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Discretizations of nonlinear differential equations using explicit nonstandard methods

Ronald E. Mickens *

Department of Physics, Clark Atlanta University, Atlanta, GA 30314, USA

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Abstract

In a recent paper, Chen and Solis investigated the appearance of spurious solutions when first-order ODEs are discretized using Runge–Kutta schemes. They concluded that the reliability of the numerical solutions to a particular ODE could be verified only by constructing several discrete models and comparing their numerical results with the known properties of the exact solutions. We demonstrate that by using nonstandard schemes, all the difficulties found by Chen and Solis can be eliminated, and that qualitatively correct numerical solutions are obtained for all values of the step size. We illustrate these issues by applying nonstandard finite-difference techniques to the logistic, sine, cubic, and Monod equations.

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A major difficulty in the numerical integration of ordinary differential equations (ODE) is the existence of numerical instabilities [2,5]. These are solutions to the discrete equations that do not correspond to any of the solutions to the original ODEs. In a recent paper, Chen and Solis [1] investigated the appearance of such spurious solutions when first-order equations are discretized using Runge–Kutta schemes. They concluded that the reliability of the numerical solutions for a particular ODE can be verified only by constructing several discrete models of the equation and then comparing these results with known properties of the exact solutions to the ODE. While they considered only first-order ODEs with unimodal functions on the right sides, their results have general application to other types of functions.

Our purpose is to demonstrate that by using the nonstandard finite-difference schemes obtained by the procedures of Mickens [5,6], the difficulties faced by Chen and Solis can be eliminated, and

* Tel.: +1-440-880-6923; fax: +1-404-880-6258.
E-mail address: rohrs@math.gatech.edu (R.E. Mickens)

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that qualitatively correct numerical solutions exist for all positive values of the step size. This result also means that the bifurcation phenomena found by Chen and Solis does not occur.

We now give a brief summary of the nonstandard method for constructing finite-difference schemes for a first order ODE. References [5,6] provide both the general philosophy and principles of the procedure, and the particular details of the constructions.

Consider the ODE

$$\frac{dx}{dt} = F(x, \lambda), \tag{1}$$

where λ represents the various parameters appearing in the function F. The simplest nonstandard finite-difference scheme (NFDS) is constructed by making the replacements [6]

$$t \to t_n = hn, \quad x(t) \to x_n, \quad F(x) \to F(x_n), \tag{2}$$

$$\frac{dx}{dt} \to \frac{x_{n+1} - x_n}{\phi}, \tag{3}$$

where

$$\phi = \frac{1 - e^{-Rh}}{R} \tag{4}$$

and R is calculated from a knowledge of the fixed points of Eq. (1) and the derivative of $F(x, \lambda)$ evaluated at the fixed point:

$$F(\bar{x}) = 0, \qquad \{\bar{x}_i;\ i = 1, 2, \ldots, I\}, \tag{5a}$$

$$R_i = \left.\frac{dF}{dx}\right|_{x=\bar{x}_i}, \tag{5b}$$

$$R = \mathrm{Max}\{|R_i|;\ i = 1, 2, \ldots, I\}. \tag{5c}$$

When $F(x, \lambda)$ is a polynomial function of x, a more sophisticated nonstandard model can be constructed by using nonlocal [5] representations of terms that appear in F, such as ax^m, where a is a constant; for example,

$$ax^m \to \begin{cases} (a+1)(x_n)^m - (x_n)^{m-1}x_{n+1} & \text{if } a > 0, \\ -(|a|+1)(x_n)^{m-1}x_{n+1} + (x_n)^m & \text{if } a < 0. \end{cases} \tag{6}$$

We illustrate the methods by applying them to the logistic ODE

$$\frac{dx}{dt} = x - x^2. \tag{7}$$

This equation has two fixed points: $\bar{x}_1 = 0$ and $\bar{x}_2 = 1$; also, $dF(0)/dx = 1$ and $dF(1)/dx = -1$, hence $R = 1$. These results lead to the following NSFDS:

$$\frac{x_{n+1} - x_n}{(1 - e^{-h})} = x_n - (x_n)^2. \tag{8}$$

Using Eq. (6), we obtain

$$\frac{x_{n+1} - x_n}{(1 - e^{-h})} = (2x_n - x_{n+1}) - [2x_n x_{n+1} - (x_n)^2] \tag{9}$$

or

$$x_{n+1} = \left[\frac{(1+2\phi)+\phi x_n}{(1+\phi)+2\phi x_n}\right] x_n, \qquad \phi = 1 - \mathrm{e}^{-h}. \tag{10}$$

