m}$ so that if (M, g) is any compact real analytic manifold of dimension m, then there exists a positive constant $C = C(M, g)$ such that*

$$\big|a_n(M, g, \mathcal{B})\big| \leqslant \kappa_{n,m} C^n \cdot \mathrm{vol}_m(M, g) \quad \text{for any } n.$$

We note some similarity between the formulae of Eq. (1.a) and Theorem 1.1. The geometric data of (M, g) appear in C^n, whereas the prefactor is of a combinatorial nature and depends on m and n only. We can choose the constant to rescale appropriately under homotheties, i.e. so that $C(M, c^2 g) = c^{-1} C(M, g)$.

We restrict momentarily to the context of closed manifolds, i.e. compact manifolds with empty boundary. We adopt the Einstein convention and sum over repeated indices. We say that D is an *operator of Laplace type*, if in any local system of coordinates we may express D in the form:

$$D = -\left(g^{ij}\partial_{x_i}\partial_{x_j} + A^k\partial_{x_k} + B\right). \tag{1.b}$$

Let $a_n(x, D)$ be the local heat trace invariant of such an operator. We shall primarily interested in the case n even so we shall set $n = 2\bar{n}$ in what follows. If f is any smooth function on M, then

$$\mathrm{Tr}_{L^2}\left(fe^{-tD}\right) \sim (4\pi t)^{-m/2}\sum_{\bar{n}=0}^{\infty} t^{\bar{n}} \int_M a_{2\bar{n}}(x, D) f(x)\,\mathrm{dvol}_m . \tag{1.c}$$

The following result shows that the factorial growth conjectured by Berry and Howls for planar domains pertains in this setting as well as regards the local heat trace invariants on closed manifolds.

Theorem 1.2. *Let (M, g) be a closed real analytic Riemannian manifold of dimension $m \geqslant 2$.*

(1) *Let D be a scalar real analytic operator of Laplace type on M. Then there exists a constant $C_1 = C_1(M, g, D)$ so that*

$$\left|a_{2\bar{n}}(x, D)\right| \leqslant C_1^{\bar{n}} \cdot \bar{n}! \quad \textit{for any } \bar{n} \geqslant 1.$$

(2) *Let P be a point of M. Suppose there exists a real analytic function f on M such that $df(P) \neq 0$. Then there exists a constant $C_2 = C_2(P, M, g, f) > 0$ and there exists a real analytic function h on M so that the conformally equivalent metric $g_h := e^{2h}g$ satisfies*

$$\left|a_{2\bar{n}}(P, \Delta_{g_h})\right| \geqslant C_2^{\bar{n}} \cdot \bar{n}! \quad \textit{for any } \bar{n} \geqslant 3.$$

Remark 1.1. Assertion (1) can be integrated to yield an upper bound on the heat trace asymptotics $a_{2\bar{n}}(D)$. However, assertion (2) is only valid at a single point of M. Since it in fact arises from considering a divergence term in the local expansion, we do not obtain a corresponding estimate for $a_{2\bar{n}}(D)$.

1.4. The heat trace asymptotics in the smooth category

The situation in the smooth non-real analytic setting is very different. Fix a background reference Riemannian metric h and let ∇^h be the associated Levi–Civita connection which we use to covariantly differentiate tensors of all types. If T is a tensor field on M, we define the C^k norm of T by setting:

$$\|T\|_k := \max_{P\in M}\left\{\sum_{i=0}^{k}\left|\nabla^{h,i}T\right|(P)\right\}.$$

Changing h replaces $\|T\|_k$ by an equivalent norm; we therefore suppress the dependence upon h. But as we will be changing the metric when considering the heat trace asymptotics subsequently, it is useful to have fixed h once and for all so the associated C^k norms do not change. Theorem 1.1 fails in the smooth context as we have:

Theorem 1.3. *Let $k \geqslant 3$ be given, let constants $C_{\bar{n}} > 0$ for $\bar{n} \geqslant k$ be given, and let $\epsilon > 0$ be given. Let (M, g) be a smooth compact Riemannian manifold of dimension $m \geqslant 2$ without boundary and let g_e be the usual Euclidean metric on $\mathbb{R}^{m+1}$.*

(1) *There exists a function $f \in C^\infty(M)$ with $\|f\|_{k-1} < \epsilon$ so that if $g_1 := e^{2f} g$ is the conformally related metric, then*

$$\left|a_{2\bar{n}}(M, g_1)\right| \geqslant C_{\bar{n}} \quad \text{for any } \bar{n} \geqslant k.$$

(2) *Suppose that $g = \Theta^* g_e$ where Θ is an immersion of M into $\mathbb{R}^{m+1}$. There exists an immersion Θ_1 with $\|\Theta - \Theta_1\|_{k-1} < \epsilon$ so that if $g_1 := \Theta_1^* g_e$, then*

$$\left|a_{2\bar{n}}(M, g_1)\right| \geqslant C_{\bar{n}} \quad \text{for any } \bar{n} \geqslant k.$$

1.5. Heat content asymptotics

There are analogous results for the heat content asymptotics. Let ϕ_1 be the initial temperature of the manifold and let ϕ_2 be the specific heat of the manifold. We suppose throughout that ϕ_1 and ϕ_2 are smooth. The total heat energy content of the manifold is then given by:

$$\beta(\phi_1, \phi_2, \Delta_g, \mathcal{B})(t) := \int_M u(x,t)\phi_2(x)\,\mathrm{dvol}_m .$$

As $t \downarrow 0$, there is a complete asymptotic expansion of the form

$$\beta(\phi_1, \phi_2, \Delta_g, \mathcal{B})(t) \sim \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \int_M \Delta_g^n \phi_1 \cdot \phi_2 \,\mathrm{dvol}_m + \sum_{\ell=0}^{\infty} t^{(\ell+1)/2} \beta_\ell^{\partial M}(\phi_1, \phi_2, \Delta_g, \mathcal{B}).$$

The coefficients involving integrals over M arise from the heat redistribution on the interior of the manifold and are well understood. The additional boundary terms $\beta_\ell^{\partial M}$ are the focus of our inquiry. They, like the heat trace asymptotics, are given by local formulae and have been studied extensively (see, for example [11–15,22,35,36,40,42] and the references contained therein).

Inspired by the work of Howls and Berry [31], Travěnec and Šamaj [48] investigated the asymptotic behaviour of the coefficients β_ℓ as $\ell \to \infty$ in flat space in the special case that $\phi_1 = \phi_2 = 1$ with Dirichlet boundary conditions. The interior invariants then play no role for $n \geqslant 1$ and one has, adopting the notational conventions of this paper, that

$$\beta\big(1, 1, \Delta_g, \mathcal{B}^-\big)(t) \sim \mathrm{vol}_m(M, g) + \sum_{\ell=0}^{\infty} t^{(\ell+1)/2} \beta_\ell^{\partial M}\big(1, 1, \Delta_g, \mathcal{B}^-\big).$$

After interpreting the results of [48] in our notation, they found that if M is a ball in $\mathbb{R}^m$ of radius r with m *even*, then as $\ell \to \infty$ one has:

$$\beta_\ell = 4\pi^{(m-3)/2}\Gamma(m/2)^{-1}(\ell+1)^{-1}\Gamma(\ell/2)r^{m-\ell-1}\big(1+o(1)\big). \tag{1.d}$$

The structure of Eq. (1.d) is similar to that of Eq. (1.a). There is a combinatorial coefficient in m and ℓ, while the shortest periodic geodesic appears to a suitable power. However, for m *odd* Travěnec and Šamaj obtained polynomial dependence rather than factorial dependence of $\beta_\ell^{\partial M}$ in ℓ [48]. Furthermore the two examples in Section 1.2 above provide iso-β_ℓ pairs of smooth planar domains with different shortest periodic geodesic lengths. Hence the structure of the asymptotic behaviour of the β_ℓ's in flat space remains unclear in general.

For ℓ even, the boundary term involves a fractional power of t and there is no corresponding interior term. This simplifies the control of these terms. Consequently, we shall usually set $\ell = 2\bar{\ell}$ in what follows.

1.6. The heat content asymptotics in the real analytic setting

As noted above, results of [48] showed that the heat content asymptotics on the ball in $\mathbb{R}^m$ for m even exhibit growth rates similar to that given in Theorem 1.2 for the local heat trace asymptotics. We generalize Theorem 1.2(2) to this setting to derive an estimate using conformal variations which shows that the metric on the boundary does not play a central role in the analysis:

Theorem 1.4. *Let $m \geqslant 2$.*

(1) *Let (N, g_N) be a closed Riemannian manifold of dimension $m-1$. Let $M := [0, 2\pi] \times N$. There exists a real analytic function $h(x)$ on $[0, 2\pi]$, which depends on the choice of (N, g_N), so that the conformally adjusted metric $g_M := e^{2h}\{dx^2 + g_N\}$ satisfies*:

$$\big|\beta_{2\bar{\ell}}^{\partial M}\big(1, 1, \Delta_{g_M}, \mathcal{B}^-\big)\big| \geqslant \bar{\ell}! \cdot \mathrm{vol}_{m-1}(N, g_N) \quad \textit{for any } \bar{\ell} \geqslant 3.$$

(2) *Let g_e be the standard Euclidean metric on the unit disk D^m in $\mathbb{R}^m$. There exists a radial real analytic function h on D^m, which depends on m, so that the conformally adjusted product metric $g_M := e^{2h} g_e$ satisfies*:

$$\big|\beta_{2\bar{\ell}}^{\partial M}\big(1, 1, \Delta_{g_M}, \mathcal{B}^-\big)\big| \geqslant \bar{\ell}! \cdot \mathrm{vol}_{m-1}(N, g_N) \quad \textit{for any } \bar{\ell} \geqslant 3.$$

We have estimates for the heat content asymptotics in this setting which are similar to those given in Theorem 1.1:

Theorem 1.5. *There exist universal constants $\kappa_{n,m}$ and $\tilde{\kappa}_{\ell,m}$ such that if (M, g) is a compact real analytic Riemannian manifold of dimension m and if (ϕ_1, ϕ_2) are real analytic, then there exists a positive constant $C = C(M, g, \phi_1, \phi_2, \mathcal{B})$ such that*

$$\left|\int_M \phi_1 \cdot \Delta_g^n \phi_2 \,\mathrm{dvol}_m\right| \leqslant \kappa_{n,m} C^n \cdot \mathrm{vol}_m(M, g),$$

$$\big|\beta_\ell^{\partial M}\big(\phi_1, \phi_2, \Delta_g, \mathcal{B}^\pm\big)\big| \leqslant \tilde{\kappa}_{\ell,m} C^\ell \cdot \mathrm{vol}_{m-1}(\partial M, g).$$

Remark 1.2. Again, the constant C can be chosen so that

$$C\big(M, c^2 g\big) = c^{-2} C(M, g).$$

1.7. The heat content asymptotics in the smooth setting

Theorem 1.5 fails in the smooth setting as we have:

Theorem 1.6. *Let $k \geqslant 3$ be given, let constants $C_{\bar\ell} > 0$ for $\bar\ell \geqslant k$ be given, and let $\epsilon > 0$ be given. Let $\mathcal{B} = \mathcal{B}^+$ or $\mathcal{B} = \mathcal{B}^-$. Let (M, g) be a smooth compact Riemannian manifold of dimension $m \geqslant 2$ with non-trivial boundary. Let ϕ_1 be a smooth initial temperature and let ϕ_2 be a smooth specific heat with $\mathcal{B}\phi_2 \neq 0$. There exists Φ_1 with $\|\phi_1 - \Phi_1\|_{2k-1} < \varepsilon$ such that*:

$$\beta_{2\bar\ell}^{\partial M}(\Phi_1, \phi_2, \Delta_g, \mathcal{B}) = C_{\bar\ell} \quad \textit{for any } \bar\ell \geqslant k.$$

The heat content asymptotics were originally studied for Dirichlet boundary conditions and for $\phi_1 = \phi_2 = 1$ [9,10,16]. We have the following theorem in this setting:

Theorem 1.7. *Let $k \geqslant 3$ be given, let constants $C_{\bar\ell} > 0$ for $\bar\ell \geqslant k$ be given, and let $\epsilon > 0$ be given. Let (M, g) be a smooth compact manifold Riemannian manifold of dimension $m \geqslant 2$ with non-trivial boundary. There exists a metric g_1 so $\|g - g_1\|_{2k-1} < \varepsilon$ such that*

$$\beta_{2\bar\ell}^{\partial M}\big(1, 1, \Delta_{g_1}, \mathcal{B}^-\big) = C_{\bar\ell} \quad \textit{for any } \bar\ell \geqslant k.$$

1.8. Bochner formalism for operators of Laplace type

The results given above in Theorem 1.3, in Theorem 1.6, and in Theorem 1.7 rely upon a leading term analysis of the heat trace asymptotics and of the heat content asymptotics. It is one of the paradoxes of this subject that to apply the functorial method, one must work with very general operators even if one is only interested in the scalar Laplacian, as is the case in this paper. We only consider the context of scalar operators. There is a corresponding notion for systems, i.e. operators which act on the space of smooth sections to some vector bundle. It is possible to express an operator D of Laplace type as given in Eq. (1.b) invariantly using a Bochner formalism [27]. There exists a unique connection ∇ and a unique smooth function E so that

$$D\phi = -\big(g^{uv}\phi_{;uv} + E\phi\big),$$

where we use ‘;’ to denote the components of multiple covariant differentiation with respect to ∇ and with respect to the Levi–Civita connection. Let $\Gamma_{uv}{}^w$ be the Christoffel symbols of the Levi–Civita connection and let ω be the connection 1-form of ∇. We then have

$$\begin{aligned} \omega_u &= \frac{1}{2} g_{uv}\big(A^v + g^{sw}\Gamma_{sw}{}^v \,\mathrm{Id}\big), \\ E &= B - g^{uv}\big(\partial_{x_u}\omega_v + \omega_u\omega_v - \omega_w\Gamma_{uv}{}^w\big). \end{aligned} \tag{1.e}$$

1.9. Leading term analysis

Theorem 1.8 below will play a central role in our analysis, and was established in [20,25,26]. We also refer to related work in the 2-dimensional setting [39]. It has been used by Brooks, Perry, Yang [21] and by Chang and Yang [23] to show families of isospectral metrics within a conformal class are compact modulo gauge equivalence in dimension 3. Let τ be the scalar curvature of g, let ρ be the Ricci tensor of g, and let Ω be the curvature of the connection ∇ defined by an operator of Laplace type.

Theorem 1.8. *Let D be an operator of Laplace type on a closed Riemannian manifold (M, g) and let $\bar n \geqslant 3$.*

(1) *The local heat trace asymptotics satisfy*:

$$a_{2\bar n}(P, \Delta_g) = \frac{(-1)^{\bar n}\bar n!}{(2\bar n+1)!}\left\{-\bar n\Delta^{\bar n-1}\tau - (4n+2)\Delta^{\bar n-1}E\right\} \\ + \textit{lower order derivative terms}.$$

(2) *The global heat trace asymptotics satisfy*:

$$\begin{aligned} a_{2\bar n}(D) = \frac{1}{2}\frac{(-1)^{\bar n}\bar n!}{(2\bar n+1)!}\int_M \Big\{&\big(\bar n^2-\bar n-1\big)\big|\nabla^{\bar n-2}\tau\big|^2 + 2\big|\nabla^{\bar n-2}\rho\big|^2 \\ &+ 4(2\bar n+1)(\bar n-1)\nabla^{(\bar n-2)}\tau\cdot\nabla^{(\bar n-2)}E + 2(2\bar n+1)\big|\nabla^{(\bar n-2)}\Omega\big|^2 \\ &+ 4(2\bar n-1)(2\bar n+1)\big|\nabla^{\bar n-2}E\big|^2 + \textit{lower order terms}\Big\}\,\mathrm{dvol}_m\,. \end{aligned}$$

In this paper, we will establish a corresponding leading term analysis for the heat content asymptotics. We shall always assume ℓ is even; thus the lack of symmetry in the way we have written the interior contributions plays no role. Let ∇ be the connection defined by D as discussed in Section 1.8. Let D^* be the formal adjoint of D; the associated connection ∇^* defined by D^* is then the connection dual to ∇ defined by the relation

$$\nabla\phi_1\cdot\phi_2 + \phi_1\cdot\nabla^*\phi_2 = d(\phi_1\cdot\phi_2).$$

Let

$$\phi_1^{(\ell)} := \nabla_\nu^\ell\phi_1|_{\partial M} \quad \text{and} \quad \phi_2^{(\ell)} := \big(\nabla_\nu^*\big)^\ell\phi_2|_{\partial M}$$

be the normal covariant derivatives of order ℓ. By using the inward geodesic flow, we can always choose coordinates (y, r) near the boundary so that $\partial_r = \nu$; consequently

$$\phi^{(\ell)} = \partial_r^\ell\phi|_{\partial M} \quad \text{if } D = \Delta_g.$$

Let S be a smooth function on the boundary. The Robin boundary operator in this more general setting is defined by the identity:

$$\mathcal{B}_S^+\phi := \big(\phi^{(1)} + S\phi\big)\big|_{\partial M}.$$

Let $\rho_{mm}^{(\ell)} := R_{amma;m\dots m}$ be the ℓth covariant derivative of ρ_{mm} restricted to ∂M. Define Ξ_ℓ recursively for ℓ even by setting:

$$\Xi_2 = -2\pi^{-1/2}\frac{2}{3} \quad \text{and} \quad \Xi_\ell = \frac{2}{\ell+1}\Xi_{\ell-2} \quad \text{if } \ell \geqslant 4.$$

Theorem 1.9. *Let $\ell \geqslant 6$ be even. Modulo lower order terms we have:*

(1)

$$\begin{aligned}\beta_\ell^{\partial M}\big(\phi_1,\phi_2,D,\mathcal{B}^-\big) = \int_{\partial M} \big\{ &\Xi_\ell\big(\phi_1^{(\ell)}\phi_2 + \phi_1\phi_2^{(\ell)}\big) + \ell\cdot\Xi_\ell\phi_1\phi_2 E^{(\ell-2)} \\ &+ 0\cdot\big(\phi_1^{(\ell-1)}\phi_2^{(1)} + \phi_1^{(1)}\phi_2^{(\ell-1)}\big) \\ &+ (\ell-2)\Xi_\ell\big(\phi_1^{(1)}\phi_2 + \phi_1\phi_2^{(1)}\big)E^{(\ell-3)} \\ &+ 0\cdot\phi_1^{(1)}\phi_2^{(1)}E^{(\ell-4)} + \frac{1}{2}(\ell-2)\Xi_\ell\phi_1\phi_2\rho_{mm}^{(\ell-2)} + \cdots\big\}\,\mathrm{dvol}_{m-1}.\end{aligned}$$

(2)

$$\begin{aligned}\beta_\ell^{\partial M}\big(\phi_1,\phi_2,D,\mathcal{B}_S^+\big) = \int_{\partial M} \big\{ &0\big(\phi_1^{(\ell)}\phi_2 + \phi_1\phi_2^{(\ell)}\big) + 0\cdot\phi_1\phi_2 E^{(\ell-2)} \\ &- \Xi_\ell\big(\phi_1^{(\ell-1)}\phi_2^{(1)} + \phi_1^{(1)}\phi_2^{(\ell-1)}\big) - \Xi_\ell\big(\phi_1^{(1)}\phi_2 + \phi_1\phi_2^{(1)}\big)E^{(\ell-3)} \\ &+ (2-\ell)\Xi_\ell\phi_1^{(1)}\phi_2^{(1)}E^{(\ell-4)} - \Xi_\ell S\big(\phi_1^{(\ell-1)}\phi_2 + \phi_1\phi_2^{(\ell-1)}\big) \\ &- \Xi_\ell S\big(\phi_1^{(\ell-2)}\phi_2^{(1)} + \phi_1^{(1)}\phi_2^{(\ell-2)}\big) - 2\cdot\Xi_\ell S\big(\phi_1\phi_2^{(1)} + \phi_1^{(1)}\phi_2\big)E^{(\ell-4)} \\ &+ 0\cdot\phi_1\phi_2\rho_{mm}^{(\ell-2)} + \cdots\big\}\,\mathrm{dvol}_{m-1}.\end{aligned}$$

1.10. Outline of the paper

In Section 2 we will prove Theorem 1.1 and Theorem 1.5. In Section 3, we use Theorem 1.9 to establish Theorem 1.6 and Theorem 1.7. In Section 4, we use Theorem 1.8 to demonstrate Theorem 1.3. Theorem 1.9 is new and is proved in Section 5 by extending functorial methods employed in [11,12]. In Section 6, we establish Theorem 1.2. We conclude the paper in Section 7 by demonstrating Theorem 1.4.

2. Local invariants in the real analytic setting

Let $\alpha := (\alpha_1,\dots,\alpha_m)$ be a non-trivial multi-index. We define:

$$|\alpha| := \alpha_1 + \cdots + \alpha_m, \qquad \partial_x^\alpha := (\partial_{x_1})^{\alpha_1}\dots(\partial_{x_m})^{\alpha_m}, \qquad g_{ij/\alpha} := \partial_x^\alpha g_{ij} \quad \text{for } |\alpha| > 0.$$

In any local system of coordinates, the Riemannian volume form on M is given by:

$$\mathrm{dvol}_m = g\,dx, \quad \text{where } g := \sqrt{\det(g_{ij})}.$$

Let g^{ij} be the inverse matrix; this gives the components of the dual metric on the cotangent bundle. Since the heat trace and heat content asymptotics are given by suitable local formulae, Theorem 1.1 and Theorem 1.5 will follow from the following result:

Theorem 2.1. *Let $\mathcal{E}_n$ be a local interior invariant which is homogeneous of degree n in the jets of the metric and a finite (possibly empty) collection $\{\phi_1, \ldots\}$ of additional smooth functions. Let $\mathcal{F}_{n-1}$ be a local boundary invariant which is homogeneous of degree $n-1$ in the jets of the metric and a finite (possibly empty) collection $\{\phi_1, \ldots\}$ of additional smooth functions. Let (M, g) be a compact real analytic manifold of dimension m with real analytic (possibly empty) boundary ∂M so that the metric g is real analytic and so that the collection $\{\phi_1, \ldots\}$ is real analytic. There exists a constant $C = C(M, g, \phi_1, \ldots) > 0$ (which is independent of the choice of $\mathcal{E}_n$ and of $\mathcal{F}_n$) and there exist constants $\kappa(\mathcal{E}_n) > 0$ and $\kappa(\mathcal{F}_{n-1}) > 0$ (which are independent of the choice of $(M, g, \phi_1, \ldots)$) so that*

$$\left|\int_M \mathcal{E}_n(x, g, \phi_1, \ldots)\,\mathrm{dvol}_m\right| \leqslant \kappa(\mathcal{E}_n)C^n \cdot \mathrm{vol}_m(M, g),$$

$$\left|\int_{\partial M} \mathcal{F}_{n-1}(y, g, \phi_1, \ldots)\,\mathrm{dvol}_{m-1}\right| \leqslant \kappa(\mathcal{F}_{n-1})C^{n-1} \cdot \mathrm{vol}_{m-1}(\partial M, g).$$