It is straightforward to show [5,6] for any $h > 0$ and $x_0 > 0$, the solutions to the 1-dim map, given by Eq. (10), have exactly the same qualitative behavior as that of the corresponding solutions to Eq. (7). In particular, Eq. (10) has only two fixed points, $\bar{x}^{(1)} = 0$ and $\bar{x}^{(2)} = 1$, and the first is unstable, while the second is stable. Further, if $x_0 > 1$, then x_k monotonically decreases to $\bar{x}^{(2)} = 1$, while if $0 < x_0 < 1$, x_k monotonically increases to $\bar{x}^{(2)} = 1$. This is exactly the type of behavior that the solutions to the logistic ODE exhibit. Of critical importance is the fact that these results hold good for any value of the time step $h > 0$ [5]. Finally, it should be noted that while Eq. (10) is not an "exact" scheme [5] for Eq. (7), i.e.,

$$\frac{x_{n+1} - x_n}{(\mathrm{e}^h - 1)} = x_n - x_{n+1}x_n, \tag{11}$$

it possesses all the relevant features of this scheme with regard to the properties of its solutions. The significance of this fact, when properly generalized, is that NFDS, as formulated by Mickens [5], always leads to discrete models of first-order ODEs for which the numerical solutions have the same qualitative features as those of the corresponding solutions to the ODEs. The mathematical analysis behind this assertion and the related numerical confirmations are given in Chapters 3 and 4 of Mickens [5].

We now present, without giving the technical details, the NSFDS for three other first-order ODEs investigated by Chen and Solis [1]. Note that we have rescaled the time variable such that the parameter λ does not appear in the sine and cubic equations.

Sine equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sin(\pi x), \tag{12a}$$

$$\frac{x_{n+1} - x_n}{\phi} = \sin(\pi x_n), \qquad \phi = \frac{1 - \mathrm{e}^{-\pi h}}{\pi}. \tag{12b}$$

Cubic equation

By means of a linear-dependent variable transformation, the cubic equation studied by Chen and Solis can be rewritten as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(1 - x^2). \tag{13a}$$

A NSFDS that incorporates the maximum symmetry in modeling the nonlinear term is

$$x_{n+1} = \left[\frac{(2+\phi)+\phi(x_n)^2}{(2-\phi)+3\phi(x_n)^2}\right] x_n, \qquad \phi = \frac{1 - \mathrm{e}^{-2h}}{2}. \tag{13b}$$

See [4,5, pp. 115–116] for details.