The constant $C(M, g, \phi_1, \ldots)$ may be chosen so that

$$C\big(M, c^2 g, \phi_1, \ldots\big) = c^{-n} C(M, g, \phi_1, \ldots).$$

Proof. Suppose first that the boundary of M is empty. For each point P of M, there exists $\varepsilon(P) > 0$ so the exponential map defines a real analytic geodesic coordinate ball of radius $\varepsilon(P)$ about P. Let $\mathcal{K}$ be a compact neighborhood of the identity in the space of all symmetric $m \times m$ matrices. Since $g_{ij} = \delta_{ij}$ at the center of such a geodesic coordinate ball, by shrinking $\varepsilon(P)$ if necessary, we may assume that the matrix (g_{ij}) belongs to $\mathcal{K}$ for any point of the coordinate ball of radius $\varepsilon(P)$. Since we are working in the real analytic category and since $\{g_{ij}, \phi_1, \ldots\}$ are real analytic near P there exists a $C = C(P, M, g, \phi_1, \ldots)$ so that again by shrinking $\varepsilon(P)$ if necessary we have that

$$\big|d_x^\alpha g_{ij}\big| \leqslant C^{|\alpha|}|\alpha|! \quad \text{and} \quad \big|d_x^\alpha \phi_\mu\big| \leqslant C^{|\alpha|}|\alpha|! \quad \text{on } B_{\varepsilon(P)}(P) \tag{2.a}$$

for any multi-index α. We cover M by a finite number of such coordinate balls about points $(P_1, \ldots)$ and set $C(M, g, \phi_1, \ldots) = \max_\nu C(P_\nu, M, g, \phi_1, \ldots)$. Since $\mathcal{E}$ is a local invariant, we may expand:

$$\mathcal{E}(x, g) = \sum e_{\vec{\alpha}, \vec{\beta}}\big(g_{ij}(x)\big)\big(\partial_x^{\alpha_1} g_{i_1 j_1}\big) \ldots \big(\partial_x^{\alpha_a} g_{i_a j_a}\big) \cdot \big(d_x^{\beta_1}\phi_{k_1}\big) \ldots \big(d_x^{\beta_b}\phi_{k_b}\big) \tag{2.b}$$

where in this sum we have the relations:

$$|\alpha_1| + \cdots |\alpha_a| + |\beta_1| + \cdots + |\beta_b| = n, \quad 0 < |\alpha_1|,\ \ldots,\ 0 < |\alpha_a|.$$

Since $e_{\vec{\alpha},\vec{\beta}}$ is continuous on the compact neighborhood $\mathcal{K}$ of the identity δ, we may bound

$$\left|e_{\vec{\alpha},\vec{\beta}}\big(g_{ij}(x)\big)\right| \leqslant E_{\vec{\alpha},\vec{\beta}} \quad \text{uniformly on } \mathcal{K}.$$

Combining the estimates of Eq. (2.a) with the estimates given above and summing over $(\vec{\alpha},\vec{\beta})$ in Eq. (2.b) yields an estimate of the desired form after integration. Since $\mathcal{E}_n$ is homogeneous of degree n, it follows that

$$\mathcal{E}_n\big(x, c^2 g, \phi_1, \ldots\big) = c^{-n}\mathcal{E}_n(x, g, \phi_1, \ldots).$$

The desired rescaling behaviour of the constant $C(M, g, \phi_1, \ldots)$ now follows.

If the boundary of M is non-empty, we must also choose suitable coordinate charts near ∂M. If $Q \in \partial M$, we consider the geodesic ball $B_\varepsilon^{\partial M}(Q)$ of radius ε in ∂M about Q relative to the restriction of the metric to the boundary and we shall let $\tilde{B}_{\varepsilon,\iota}(Q) := [0,\iota) \times B_{\varepsilon(Q)}^{\partial M}(Q)$ for some $\iota > 0$ be defined using the inward geodesic flow so that the curves $r \to (r, Q)$ are unit speed geodesics perpendicular to the boundary. Again, by shrinking ε and ι, we may achieve the estimates of Eq. (2.a) uniformly on $\tilde{B}_{\varepsilon,\iota}(Q)$. We cover M by a finite number of coordinate charts $B_\varepsilon(P)$ for $P \in \operatorname{int}(M)$ and $\tilde{B}_{\iota,\varepsilon}(Q)$ for $Q \in \partial M$. The desired estimate for $\mathcal{E}_n$ now follows. To study the invariant $\mathcal{F}_{n-1}$, we cover ∂M by a finite number of coordinate charts $\tilde{B}_{\iota,\varepsilon}(Q)$ for $Q \in \partial M$ and argue as above. □

3. Leading terms in the heat content asymptotics

We shall omit the proof of the following result as it is well known.

Lemma 3.1.

(1) *Let $k \geqslant 1$ be given, let constants $\gamma_\ell > 0$ for $\ell \geqslant k$ be given, and let $\epsilon > 0$ be given. Let (M, g) be a smooth Riemannian manifold with non-empty boundary ∂M. There exists a smooth function Φ on M so that $\|\Phi\|_{k-1} < \varepsilon$ and so that*

$$\Phi^{(\ell)} = \psi(y)\gamma_\ell \quad \text{for } \ell \geqslant k.$$

(2) *Let $k \geqslant 1$ be given, let $C > 0$ be given, and let $\epsilon > 0$ be given. There exists a smooth function f on $M := [0,1]$ with $\|f\|_{k-1} < \varepsilon$ and $\int_M |\partial_x^k f|^2\,dx \geqslant C$.*

Proof of Theorem 1.6 and of Theorem 1.7. Let $k \geqslant 3$ be given, let constants $C_{\bar{\ell}} > 0$ for $\bar{\ell} \geqslant k$ be given, and let $\epsilon > 0$ be given. Let (M, g) be a smooth compact Riemannian manifold of dimension $m \geqslant 2$ with non-trivial boundary. We first take $\mathcal{B} = \mathcal{B}^-$ to consider Dirichlet boundary conditions. Let ϕ_1 be a smooth initial temperature and let ϕ_2 be a smooth specific heat with $\mathcal{B}^-\phi_2 \neq 0$. Since ϕ_2 does not vanish identically on the boundary, there exists a smooth function ψ on ∂M so

$$\int_{\partial M} \psi\phi_2 \,\mathrm{dvol}_{m-1} = 1.$$

Let $\{\gamma_1, \ldots\}$ be a sequence of constants, to be determined presently. For $\nu \geqslant k$, let

$$\Phi_\nu(y, r) = \sum_{j=k}^{\nu} \frac{r^{2j}}{(2j)!} \gamma_j \psi(y) \quad \text{near } \partial M.$$

Since $\beta_{2\bar{\ell}}$ is given by a local formula of degree $2\bar{\ell}$, only the constants $\gamma_1, \ldots, \gamma_{\bar{\ell}}$ play a role in the computation of $\beta_{2\bar{\ell}}^{\partial M}$, i.e.

$$\beta_{2\bar{\ell}}^{\partial M}(\Phi_\mu + \phi_1, \phi_2, \Delta_g, \mathcal{B}) = \beta_{2\bar{\ell}}^{\partial M}(\Phi_{\bar{\ell}} + \phi_1, \phi_2, \Delta_g, \mathcal{B}) \quad \text{if } \mu \geqslant \bar{\ell}.$$

We take $\Phi_{k-1} = 0$. Since $\Xi_{2\bar{\ell}} \neq 0$, we can recursively choose the constants $\gamma_{\bar{\ell}}$, and hence the functions $\Phi_{\bar{\ell}}$, for $\bar{\ell} \geqslant k$ so

$$\Xi_{2\bar{\ell}} \cdot \gamma_{\bar{\ell}} = C_{\bar{\ell}} - \beta_{2\bar{\ell}}^{\partial M}(\Phi_{\bar{\ell}-1} + \phi_1, \phi_2, \Delta_g, \mathcal{B}) \quad \text{for } \bar{\ell} \geqslant k$$

and apply Theorem 1.9 to see:

$$\beta_{2\bar{\ell}}^{\partial M}\big(\Phi_{2\bar{\ell}} + \phi_1, \phi_2, \Delta_g, \mathcal{B}^-\big) = C_{\bar{\ell}}.$$

We complete the proof of Theorem 1.6(1) by using Lemma 3.1 to choose Φ with $\|\Phi\|_{2k-1} < \varepsilon$ such that

$$\Phi^{(j)} = \left\{ \begin{array}{ll} 0 & \text{if } j < 2k \text{ or if } j \text{ is odd} \\ \gamma_{\bar{\ell}} & \text{if } j = 2\bar{\ell} \text{ for } \bar{\ell} \geqslant k \end{array} \right\}.$$

To prove assertion (2) of Theorem 1.6, we use assertion (2) of Theorem 1.9 and examine the term $-\Xi_{2\bar{\ell}} \phi_1^{(2\bar{\ell}-1)} \phi_2^{(1)}$; to prove Theorem 1.7, we apply assertion (1) of Theorem 1.9 and examine the term $\frac{1}{2}(2\bar{\ell} - 2)\Xi_{2\bar{\ell}} \phi_1 \phi_2 \rho_{mm}^{(2\bar{\ell}-2)}$. As apart from these minor changes the proof is exactly the same as that given above, we shall omit details in the interests of brevity. □

4. Leading terms in the heat trace asymptotics

4.1. *Proof of Theorem 1.3(1)*

We set $E = 0$ and $\Omega = 0$ in Theorem 1.8 to study the Laplacian and see thereby that there exists a non-zero constant d_n so:

$$a_{2\bar{n}}(\Delta_g) = d_{\bar{n}} \int_M \big\{ \big(\bar{n}^2 - \bar{n} - 1\big) \big|\nabla^{\bar{n}-2} \tau\big|^2 + 2 \big|\nabla^{\bar{n}-2} \rho\big|^2 + Q_{\bar{n},m}\big(R, \nabla R, \ldots, \nabla^{\bar{n}-3} R\big) \big\} \, \mathrm{dvol}_m .$$

Let $\varepsilon > 0$ be given. We restrict to a single geodesic ball B of radius 3δ for some $\delta > 0$ about a point P. Let θ be a plateau function so that $\theta = 1$ for $|x| < \delta$ and $\theta = 0$ for $|x| > 2\delta$. We shall define the functions $f_k, f_{k+1}, \ldots$ recursively and consider the conformal deformation:

$$g_\mu := e^{\theta(x)(2 f_k(x_1) + \cdots + 2 f_\mu(x_1))} g.$$

Let $k \geqslant 3$. Choose $0 < \delta_\mu^1$ for $k \leqslant \mu$ so that $\|f_\mu\|_{\mu-1} \leqslant \delta_\mu^1$ for $k \leqslant \mu$ implies:

Constraint 4.1.

(1) $f_\infty := \lim_{\mu\to\infty}\{f_k + \cdots + f_\mu\}$ converges in the C^ℓ topology for any ℓ.
(2) $g_\infty := \lim_{\mu\to\infty} g_\mu$ converges in the C^ℓ topology for any ℓ.
(3) $\|f\|_{k-1} < \varepsilon$.
(4) $\|g_\mu - g_{\mu+1}\|_\mu < 2^{-\mu}\varepsilon$ for any μ.

A priori, one must consider jets of degree $2\bar{n}$ in computing $a_{2\bar{n}}(\Delta_g)$ (and in fact this is the case when considering the local heat asymptotic coefficients of Eq. (1.c)). However, by Theorem 1.8, only the jets of the metric to degree $\bar{n}$ play a role in the computation of the integrated invariants, $a_{2\bar{n}}$.

Constraint 4.2. Choose $0 < \delta_\mu^2 < \delta_\mu^1$ for $k \leqslant \mu$ so $\|f_\mu\|_{\mu-1} \leqslant \delta_\mu^2$ for $k \leqslant \mu$ implies:

(1) $|a_{2\bar{n}}(\Delta_{g_{\mu-1}}) - a_{2\bar{n}}(\Delta_{g_\mu})| < 2^{-\mu}$ for $3 \leqslant k \leqslant \bar{n} < \mu$.
(2) $|a_{2\bar{n}}(\Delta_{g_\mu})| - 1 \leqslant |a_{2\bar{n}}(\Delta_{g_\infty})|$ for $3 \leqslant k \leqslant \bar{n}$.

The polynomial $Q_{\bar{n},m}(\cdot)$ involves lower order derivatives of the metric.

Constraint 4.3. Choose $0 < \delta_\mu^3 < \delta_\mu^2$ for $k \leqslant \mu$ so that $\|f_\mu\|_{\mu-1} \leqslant \delta_\mu^3$ for $k \leqslant \mu$ implies there are constants $C_\mu^1 = C_\mu^1(f_k, \ldots, f_{\mu-1})$ depending only on the choices made previously so

$$\begin{aligned}\left|a_{2\mu}(\Delta_{g_\mu})\right| &\geqslant |d_\mu| \int_M \left\{\left|2\nabla^{\mu-1}\tau_{g_\mu}\right|^2 + \left(\mu^2 - \mu - 1\right)|\nabla^{n-1}\rho|^2\right\} \mathrm{dvol}_m - C_\mu^1 \\ &\geqslant |d_\mu| \int_{B_\delta} \left\{\left|2\nabla^{\mu-1}\tau_{g_\mu}\right|^2\right\} \mathrm{dvol}_m - C_\mu^1.\end{aligned}$$

On B_δ, the plateau function θ is identically 1 and we have:

$$g_\mu = e^{2f_\mu} g_{\mu-1}.$$

From this it follows that

$$\nabla^{\bar{n}-2}\tau = (m-1)\partial_{x_1}^{\bar{n}} f_\mu + \text{lower order terms}.$$

Since g_{ij} is in a compact neighborhood of δ_{ij}, we may estimate:

$$\left\|\nabla^{\bar{n}-2}\tau_{g_n}\right\|^2(P) \geqslant \left|\partial_{x_1}^{\bar{n}-2}\tau\right|^2 = \left|\partial_{x_1}^{\bar{n}} f_{\bar{n}}\right|^2 + \text{lower order terms}. \tag{4.a}$$

Constraint 4.4. Choose $0 < \delta_\mu^4 < \delta_\mu^3$ for $k \leqslant \mu$ where $\delta_\mu^4 = \delta_\mu^4(f_k, \dots, f_{\mu-1})$ depends on the choices made previously so that $\|f_\mu\|_{\mu-1} \leqslant \delta_\mu^4$ for $k \leqslant \mu$ implies there are constants $C_\mu^2 = C_\mu^2(f_k, \dots, f_{\mu-1})$ depending only on the choices made previously so

$$\int_{B_{\delta_\mu^4}} \left|\nabla^{n-2}\tau_{g_\mu}\right|^2 \mathrm{dvol}_m \geqslant \int_{B_{\delta_\mu^4}} \left|\partial_{x_1}^\mu f_\mu\right|^2 \mathrm{dvol}_m - C_\mu^2.$$

Theorem 1.1(1) now follows from Lemma 3.1(2). We can choose recursively f_μ subject to the constraints given above so that $\|f_\mu\|_{\mu-1}$ is arbitrarily small and so that $\int_{B_{\delta_\mu^4}} |\partial_{x_1}^\mu f_\mu|^2 \mathrm{dvol}_m$ is arbitrarily large.

4.2. The proof of Theorem 1.1(2)

Let (M, g) be a hypersurface in $\mathbb{R}^{m+1}$. We fix $P \in M$. After applying a rigid body motion, we may assume that $P = 0$ and that the normal to M at P is given by $e_{m+1} := (0, \dots, 0, 1)$. Thus we may write M as a graph over the ball $B_{3\delta}$ in $\mathbb{R}^m$ in the form $x \to (x, f_0(x))$ where $f_0(P) = 0$ and $df_0(P) = 0$. Let θ be a plateau function which is 1 for $|x| \leqslant \delta$ and 0 for $|x| \geqslant \delta$. We shall consider the perturbed hypersurface defined near P by $x \to (x, f_0(x) + \theta(x)(f_k(x) + \cdots))$ where $f_\mu(P) = 0$ and $df_\mu(P) = 0$. This hypersurface agrees with the original hypersurface away from P. We shall need to establish an analogue of Eq. (4.a). The remainder of the analysis will be similar to that performed in the proof of Theorem 1.1(1), and will therefore be omitted.

Suppose we have a hypersurface in the form $\Psi(x) := (x, F(x))$ where $F(0) = 0$ and $dF(0) = 0$. Let $F_i := \partial_{x_i} F$, $F_{ij} := \partial_{x_i}\partial_{x_j} F$, and so forth. We compute:

$$\begin{aligned}
&\Psi_*(\partial_{x_i}) = e_i + F_i e_{m+1},\\
&g_{ij} = \delta_{ij} + F_i F_j,\\
&\Gamma_{jkl} = \frac{1}{2}\{F_{jk}F_l + F_{jl}F_k + F_{jk}F_l + F_{kl}F_j - F_{jl}F_k - F_{kl}F_j\} = F_{jk}F_l,\\
&\Gamma_{jk}{}^l = g^{ln}F_{jk}F_n,\\
&R_{ijk}{}^l = g^{ln}\{F_{jk}F_{in} - F_{ik}F_{jn}\} + \text{lower order terms},
\end{aligned}$$

where the lower order terms are either 4th order in the 1-jets or linear in the 2-jets and quadratic in the 1-jets. We suppose $F = F_{\mu-1} + f_\mu$ where we set $f_\mu = \varepsilon_\mu \cos(a_\mu x^1)\cos(b_\mu x^2)$,

$$\begin{aligned}
&\tau = 4\varepsilon_\mu a_\mu^2 b_\mu^2 \left\{\cos^2(a_\mu x^1)\cos^2(b_\mu x^1) - \sin^2(a_\mu x^1)\sin^2(b_\mu x^1)\right\} + \cdots,\\
&\left|\nabla^{\mu-2}\tau\right|^2 = 4\varepsilon_\mu a_\mu^4 b_\mu^\mu \left|\cos^2(a_\mu x^1)\cos^2(b_\mu x^1) - \sin^2(a_\mu x^1)\sin^2(b_\mu x^1)\right|^2 + \cdots,
\end{aligned}$$

where we have omitted lower order terms either involving ε^2 or not multiplied by the appropriate power of $a_\mu^4 b_\mu^\mu$. To simplify matters, we suppose $\delta = \pi$ and that a_μ and b_μ are non-zero integers. We use the fact that we are dealing with periodic functions to compute:

$$\int_{x^1=-\pi}^{\pi}\int_{x^2=-\pi}^{\pi}\left|\cos^2(a_\mu x^1)\cos^2(b_\mu x^2)-\sin^2(a_\mu x^1)\sin^2(b_\mu x^2)\right|^2 dx^2\,dx^1$$
$$=a_\mu^{-1}b_\mu^{-1}\int_{x^1=-a_\mu\pi}^{a_\mu\pi}\int_{x^2=-b_\mu\pi}^{b_\mu\pi}\left|\cos^2(x^1)\cos^2(x^2)-\sin^2(x^1)\sin^2(x^2)\right|^2 dx^2\,dx^1$$
$$=a_\mu^{-1}b_\mu^{-1}a_\mu b_\mu\int_{x^1=-\pi}^{\pi}\int_{x^2=-\pi}^{\pi}\left|\cos^2(x^1)\cos^2(x^2)-\sin^2(x^1)\sin^2(x^2)\right|^2 dx^2\,dx^1$$
$$=(2\pi)^2.$$

We shall take $b_\mu = a_\mu^\mu$, take a_μ large, and take ε_μ appropriately small to complete the proof.

5. Leading terms in the heat content asymptotics

This section is devoted to the proof of Theorem 1.9. Let D be an operator of Laplace type on a compact smooth Riemannian manifold (M, g) with non-empty boundary. We adopt the notation established in Section 1.8 and in Section 1.9. We shall always take S to be real in defining the Robin boundary operator. One then has the symmetry

$$\beta(\phi_1,\phi_2,D,\mathcal{B})(t)=\beta\big(\phi_2,\phi_1,D^*,\mathcal{B}\big)(t). \tag{5.a}$$

If ℓ is even, the lack of symmetry in the way we expressed the interior terms plays no role and thus Eq. (5.a) yields:

$$\beta_{2\bar{\ell}}^{\partial M}(\phi_1,\phi_2,D,\mathcal{B})=\beta_{2\bar{\ell}}^{\partial M}\big(\phi_2,\phi_1,D^*,\mathcal{B}\big). \tag{5.b}$$

Let indices $\{a, b\}$ range from 1 to $m-1$ and index the tangential coordinates $(y^1,\dots,y^{m-1})$ in an adapted coordinate system such that ∂_r is the inward unit geodesic normal. We then have

$$ds^2=g_{ab}(y,r)\,dy^a\circ dy^b+dr\circ dr.$$

We define the *second fundamental* form by setting:

$$L_{ab}:=g(\nabla_{\partial_{y_a}}\partial_{y_b},\partial_r)=-\frac{1}{2}\partial_r g_{ab}.$$

Results of [11,12] yield the following formulae which will form the starting point for our analysis:

Lemma 5.1. *Adopt the notation established above. Then*

(1) $\beta_0^{\partial M}(\phi_1,\phi_2,D,\mathcal{B}^-)=-\frac{2}{\sqrt{\pi}}\int_{\partial M}\phi_1\phi_2\,\mathrm{dvol}_{m-1}$.

(2) $\beta_0^{\partial M}(\phi_1,\phi_2,D,\mathcal{B}_S^+)=0$.

$$\beta_2^{\partial M}(\phi_1,\phi_2,D,\mathcal{B}^-) = -\frac{2}{\sqrt{\pi}}\int_{\partial M}\bigg\{\frac{2}{3}\big(\phi_1^{(2)}\phi_2+\phi_1\phi_2^{(2)}\big)+\phi_1\phi_2E-\phi_{1;a}\phi_{2;a} - \frac{2}{3}L_{aa}\big(\phi_1^{(1)}\phi_2+\phi_1\phi_2^{(1)}\big) + \left(\frac{1}{12}L_{aa}L_{bb}-\frac{1}{6}L_{ab}L_{ab}-\frac{1}{6}\rho_{mm}\right)\phi_1\phi_2\bigg\}\,\mathrm{dvol}_{m-1}. \tag{3}$$

$$\beta_2^{\partial M}(\phi_1,\phi_2,D,\mathcal{B}_S^+) = \frac{2}{\sqrt{\pi}}\int_{\partial M}\frac{2}{3}\big(\phi_1^{(1)}+S\phi_1\big)\big(\phi_2^{(2)}+S\phi_2\big)\,\mathrm{dvol}_{m-1}. \tag{4}$$

We begin the proof of Theorem 1.9 by expressing $\beta_\ell^{\partial M}$, modulo lower order terms, in terms of certain invariants involving maximal derivatives with unknown but universal coefficients; the symmetry of Eq. (5.b) plays a crucial role in our analysis. Standard arguments (see [11]) show the coefficients in the following expressions are independent of the underlying dimension of the manifold:

$$\begin{aligned}
&\beta_\ell^{\partial M}(\phi_1,\phi_2,D,\mathcal{B}^-)\\
&\quad=\int_{\partial M}\big\{c_{\ell,1}^-\big(\phi_1^{(\ell)}\phi_2+\phi_1\phi_2^{(\ell)}\big)+c_{\ell,2}^-\big(\phi_1^{(\ell-1)}\phi_2^{(1)}+\phi_1^{(1)}\phi_2^{(\ell-1)}\big)\\
&\qquad+e_{\ell,1}^-\phi_1\phi_2E^{(\ell-2)}+e_{\ell,2}^-\big(\phi_1^{(1)}\phi_2+\phi_1\phi_2^{(1)}\big)E^{(\ell-3)}+e_{\ell,3}^-\phi_1^{(1)}\phi_2^{(1)}E^{(\ell-4)}\\
&\qquad+r_\ell^-\phi_1\phi_2\rho_{mm}^{(\ell-2)}+\cdots\big\}\,\mathrm{dvol}_{m-1},
\end{aligned}$$

$$\begin{aligned}
&\beta_\ell^{\partial M}(\phi_1,\phi_2,D,\mathcal{B}_S^+)\\
&\quad=\int_{\partial M}\big\{c_{\ell,1}^+(\phi_1^{(\ell)}\phi_2+\phi_1\phi_2^{(\ell)})+c_{\ell,2}^+(\phi_1^{(\ell-1)}\phi_2^{(1)}+\phi_1^{(1)}\phi_2^{(\ell-1)})\\
&\qquad+e_{\ell,1}^+\phi_1\phi_2E^{(\ell-2)}+e_{\ell,2}^+\big(\phi_1^{(1)}\phi_2+\phi_1\phi_2^{(1)}\big)E^{(\ell-3)}+e_{\ell,3}^+\phi_1^{(1)}\phi_2^{(1)}E^{(\ell-4)}\\
&\qquad+d_{\ell,1}^+S\big(\phi_1^{(\ell-1)}\phi_2+\phi_1\phi_2^{(\ell-1)}\big)+d_{\ell,2}^+S\big(\phi_1^{(\ell-2)}\phi_2^{(1)}+\phi_1^{(1)}\phi_2^{(\ell-2)}\big)\\
&\qquad+d_{\ell,3}^+S\big(\phi_1\phi_2^{(1)}+\phi_1^{(1)}\phi_2\big)E^{(\ell-4)}+d_{\ell,5}^+S\phi_1\phi_2E^{(\ell-3)}+r_\ell^+\phi_1\phi_2\rho_{mm}^{(\ell-2)}\\
&\qquad+\cdots\big\}\,\mathrm{dvol}_{m-1}.
\end{aligned}$$

We will determine all the coefficients except $d_{\ell,5}^+$ in what follows. Recall that

$$\Xi_2=-2\pi^{-1/2}\frac{2}{3}\quad\text{and}\quad\Xi_\ell=\frac{2}{\ell+1}\Xi_{\ell-2}.$$

Lemma 5.2. *Let $\ell\geqslant 4$ be even. Let $\mathcal{B}=\mathcal{B}^-$ or $\mathcal{B}=\mathcal{B}_S^+$.*

(1) *Let D be self-adjoint with respect to the boundary conditions defined by $\mathcal{B}$. If $\mathcal{B}\phi_1=0$, then $\beta_\ell^{\partial M}(\phi_1,\phi_2,D,\mathcal{B})=\frac{2}{\ell+1}\beta_{\ell-2}(\phi_1^{(2)}+E,\phi_2,D,\mathcal{B})$.*

(2) $c^-_{\ell,1} = \Xi_\ell$, $c^-_{\ell,2} = 0$, $c^+_{\ell,1} = 0$, *and* $c^+_{\ell,2} = -\Xi_\ell$.
(3) $e^-_{\ell,2} = (\ell - 2)\Xi_\ell$, $e^-_{\ell,3} = 0$, $e^+_{\ell,1} = 0$, $e^+_{\ell,2} = -\Xi_\ell$, *and* $r^+_\ell = 0$.
(4) $d^+_{\ell,1} = d^+_{\ell,2} = -\Xi_\ell$.