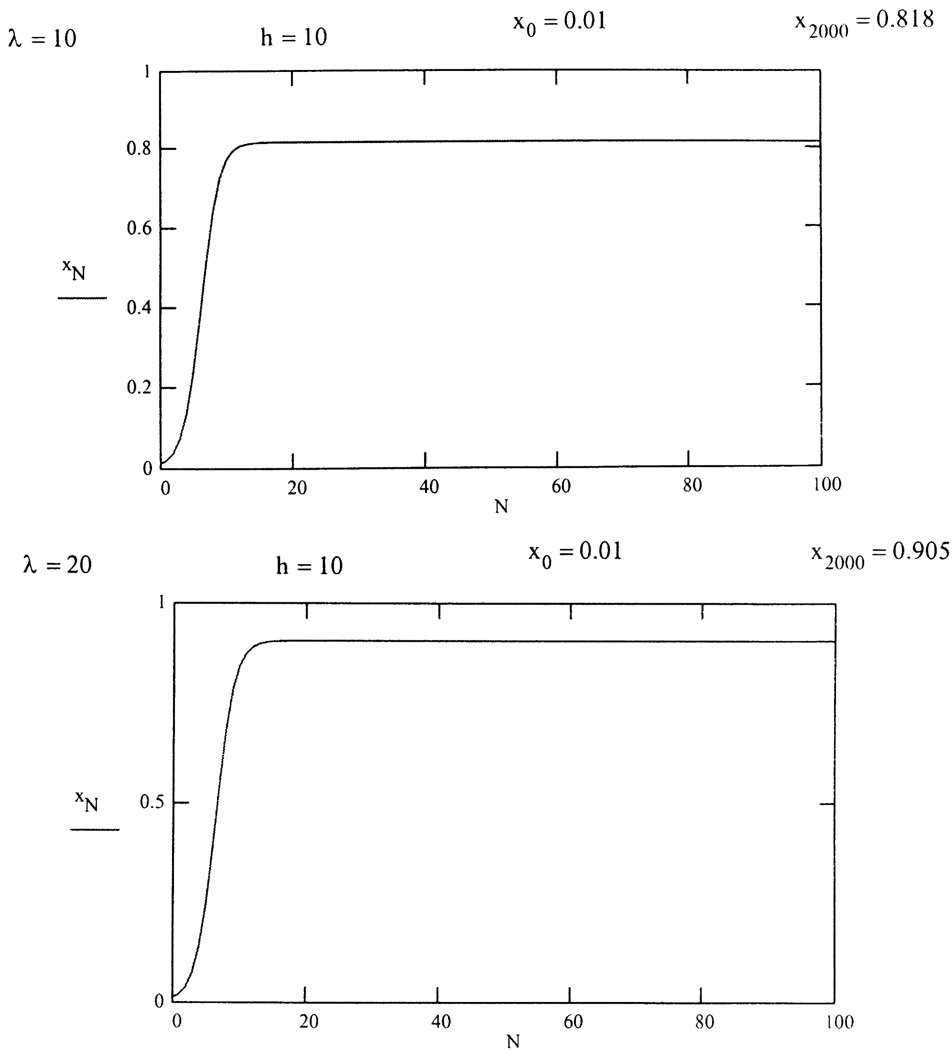

Fig. 1. Numerical solutions to Monod's equation for a step size $h = 10$ and two values of λ.

Modified Monod equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{(\lambda-1)x-(\lambda+1)x^2}{1+x}, \qquad \lambda > 1, \tag{14a}$$

where for $\lambda > 1$, the fixed point

$$\bar{x}^{(2)} = \frac{\lambda-1}{\lambda+1} \tag{14b}$$

is stable and the second fixed point $\bar{x}^{(1)} = 0$, is unstable. The NSFDS is

$$\frac{x_{n+1} - x_n}{\phi} = \left[\frac{(\lambda - 1) + (1 + \lambda)x_{n+1}}{1 + x_n}\right] x_n, \tag{14c}$$

$$\phi = \frac{1 - e^{-Rh}}{R}, \qquad R = \lambda - 1. \tag{14d}$$

We have used the above NSFDS to determine numerical solutions to the three ODEs. Values of the step size h ranged in the interval $(0, 50]$, while the parameter λ, in the modified Monod equation took values up to sixty. In addition, for all the equations, a large number of initial conditions, x_0, were selected for study. In none of these cases did spurious solutions appear in the numerical results. Fig. 1 gives typical results for the Monod equation for large values of both λ and h. All of the standard finite-difference schemes produced numerical instabilities for these parameter values. But, as can be clearly seen, our NSFDS gave the correct qualitative behavior of the solution along with convergence to the proper value of the fixed point at $\bar{x} = (\lambda - 1)/(\lambda + 1)$. In contrast to what occurs in the standard Runge–Kutta schemes [1], no bifurcation occurs as λ changes value. This situation is not unexpected given that the prime purpose for constructing NSFDS is to have discretizations for which spurious solutions are not present. Further, and this is an especially important point, our prior rigorous mathematical analysis of these schemes [5,6], based on results from the theory of 1-dim mappings [3], shows that spurious solutions can not exist.

In summary, the use of NSFDS leads to asymptotic dynamics and numerical results that are always qualitatively the same as the corresponding solutions of first order ODEs for any value of the step size, $h > 0$. The number, location, and stability properties of the fixed points are exactly preserved. Further, no spurious bifurcations take place as parameters are varied. While these NSFDS are not exact schemes [5,7], they provide close qualitative and quantitative discrete representations of the actual continuous solutions to the ODEs. Our major conclusion is that the issues raised by Chen and Solis [1] are resolved by the application of NSFDS to the numerical integration of first order ODEs.

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