Proof. We follow [11] to derive assertion (1) as follows. Let $\{\lambda_\mu, \phi_\mu\}$ be a complete spectral resolution of $D_{\mathcal{B}}$. Here $\{\phi_\mu\}$ is a complete orthonormal basis for $L^2(M)$ of smooth functions with $D\phi_\mu = \lambda_\mu \phi_\mu$ and $\mathcal{B}\phi_\mu = 0$. Let

$$\gamma^D_\mu(f) := \int_M f \phi_\mu \,\mathrm{dvol}_m$$

be the associated Fourier coefficients. Then

$$\beta(\phi_1, \phi_2, D, \mathcal{B})(t) = \sum_{\mu=1}^{\infty} e^{-t\lambda_\mu} \gamma^D_\mu(\phi_1)\gamma^D_\mu(\phi_2).$$

If $\mathcal{B}\phi_1 = 0$, then

$$\gamma^D_\mu(D\phi_1) = \int_M D\phi_1 \cdot \phi_\mu \,\mathrm{dvol}_m = \int_M \phi_1 \cdot D\phi_\mu \,\mathrm{dvol}_m = \lambda_\mu \gamma^D_\mu(\phi_1).$$

Consequently we have that:

$$\begin{aligned}
&\beta(D\phi_1, \phi_2, D, \mathcal{B})(t) \\
&\quad \sim \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \int_M D^{n+1}\phi_1 \cdot \phi_2 \,\mathrm{dvol}_m + \sum_{k=0}^{\infty} t^{(k+1)/2} \beta^{\partial M}_k(D\phi_1, \phi_2, D, \mathcal{B}) \\
&\quad = \sum_{\mu=1}^{\infty} e^{-t\lambda_\mu} \gamma^D_\mu(D\phi_1)\gamma^D_\mu(\phi_2) = \sum_{\mu=1}^{\infty} \lambda_\mu e^{-t\lambda_\mu} \gamma^D_\mu(\phi_1)\gamma^D_\mu(\phi_2) \\
&\quad = -\frac{\partial}{\partial t} \sum_{\mu=1}^{\infty} e^{-t\lambda_\mu} \gamma^D_\mu(\phi_1)\gamma^D_\mu(\phi_2) = -\frac{\partial}{\partial t} \beta(\phi_1, \phi_2, D, \mathcal{B})(t) \\
&\quad \sim \sum_{j=1}^{\infty} \frac{(-t)^{j-1}}{(j-1)!} \int_M D^j \phi_1 \cdot \phi_2 \,\mathrm{dvol}_m - \sum_{\ell=0}^{\infty} \frac{\ell+1}{2} t^{(\ell-1)/2} \beta^{\partial M}_\ell(\phi_1, \phi_2, D, \mathcal{B}).
\end{aligned}$$

The asymptotics defined by the interior integrals are the same. We note that $-D\phi_1 = \phi_1^{(2)} + E\phi_1$. We set $k = \ell - 2$ and equate the asymptotics defined by the boundary integrals to establish assertion (1).

If $\ell = 2$, then the relations of assertion (2) would follow from Lemma 5.1 modulo the caveat that we have but a single term $c^\pm_{\ell,2}\phi_1^{(1)}\phi_2^{(1)}$ rather than 2 distinct terms in that setting. This will let us apply the recursion relation of Assertion (1) even if $\ell = 4$. Let $\phi_1|_{\partial M} = \phi_1^{(1)}|_{\partial M} = 0$. We set $E = 0$ and consider $c^\pm_{\ell,1}\phi_1^{(\ell)}\phi_2$ and $c^\pm_{\ell,2}\phi_1^{(\ell-1)}\phi_2^{(1)}$. These terms arise in $\beta_{\ell-2}(\phi_1^{(2)}, \phi_2, D, \mathcal{B})$

only from the corresponding terms $c^{\pm}_{\ell-1,1}(\phi_1^{(2)})^{(\ell-2)}\phi_2$ and $c^{\pm}_{\ell-1,2}(\phi_1^{(2)})^{(\ell-3)}\phi_2^{(1)}$. Assertion (2) now follows from the recursion relation

$$c^{\pm}_{\ell,1}=\frac{2}{\ell+1}c^{\pm}_{\ell-2,1}\quad\text{and}\quad c^{\pm}_{\ell,2}=\frac{2}{\ell+1}c^{\pm}_{\ell-2,2}.$$

To prove assertion (3), we first take Dirichlet boundary conditions. Let $\ell\geqslant 4$. Let $\phi_1^{(k)}|_{\partial M}=0$ for $k\neq 1$. No information is garnered concerning $e^-_{\ell,1}$ or r^-_ℓ. The term $e^-_{\ell,2}\phi_1^{(1)}\phi_2E^{(\ell-3)}$ arises in $\beta_{\ell-2}(\phi_1^{(2)}+E\phi_1,\phi_2,D,\mathcal{B})$ only from the monomial $c^-_{\ell,1}(\phi_1^{(2)}+E\phi_1)^{(\ell-2)}\phi_2$. It now follows that

$$e^-_{\ell,2}=(\ell-2)\frac{2}{\ell+1}c^-_{\ell-2,1}=(\ell-2)\Xi_\ell.$$

Since the coefficient $c^-_{\ell-2,2}=0$, the term $\phi_1^{(1)}\phi_2^{(1)}E^{(\ell-4)}$ does not arise in the invariant $\frac{2}{\ell+1}\beta_{\ell-2}(\phi_1^{(2)}+E\phi_1,\phi_2,D,\mathcal{B}^-_S)$ and thus

$$e^-_{\ell,3}=0.$$

Next we examine Neumann boundary conditions. We take $S=0$ and suppose $\phi_1^{(k)}|_{\partial M}=0$ for $k\geqslant 1$. No information is garnered concerning $e^+_{\ell,3}$. Since $c^+_{\ell,1}=0$, the term $e^+_{\ell,1}\phi_1\phi_2E^{(\ell-2)}$ and the term $e^+_{\ell,2}\phi_1\phi_2^{(1)}E^{(\ell-3)}$ can arise in the invariant $\beta_{\ell-2}(\phi_1^{(2)}+E\phi_1,\phi_2,D,\mathcal{B})$ only from the term $c^+_{\ell,2}(\phi_1^{(2)}+E\phi_1)^{(\ell-3)}\phi_2^{(1)}$. We conclude

$$e^+_{\ell,1}=0\quad\text{and}\quad e^+_{\ell,2}=\frac{2}{\ell+1}c^+_{\ell,2}=-\Xi_\ell.$$

The argument that $r^+_\ell=0$ is similar and is therefore omitted. This establishes assertion (3).

To examine assertion (4), we assume $\phi_1|_{\partial M}=\phi_1^{(1)}|_{\partial M}=0$. Again, we set $E=0$. We study the terms $d^+_{\ell,1}S\phi_1^{(\ell-1)}\phi_2$ and $d^+_{\ell,2}S\phi_1^{(\ell-2)}\phi_2^{(1)}$. The case $\ell=4$ is a bit exceptional as these terms arise in $\beta_2(\phi_1^{(2)},\phi_2,D,\mathcal{B})$ only from $2\pi^{-1/2}\frac{2}{3}S(\phi_1^{(2)})^{(1)}\phi_2$ and from $2\pi^{-1/2}\frac{2}{3}S(\phi_1^{(2)})\phi_2^{(1)}$. This shows that

$$d^+_{4,1}=d^+_{4,2}=\frac{2}{5}\cdot\frac{2}{3}\cdot 2\pi^{-1/2}=-\Xi_4.$$

For $\ell\geqslant 6$, these terms decouple and the recursion relation proceeds without complication to show

$$d^+_{\ell,1}=\frac{2}{\ell+1}d^+_{\ell-2,1}=-\Xi_\ell\quad\text{and}\quad d^+_{\ell,2}=\frac{2}{\ell+1}d^+_{\ell-2,2}=-\Xi_\ell.\qquad\square$$

We can relate Neumann and Dirichlet boundary conditions. Let $M:=[0,1]$ and let $b\in C^\infty(M)$. Let $\varepsilon\partial_r$ be the inward unit normal; $\varepsilon(0)=1$ and $\varepsilon(1)=-1$. Define:

$$A := \partial_r + b, \qquad A^* := -\partial_r + b, \qquad D_1 := A^*A, \qquad D_2^* := AA^*,$$
$$S := \varepsilon b, \qquad \mathcal{B}_S^+ := \varepsilon A, \qquad E_1 := b' - b^2, \qquad E_2 := -b' - b^2. \tag{5.c}$$

Then $\mathcal{B}_S^+ \phi = 0$ simply means $A\phi|_{\partial M} = 0$. Furthermore E_i is the endomorphism defined by D_i.

Lemma 5.3. *Adopt the notation established above. Let $\ell \geqslant 6$ be even.*

(1) $\beta_\ell^{\partial M}(\phi_1, \phi_2, D_1, \mathcal{B}_S^+) = -\frac{2}{\ell+1}\beta_{\ell-2}(A\phi_1, A\phi_2, D_2, \mathcal{B}^-)$.
(2) $e_{\ell,1}^- = \ell \cdot \Xi_\ell$, $e_{\ell,3}^+ = (2-\ell)\Xi_\ell$, $d_{\ell,3}^+ = -2 \cdot \Xi_\ell$.

Proof. Again, we follow [11] to prove the first assertion. Let $\{\lambda_\mu, \phi_\mu\}$ be a complete spectral resolution of $(D_1)_{\mathcal{B}_S^+}$. We obtain as above that

$$-\partial_t \beta\big(\phi_1, \phi_2, D_1, \mathcal{B}_S^+\big)(t) = \sum_\mu \lambda_\mu e^{-t\lambda_\mu} \gamma_\mu^{D_1}(\phi_1)\gamma_\mu^{D_1}(\phi_2).$$

We restrict henceforth to $\lambda_\mu > 0$ since the contribution of zero eigenvalues to the above sum is zero. Let

$$\psi_\mu := \frac{A\phi_\mu}{\sqrt{\lambda_\mu}}.$$

Then $\{\lambda_\mu, \psi_\mu\}$ is a spectral resolution of D_2 on $\operatorname{Range}(A) = \ker(D_2)^\perp$ with Dirichlet boundary conditions. Since $A\phi_\mu|_{\partial M} = 0$, the boundary terms vanish and we may express:

$$\begin{aligned}\gamma_\mu^{D_2}(Af) &= \int_M \langle Af, \psi_\mu\rangle \,\mathrm{dvol}_m = \frac{1}{\sqrt{\lambda_\mu}} \int_M \langle Af, A\phi_\mu\rangle \,\mathrm{dvol}_m \\ &= \frac{1}{\sqrt{\lambda_\mu}} \int_M \big\langle f, A^*A\phi_\mu\big\rangle \mathrm{dvol}_m = \sqrt{\lambda_\mu}\gamma_\mu^{D_1}(f).\end{aligned}$$

This then permits us to express

$$\beta\big(A\phi_1, A\phi_1, D_2, \mathcal{B}^-\big)(t) = \sum_\mu \lambda_\mu e^{-t\lambda_\mu} \gamma_\mu^{D_1}(\phi_1)\gamma_\mu^{D_1}(\phi_2)$$

which yields the identity

$$-\partial_t \beta\big(\phi_1, \phi_2, D_1, \mathcal{B}_S^+\big)(t) = \beta\big(A\phi_1, A\phi_2, D_2, \mathcal{B}^-\big)(t).$$

Assertion (1) now follows by equating terms in the asymptotic expansion in exactly the same fashion as was used to establish assertion (1) of Lemma 5.2 (the extra negative sign cannot be absorbed into D).

We apply the relations of Eq. (5.c) and use the fact that $e_{\ell,1}^+ = c_{\ell-2,2}^- = 0$ to examine

$$\big\{\phi_1^{(1)}\phi_2 b^{(\ell-2)},\ \phi_1^{(1)}\phi_2 b b^{(\ell-3)},\ \phi_1^{(1)}\phi_2^{(1)} b^{(\ell-3)},\ \phi_1^{(1)}\phi_2^{(1)}\big(b^2\big)^{(\ell-4)}\big\}.$$

The assumption that $\ell \geqslant 6$ is employed to ensure that $S^2(\phi_1^{(\ell-3)}\phi_2^{(1)} + \phi_1^{(1)}\phi_2^{(\ell-3)})$ does not produce such a term. We compute at the boundary component $x = 0$:

$$e_{\ell,2}^{+}\phi_1^{(1)}\phi_2 E_1^{(\ell-3)} = -\Xi_\ell \phi_1^{(1)}\phi_2 b^{(\ell-2)} + 2\cdot \Xi_\ell \phi_1^{(1)}\phi_2 b b^{(\ell-3)} + \cdots,$$
$$e_{\ell,3}^{+}\phi_1^{(1)}\phi_2^{(1)} E_1^{(\ell-4)} = e_{\ell,3}^{+}\phi_1^{(1)}\phi_2^{(1)} b^{(\ell-3)} - e_{\ell,3}^{+}\phi_1^{(1)}\phi_2^{(1)}\big(b^2\big)^{(\ell-4)} + \cdots,$$
$$d_{\ell,3}^{+} S\phi_1^{(1)}\phi_2 E_1^{(\ell-4)} = d_{\ell,3}^{+}\phi_1^{(1)}\phi_2 b b^{(\ell-3)} + \cdots,$$
$$\begin{aligned}&-\frac{2}{\ell+1}c_{\ell-2,1}^{-}\big\{\big(\phi_1^{(1)} + b\phi_1\big)^{(\ell-2)}\big(\phi_2^{(1)} + b\phi_2\big) + \big(\phi_1^{(1)} + b\phi_1\big)\big(\phi_2^{(1)} + b\phi_2\big)^{(\ell-2)}\big\}\\ &\quad = -\Xi_\ell\phi_1^{(1)}\phi_2 b^{(\ell-2)} - \Xi_\ell(\ell-2)\phi_1^{(1)}\phi_2 b b^{(\ell-3)} - 2(\ell-2)\Xi_\ell\phi_1^{(1)}\phi_2^{(1)} b^{(\ell-3)} + \cdots,\\ &-\frac{2}{\ell+1}e_{\ell-2,1}^{-}\big(\phi_1^{(1)} + b\phi_1\big)\big(\phi_2^{(1)} + b\phi_2\big)E_2^{(\ell-4)}\\ &\quad = -\frac{2}{\ell+1}e_{\ell-2,1}^{-}\big\{-\phi_1^{(1)}\phi_2 b b^{(\ell-3)} - \phi_1^{(1)}\phi_2^{(1)} b^{(\ell-3)} - \phi_1^{(1)}\phi_2^{(1)}(b^2)^{(\ell-4)}\big\} + \cdots.\end{aligned}$$

This gives us the following relations:

(a) $\phi_1^{(1)}\phi_2 b^{(\ell-2)}$: $-\Xi_\ell = -\Xi_\ell$,
(b) $\phi_1^{(1)}\phi_2 b b^{(\ell-3)}$: $2\cdot\Xi_\ell + d_{\ell,3}^{+} = -\Xi_\ell(\ell-2) + \frac{2}{\ell+1}e_{\ell-2,1}^{-}$,
(c) $\phi_1^{(1)}\phi_2^{(1)} b^{(\ell-3)}$: $e_{\ell,3}^{+} = -2(\ell-2)\Xi_\ell + \frac{2}{\ell+1}e_{\ell-2,1}^{-}$,
(d) $\phi_1^{(1)}\phi_2^{(1)}(b^2)^{(\ell-4)}$: $-e_{\ell,3}^{+} = \frac{2}{\ell+1}e_{\ell-2,1}^{-}$.

This then yields the following 3 relations:

(1) (c) + (d): $0 = -2(\ell-2)\Xi_\ell + 2\cdot\frac{2}{\ell+1}e_{\ell-2,1}^{-}$ so $e_{\ell-2,1}^{-} = (\ell-2)\frac{\ell+1}{2}\cdot\Xi_\ell = (\ell-2)\Xi_{\ell-2}$.
(2) (d) − (c): $-2e_{\ell,3}^{+} = 2(\ell-2)\Xi_\ell$ so $e_{\ell,3}^{+} = (2-\ell)\Xi_\ell$.
(3) (c) − (b): $-d_{\ell,3}^{+} + e_{\ell,3}^{+} - 2\cdot\Xi_\ell = -(\ell-2)\Xi_\ell$ so $d_{\ell,3}^{+} = e_{\ell,3}^{+} + (\ell-4)\Xi_\ell = -2\cdot\Xi_\ell$. □

We now work in dimension $m \geqslant 2$ to examine

$$\begin{aligned}&\beta_\ell^{\partial M}\big(\phi_1,\phi_2,D,\mathcal{B}^-\big)\\ &\quad = \int_{\partial M}\big\{c_{\ell,1}^{-}\phi_1^{(\ell)}\phi_2 + e_{\ell,1}^{-}\phi_1\phi_2 E^{(\ell-2)} + r_{\ell,1}^{-}\phi_1\phi_2\rho_{mm}^{(\ell-2)} + \cdots\big\}\,\mathrm{dvol}_{m-1}.\end{aligned}$$

Let $M_1 := [0,1]$ and $\alpha \in C^\infty(M_1)$ satisfy $\alpha|_{\partial M_1} = 0$. Let

$$D_1 := -\partial_r^2, \qquad M_2 := M_1 \times S^1, \qquad D_2 := D_1 - e^{-2\alpha(r)}\partial_\theta^2.$$

Lemma 5.4.

(1) *If* $\ell \geqslant 2$, *then* $0 = \beta_\ell^{\partial M}(1, e^{\alpha(r)}, -\partial_r^2, \mathcal{B}^-)$.
(2) $r_\ell^{-} = \frac{1}{2}(\ell-2)\Xi_\ell$.

Proof. We follow the treatment in [11] to prove assertion (1). We consider the function $u(r,t) = e^{-tD_{1,\mathcal{B}^-}}1$. This solves the equations

$$(\partial_t + D_1)u = 0, \qquad \lim_{t\to 0} u(\cdot,t) = 1 \quad \text{in } L^2(M_1), \qquad \mathcal{B}^- u = 0.$$

Since u also solves the equations

$$(\partial_t + D_2)u = 0, \qquad \lim_{t\to 0} u(\cdot,t) = 1 \quad \text{in } L^2(M_2), \qquad \mathcal{B}^- u = 0,$$

we also have that $u(\cdot,t) = e^{-tD_{2,\mathcal{B}}}1$ as well. Since $\mathrm{dvol}_{M_2} = e^\alpha\, dr\, d\theta$,

$$\begin{aligned}\beta_{M_2}\big(1, e^{-\alpha}, D_2, \mathcal{B}^-\big)(t) &= \int_{r=0}^{1}\int_{\theta=0}^{2\pi} u(r,t)e^{-\alpha(r)}e^{\alpha(r)}\, d\theta\, dr \\ &= 2\pi \int_{r=0}^{1} u(r,t)\, dr = 2\pi \beta_{M_1}\big(1, 1, D_1, \mathcal{B}^-\big)(t).\end{aligned}$$

Since the structures are flat on M_1, $\beta_\ell^{\partial M_1}(1,1,D_1,\mathcal{B}^-) = 0$ for $\ell > 0$ and $\Delta_{M_1}^k 1 = 0$. We equate terms in the asymptotic expansion to see $\beta_\ell^{\partial M_2}(1, e^{-\alpha(r)}, D_2, \mathcal{B}^-) = 0$ for $\ell > 0$ as well.

We apply assertion (1). We use the formalism of Eq. (1.e). We have $ds^2_{M_2} = dr^2 + e^{2\alpha(r)}\, d\theta^2$ where $\alpha(0) = 0$ and $\alpha(r) = 0$ near $\alpha = 1$. We compute:

$$\begin{aligned}
&\Gamma_{122} = \Gamma_{212} = -\Gamma_{221} = e^{2\alpha}\alpha^{(1)}, && \omega_1 = \frac{1}{2}e^{-2\alpha}\Gamma_{221} = -\frac{1}{2}\alpha^{(1)},\\
&\omega_2 = 0, && E^{(\ell-2)} = \frac{1}{2}\alpha^{(\ell)} + \cdots,\\
&\phi_1^{(\ell)} = 0 + \cdots, && \phi_2^{(\ell)} = -\alpha^{(\ell)} + \cdots,\\
&\rho_{mm}^{(\ell-2)} = -\alpha^{(\ell)} + \cdots.
\end{aligned}$$

We examine the coefficient of $\alpha^{(\ell)}$ in β_ℓ for ℓ even:

$$\begin{aligned}
c_{\ell,1}^- \phi_1\phi_2^{(\ell)} &= -\Xi_\ell \alpha^{(\ell)} + \cdots,\\
e_{\ell,1}^- \phi_1\phi_2 E^{(\ell-2)} &= \frac{1}{2}\ell\cdot \Xi_\ell \alpha^{(\ell)} + \cdots,\\
r_\ell^- \phi_1\phi_2\rho_{mm}^{(\ell-2)} &= -r_\ell^- \alpha^{(\ell)} + \cdots.
\end{aligned}$$

It now follows from assertion (1) that $r_\ell^- = \frac{1}{2}(\ell-2)\Xi_\ell$. This completes the proof of Lemma 5.4 and thereby completes the proof of Theorem 1.9 as well. □

6. Estimating the heat trace asymptotics on a closed manifold

In this section, we shall prove Theorem 1.2. We shall proceed purely formally and shall use the discussion in Sections 1.7-1.8 of [27] (which is based on the Seeley calculus [44,45]) to justify our formal procedures. As in Eq. (1.b), let

$$D = -g^{ij}\partial_{x_i}\partial_{x_j} - A^k\partial_{x_k} - B$$

be an operator of Laplace type. Throughout this section, $C = C(M, g, D)$ will denote a generic constant which depends only on (M, g, D) (and hence also implicitly on m) but not on n; $c(m)$ will denote a generic constant which only depends on m. If we take $D = \Delta_g$, then $C = C(M, g)$.

We introduce coordinates $\xi = (\xi_1, \dots, \xi_m)$ on the cotangent bundle to express a covector in the form $\xi = \xi_i dx^i$. The symbol of D is $p_2(x,\xi) + p_1(x,\xi) + p_0(x)$ where:

$$p_2(x,\xi) := g^{ij}(x)\xi_i\xi_j, \qquad p_1(x,\xi) := A^k(x)\xi_k, \quad \text{and} \quad p_0 = B.$$

There are suitable normalizing constants involving factors of $\sqrt{-1}$ which we ignore in the interests of simplicity henceforth since they play no role in the estimates we shall be deriving. Let $\mathcal{C} := \mathbb{C} - [0,\infty)$ be the slit complex plane and let $\lambda \in \mathcal{C}$. Following the discussion in Lemma 1.7.2 of [27], one defines inductively:

$$\begin{aligned} r_0(x,\xi,\lambda) &:= \left(|\xi|^2 - \lambda\right)^{-1}, \\ r_n(x,\xi,\lambda) &:= -r_0(x,\xi,\lambda) \cdot \sum_{|\alpha|+j+2-k=n,\, j<n} d_\xi^\alpha p_k(x,\xi) \cdot d_x^\alpha r_j(x,\xi,\lambda)/\alpha!. \end{aligned} \tag{6.a}$$

In this sum $k = 0, 1, 2$ and $|\alpha| \leqslant 2 - k$. The symbol of e^{-tD} is given by:

$$e_0(x,\xi,t) + \cdots + e_n(x,\xi,t) + \cdots$$

where, following Eq. (1.8.4) of [27], one sets:

$$e_n(x,\xi,t) := \frac{1}{2\pi\sqrt{-1}} \int_\gamma e^{-t\lambda} r_n(x,\xi,\lambda)\, d\lambda;$$

here γ is a suitable contour about the positive real axis in the complex plane. Then, following Eq. (1.8.3) of [27], one may obtain the local heat trace invariants of Eq. (1.c) by setting:

$$a_n(x,D) = \left(\sqrt{\det(g_{ij})}\right)^{-1} \int_{\mathbb{R}^m} e_n(x,\xi,1)\, d\xi. \tag{6.b}$$

To measure the degree of an expression in the derivatives of the symbol, we set:

$$\operatorname{degree}\left(d_x^\alpha g^{ij}\right) = |\alpha|, \qquad \operatorname{degree}\left(d_x^\alpha A^k\right) = |\alpha| + 1, \qquad \operatorname{degree}\left(d_x^\alpha B\right) = |\alpha| + 2.$$

Note that if D is the scalar Laplacian, then $B = 0$ and $A^k = g^{-1}\partial_{x_i} g^{ij} g$ has degree 1 in the derivatives of the metric so this present definition is consistent with our previous definition in this special case. It is immediate from the definition that r_0 is of total degree 0 in the jets of the symbol of D. Furthermore, since

$$\mathrm{degree}\big(d_\xi^\alpha p_k\big) = 2-k \quad \text{and} \quad \mathrm{degree}\big(d_x^\alpha r_j\big) = |\alpha| + \mathrm{degree}(r_j),$$

we have by induction that

$$\mathrm{degree}(r_n) = n. \tag{6.c}$$

There is a similar grading on the variables (ξ, λ). One defines:

$$\mathrm{weight}(\xi_i) = 1 \quad \text{and} \quad \mathrm{weight}(\lambda) = 2.$$

It is then immediate that r_0 has weight -2 in (ξ, λ). Clearly

$$\mathrm{weight}\big(d_\xi^\alpha p_k\big) = k - |\alpha| \quad \text{and} \quad \mathrm{weight}\big(d_x^\alpha r_j\big) = \mathrm{weight}(r_j).$$

Thus it then also follows by induction from Eq. (6.a) that

$$\mathrm{weight}(r_n) = -2-n. \tag{6.d}$$

Let n be odd. Since the weight of $r_n(x,\xi,\lambda)$ is $-n-2$ in (ξ,λ), it follows that $e_n(x,\xi,1)$ is an odd function of ξ and hence the integral in Eq. (6.b) vanishes in this instance. This yields $a_n(x,D) = 0$ for n odd. Let $[\cdot]$ be the greatest integer function.

Lemma 6.1.

(1) *We may expand r_n in the form*:

$$r_n(x,\xi,\lambda) = \sum_{j=[\frac{1}{2}n]+1}^{2n+1} \sum_{|\beta|=2j-n-2} q_{n,m,j,\beta}(x,g)\xi^\beta r_0^j(x,\xi,\lambda).$$

(2) *There exists a constant $C(M,g)$ so that if $n = 2\bar{n} > 0$ and if $|\beta| = 2j-n-2$, then*

$$\left| \int_{\mathbb{R}^m} \int_\gamma e^{-\lambda} r_0^j(x,\xi,\lambda)\xi^\beta \, d\lambda \, d\xi \right| \leqslant \frac{C(M,g)^n}{\bar{n}!}.$$

Proof. We apply the recursive scheme of Eq. (6.a) to obtain an expression for r_n of the form given in assertion (1). By Eq. (6.c), r_n has degree n in the derivatives of the symbol of D. Thus there are at most n x-derivatives of r_0 which are involved in the process. Each x-derivative of r_0 adds one power of r_0 (other variables can be differentiated as well of course so we are obtaining an upper bound not a sharp estimate). Each step in the induction process adds 1 power of r_0. Thus $j \leqslant 2n+1$. By Eq. (6.d), r_n is homogeneous of weight $-n-2$ in (ξ,λ). Since $|\beta| - 2j = -n-2$ and $|\beta| \geqslant 0$, we may conclude that $j \geqslant 1 + \frac{1}{2}n \geqslant [\frac{1}{2}n] + 1$. Assertion (1) now follows.

We use the Cauchy integral formula to estimate:

$$\left|\int\limits_{\mathbb{R}^m}\int\limits_{\gamma} e^{-\lambda}\big(|\xi|^2-\lambda\big)^{-j}\xi^\beta\,d\lambda\,d\xi\right| \leqslant \frac{1}{(j-1)!}\left|\int\limits_{\mathbb{R}^m} e^{-|\xi|^2}\xi^\beta\,d\xi\right|.$$

The quadratic form g^{ij} is positive definite. Thus we may estimate $|\xi|^2 \geqslant \varepsilon|\xi|_e^2$ for some $\varepsilon = \varepsilon(M,g) > 0$ where $|\xi|_e^2 = \xi_1^2 + \cdots + \xi_m^2$ is the usual Euclidean length. Note that $|\xi^\beta| \leqslant |\xi|_e^{|\beta|}$. Since $e^{-|\xi|^2} \leqslant e^{-\varepsilon|\xi|_e^2}$, we may use spherical coordinates to estimate:

$$\left|\int\limits_{\mathbb{R}^m}\int\limits_{\gamma} e^{-\lambda}\big(|\xi|^2-\lambda\big)^{-j}\xi^\beta\,d\lambda\,d\xi\right| \leqslant \frac{1}{(j-1)!}\int\limits_{r=0}^{\infty} e^{-\varepsilon r^2} r^{|\beta|+m}\,dr\,\mathrm{vol}_{m-1}\big(S^{m-1}, g_{S^{m-1}}\big).$$

Since $|\beta| \leqslant 2j \leqslant 4n+4$ is uniformly and linearly bounded in n, we may rescale to remove ε in $e^{-\varepsilon r^2}$ at the cost of introducing a suitable multiplicative constant. We may then evaluate the integral to estimate:

$$\left|\int\limits_{\mathbb{R}^m}\int\limits_{\gamma} e^{-\lambda}\big(|\xi|^2-\lambda\big)^{-j}\xi^\beta\,d\lambda\,d\xi\right| \leqslant C(M,g)^n\frac{(\frac{|\beta|+m}{2})!}{(j-1)!}.$$

Since $j-1-\frac{1}{2}|\beta| = \bar{n}$ the desired estimate follows; the shift by m can be absorbed into $C(M,g)^n$ since we have restricted to $n>0$. □

Let $D_\varepsilon^{\mathbb{C}} \subset \mathbb{C}^m$ be the complex polydisk of radius ε of real dimension $2m$ about the origin in $\mathbb{C}^m$ given by setting:

$$D_\varepsilon^{\mathbb{C}} := \big\{\vec{z} = (z_1,\dots,z_m) \in \mathbb{C}^m\colon\ |z_i| \leqslant \varepsilon \text{ for } 1 \leqslant i \leqslant m\big\}.$$

We let $D_\varepsilon^{\mathbb{R}} = D_\varepsilon^{\mathbb{C}} \cap \mathbb{R}^m$ be the corresponding real polydisk. We also consider the submanifold S_ε of real dimension m in $\mathbb{C}^m$ (which is not the boundary either of the complex polydisk $D_\varepsilon^{\mathbb{C}}$ or of the real polydisk $D_\varepsilon^{\mathbb{R}}$) given by:

$$S_\varepsilon := \big\{\vec{z} \in \mathbb{C}^m\colon\ |z_i| = \varepsilon \quad \text{for } 1 \leqslant i \leqslant m\big\}.$$

We consider the holomorphic m-form

$$dw = (2\pi\sqrt{-1}\,)^{-m}\,dw_1\dots dw_m.$$

Let f be a holomorphic function on the interior of $D_\varepsilon^{\mathbb{C}}$ which extends continuously to all of $D_\varepsilon^{\mathbb{C}}$ and let α is a multi-index. If z belongs to the interior of the polydisk $D_\varepsilon^{\mathbb{C}}$, then we shall define:

$$\mathcal{I}_\alpha(f)(z) := \int\limits_{w\in S_\varepsilon} f(w)(w_1-z_1)^{-1-\alpha_1}\dots(w_m-z_m)^{-1-\alpha_m}\,dw.$$

We may then use the Cauchy integral formula to represent:

$$\partial_z^\alpha f(z) = \alpha! \mathcal{I}_\alpha(f) \quad \text{for } z \in \operatorname{int}\bigl(D_\varepsilon^{\mathbb{C}}\bigr).$$

Let $\beta = \beta(i, \alpha)$ be the multi-index $(\alpha_1, \dots, \alpha_{i-1}, \alpha_i + 1, \alpha_{i+1}, \dots, \alpha_m)$. We then have:

$$\partial_{x_i} \mathcal{I}_\alpha(f)(x) = (\alpha_i + 1) \cdot \mathcal{I}_\beta(f)(x). \tag{6.e}$$

We introduce variables $\{f_\nu\}$ for the $\{g^{ij}, A^k, B\}$ variables; we have a total of $\frac{1}{2}m(m-1) + m + 1$ such variables. Since we are in the real analytic setting, we can choose real analytic coordinates about each point P of M which are real analytically equivalent to the polydisk $D_2^{\mathbb{R}}(P)$ of radius 2 in such a way that the variables $\{f_\nu\}$ extend continuously to $D_2^{\mathbb{C}}(P)$ with f_ν holomorphic on the interior of $D_2^{\mathbb{C}}(P)$. The functions $|f_\nu|$ are uniformly bounded on $D_2^{\mathbb{C}}(P)$. If $z \in D_1^{\mathbb{R}}(P)$ and $|w| \in S_2^{\mathbb{C}}(P)$, then $|z_i - w_i| \geqslant 1$ and thus we have uniform estimates

$$\bigl|\mathcal{I}_\alpha(f_\nu)(z)\bigr| \leqslant C(M, D) \quad \text{for any } \nu, \alpha. \tag{6.f}$$

We decompose r_n in terms of monomials of the form

$$r_0^j \xi^\beta \cdot g^{i_1 j_1} \cdot \ldots \cdot g^{i_a j_a} \cdot I_{\alpha_1}(f_{\nu_1}) \cdot \ldots \cdot I_{\alpha_b}(f_{\nu_b}). \tag{6.g}$$

Here we assume degree$\{\partial_\alpha^x f_{\nu_i}\} > 0$ since we have made explicit the dependence on the variables of degree 0. Thus $b \leqslant n$ since, by Eq. (6.c), r_n is homogeneous of degree n in the jets of the symbol. There are no g^{ij} variables in r_0. Each multiplication by $\partial_\xi^\alpha p_2$ can add at most one g^{ij} variable; each multiplication by $\partial_{\xi_i}^\alpha p_1$ or p_0 adds no g^{ij} variable. Each application of ∂_x^α to r_j does not add a g^{ij} variable (and can in fact reduce the number of g^{ij} variables if they are differentiated). Thus the number of g^{ij} variables is at most n. Thus in considering monomials of the form given in Eq. (6.g), we may assume $a \leqslant n$. We summarize these constraints:

$$j \leqslant 2n + 1, \qquad -n - 2 = |\beta| - 2j, \qquad a \leqslant n, \quad \text{and} \quad b \leqslant n. \tag{6.h}$$

Lemma 6.2. *Let $c(m) := 50m^2$. We can decompose r_n as the sum of at most $c(m)^n n!$ monomials of the form given in* Eq. (6.g) *satisfying the constraints of* Eq. (6.h) *where the coefficient of each monomial has absolute value at most* 1.

Proof. Since r_0 can be written as a single monomial with coefficient 1, we proceed by induction.

(1) Consider $-r_0 \partial_{\xi_k} p_2 \cdot \partial_{x_k} r_{n-1}$. Each k generates m terms so there are m^2 terms generated in this way. Differentiating r_0^j generates at most $3n$ terms since $j \leqslant 3n$ by Eq. (6.h). Differentiating the g^{ij} variables generates at most n terms since $a \leqslant n$. Differentiating the $\mathcal{I}$ variables generates at most $b + \sum |\alpha_i| \leqslant 2n$ terms by Eq. (6.e). Thus we generate at most $m^2(3n + n + 2n) = 6m^2 n$ terms from each monomial of r_{n-1}. This can be written in terms of at most

$$6m^2 n \cdot c(m)^{m-1}(n-1)! = 6m^2 c(m)^{m-1} n! \quad \text{monomials.}$$

(2) Consider $-r_0\partial_{\xi_{k_1}}\partial_{\xi_{k_2}}p_2\cdot\partial_{x_{k_1}}\partial_{x_{k_2}}r_{n-2}$. A similar argument shows this generates at most $m^2(6n)(6(n-1))$ new terms from each monomial of r_{n-2}. This can be written in terms of at most

$$36m^2n(n-1)\cdot c(m)^{n-2}(n-2)!\leqslant 36m^2\cdot c(m)^{n-1}n!\quad\text{monomials.}$$

(3) Consider $-r_0A^k\xi_kr_{n-1}$. This can be written in terms of at most

$$m\cdot c(m)^{m-1}(n-1)!\leqslant m^2c(m)^{m-1}n!\quad\text{monomials.}$$

(4) Consider $-r_0A^k\partial_{x_k}r_{n-2}$. This can be written in terms of at most

$$6mn\cdot c(m)^{n-2}(n-2)!\leqslant 6m^2c(m)^{n-1}n!\quad\text{monomials.}$$

(5) Consider $-r_0Br_{n-2}$. This can be written in terms of at most

$$c(m)^{n-2}(n-2)!\leqslant m^2c(m)^{n-1}n!\quad\text{terms.}$$

The above argument shows that r_n can be decomposed as the sum of at most of $50m^2\cdot c(m)^{n-1}n!=c(m)^n\cdot n!$ monomials each of which has a coefficient of absolute value at most 1. □

Proof of Theorem 1.2(1). We consider monomials where the coefficient has absolute value at most 1. We have shown that there exists a constant $c(m)$ so that r_n can be written in terms of at most $c(m)^nn!$ such monomials. We may then use the constraints of Eq. (6.h), the estimates of Eq. (6.f), and the estimate of Lemma 6.1 to construct a new constant $\tilde{C}(M,g)$ and complete the proof of Theorem 1.2(1) by bounding:

$$\left|a_n(x,D)\right|\leqslant c(m)^nn!\cdot C(M,g,D)^{2n}\cdot C(M,g)^n\frac{1}{\bar{n}!}\leqslant\tilde{C}(M,g,D)^n\bar{n}!.\qquad\square$$

Proof of Theorem 1.2(2). Let P be a point of a closed real analytic Riemannian manifold (M,g). Let f be a real analytic function on M so that $df(P)\neq 0$. Since f is continuous and M is compact, $|f|$ is bounded. By rescaling and shifting f, we may suppose without loss of generality that $f(P)=0$ and that $|f(x)|\leqslant 1$ for all points x of M. We make a real analytic change of coordinates to assume that $g^{ij}(P)=\delta_{ij}$ and that $f(x)=c_f\cdot x_1$ near P. We shall choose $\varepsilon_k=\pm 1$ recursively and define:

$$h(x)=\sum_{k=3}^{\infty}\varepsilon_k2^{-k}f(x)^{2k}.$$

This series converges uniformly in the real analytic topology so h is real analytic. Let $\mathcal{E}_{\bar{n}}(\cdot)$ be a generic invariant which only depends on the parameters indicated. Let $g_h=e^{2h}g$. Let $\bar{n}\geqslant 3$. We use Theorem 1.8 to see that:

$$
\begin{aligned}
&\big(\partial_{x_1}^{2\bar n}h\big)(P)=\varepsilon_{\bar n}2^{-\bar n}c_f^{2\bar n}(2\bar n)!+\mathcal{E}_{\bar n}^1(\varepsilon_1,\dots,\varepsilon_{\bar n-1}),\\
&\tau_{g_h}(P)=c_m\big(\partial_{x_1}^2 h\big)(P)+\text{lower order terms}\quad\text{for some } |c_m|\geqslant 1,\\
&(-1)^{\bar n-1}\Delta_{g_h}^{\bar n-1}\tau_{g_n}(P)=\varepsilon_{\bar n}c_m2^{-\bar n}c_f^{2\bar n}(2\bar n)!+\mathcal{E}_{\bar n}^2(\varepsilon_1,\dots,\varepsilon_{\bar n-1},g),\\
&a_{2\bar n}(P,\Delta_g)=(-1)^{\bar n-1}\frac{\bar n\cdot\bar n!}{(2\bar n+1)!}\Delta^{\bar n-1}\tau+\text{lower order terms}\\
&\qquad=c_m\frac{\bar n\cdot\bar n!}{(2\bar n+1)!}c_f^{2\bar n}\varepsilon_{\bar n}2^{-\bar n}(2\bar n)!+\mathcal{E}_{\bar n}^3(\varepsilon_1,\dots,\varepsilon_{\bar n-1},g).
\end{aligned}
$$

We set

$$
\varepsilon_{\bar n}:=\left\{\begin{array}{ll}+1 & \text{if } c_m\mathcal{E}_{\bar n}^3(\varepsilon_1,\dots,\varepsilon_{\bar n-1},g)\geqslant 0\\ -1 & \text{if } c_m\mathcal{E}_{\bar n}^3(\varepsilon_1,\dots,\varepsilon_{\bar n-1},g)<0\end{array}\right\}.
$$

With this choice of $\varepsilon_{\bar n}$, there is no cancellation. As $\frac{1}{2}\frac{\bar n}{2\bar n+1}\geqslant\frac{3}{14}$ for $\bar n\geqslant 3$, we obtain the desired estimate:

$$
\big|a_{2\bar n}(P,\Delta_g)\big|\geqslant c_m\frac{\bar n\cdot\bar n!}{(2\bar n+1)!}c_f^{2\bar n}2^{-\bar n}(2\bar n)!\geqslant\frac{\bar n}{2\bar n+1}c_f^{2\bar n}2^{-\bar n}\cdot\bar n!\geqslant\left(\frac{3}{14}c_f^2\right)^{\bar n}\bar n!.\qquad\square
$$

7. Growth of heat content asymptotics

This section is devoted to the proof of Theorem 1.4. We first examine a product manifold $[0,1]\times N$. Let $\{\varepsilon_{\bar\ell}\}$ be a sequence of signs to be chosen recursively. We replace the function $f(x)$ of the previous section by $\sin(x)$ and define:

$$
h(x):=\sum_{\nu=1}^{\infty}\varepsilon_\nu 2^{-\nu}\sin(x)^{2\nu}.
$$

This series converges in the real analytic topology to a real analytic function h which is periodic with period 2π and which satisfies $h(0)=h(2\pi)=0$. We set

$$
g_M:=e^{2h}\big(dx^2+g_N\big).
$$

The inward unit normal is given at 0 by $\nu(0)=\partial_x$ and at 2π by $\nu(2\pi)=-\partial_x$. If j is odd, then $\{\partial_x^j h\}(0)=\{\partial_x^j h\}(2\pi)=0$ since h is an even function. And clearly we have that $\{(\partial_x^j)h\}(0)=\{(-\partial_x)^j h\}(2\pi)$ if j is even. Consequently

$$
h^{(j)}(0)=h^{(j)}(2\pi)\quad\text{for any } j.
$$

This ensures that the behaviour of h is the same on the boundary components and gives rise to the factor of $2\,\mathrm{vol}_{m-1}(N,g_N)$ in Eq. (7.a) below. We have:

$$
h^{(2\bar\ell)}(0)=\varepsilon_{\bar\ell}\cdot 2^{-\bar\ell}(2\bar\ell)!+\mathcal{E}_{\bar\ell}^4(\varepsilon_1,\dots,\varepsilon_{\bar\ell-1}).
$$

Since $m \geqslant 2$, there is a non-zero constant c_m with $|c_m| \geqslant 1$ which only depends on m and not on $\bar{\ell}$ so that:

$$\rho_{mm}^{(2\bar{\ell}-2)}(0) = \varepsilon_{\bar{\ell}} \cdot c_m 2^{-\bar{\ell}}(2\bar{\ell})! + \mathcal{E}_{\bar{\ell}}^5(\varepsilon_1, \ldots, \varepsilon_{\bar{\ell}-1}, g_N).$$

We may then apply Theorem 1.9 to express:

$$\beta_{2\bar{\ell}}^{\partial M}\big(1, 1, \Delta_{M,g_M}, \mathcal{B}^-\big) = \varepsilon_\ell \left\{ \frac{1}{2}(2\bar{\ell} - 2)\Xi_{2\bar{\ell}} c_m 2^{-\bar{\ell}}(2\bar{\ell})! \cdot 2\operatorname{vol}_{m-1}(N, g_N) \right\} + \mathcal{E}_{2\bar{\ell}}^6(\varepsilon_1, \ldots, \varepsilon_{\bar{\ell}-1}, g_N). \tag{7.a}$$

Set

$$\varepsilon_{\bar{\ell}} := \left\{ \begin{array}{ll} +1 & \text{if } \mathcal{E}_{2\bar{\ell}}^6(\varepsilon_1, \ldots, \varepsilon_{\bar{\ell}-1}, g_N) > 0 \\ -1 & \text{if } \mathcal{E}_{2\bar{\ell}}^6(\varepsilon_1, \ldots, \varepsilon_{\bar{\ell}-1}, g_N) \leqslant 0 \end{array} \right\}.$$

Since there is no cancellation in Eq. (7.a), we may estimate:

$$\big|\beta_{2\bar{\ell}}^{\partial M}\big(1, 1, \Delta_{M,g_M}, \mathcal{B}^-\big)\big| \geqslant \frac{1}{2}(2\bar{\ell} - 2)\Xi_{2\bar{\ell}} c_m (2\bar{\ell})! \varepsilon_{\bar{\ell}} 2^{-\bar{\ell}} \cdot 2\operatorname{vol}_{m-1}(N, g_N).$$

The desired estimate in assertion (1) of Theorem 1.4 now follows since:

$$\begin{aligned} \left| \frac{1}{2}(2\bar{\ell} - 2)\Xi_{2\bar{\ell}} c_m (2\bar{\ell})! \varepsilon_{\bar{\ell}} 2^{-\bar{\ell}} \right| &\geqslant (2\bar{\ell} - 2)\frac{2}{2\bar{\ell}+1} \cdots \frac{2}{3}\frac{2}{\sqrt{\pi}} 2^{-\bar{\ell}} 1 \cdot 2 \cdot 3 \ldots \cdot 2\bar{\ell} \\ &= \frac{2\bar{\ell} - 2}{2\bar{\ell}+1} 2 \cdot 4 \cdot \cdots \cdot 2\bar{\ell} \geqslant \frac{4}{14} 2^{\bar{\ell}} \bar{\ell}! \geqslant \bar{\ell}! \quad \text{for } \bar{\ell} \geqslant 3. \end{aligned}$$

We now turn to the case of the ball and apply a similar analysis to establish assertion (2) of Theorem 1.4. The functions $\sin(x)$ is now replaced by the function $(x_1^2 + \cdots + x_m^2 - 1)^{2\nu}$, the operator ∂_x is replaced by the radial derivative ∂_r, and the boundary components $x = 0$ and $x = 2\pi$ are replaced by the single boundary component $r = 1$. The remainder of the argument is the same and is therefore omitted. □

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