NORTH-HOLLAND

Constructing the Polynomial Identities and Central Identities of Degree < 9 of 3×3 Matrices

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ABSTRACT

We present an algorithm for computing an independent generating set for the multilinear identities and the multilinear central identities of the $m \times m$ matrices over a field ϕ of characteristic zero or a large enough prime. Then we use it to construct all the multilinear identities and all the multilinear central identities of degree < 9 for $M_3(\phi)$. © Elsevier Science Inc., 1997

1. INTRODUCTION

Let R be an algebra over a field ϕ, and $X = \{x_1, x_2, \dots\}$ be a countable set of symbols. A polynomial $f(x_1, \dots, x_n)$ from the free associative algebra $\phi\langle X \rangle$ is said to be a *polynomial identity* of R if $f(x_1, \dots, x_n) = 0$ for all $x_1, \dots, x_n \in R$. It is called a *polynomial central identity* of R if f is not an identity and $f(x_1, \dots, x_n) \in C(R)$ for all $x_1, \dots, x_n \in R$, where $C(R)$ denotes the center of R. We usually just say simply *identity* and *central identity*. The *degree* of each is the degree of the polynomial.

Let $M_n(\phi)$ be the ring of $n \times n$ matrices with entries from a field ϕ, and Sym(n) be the symmetric group of n objects. Define $[x_1, x_2, \dots, x_{2n}]_\pi := x_{(1)\pi^{-1}} x_{(2)\pi^{-1}} \cdots x_{(2n)\pi^{-1}}$, where $(i)\pi^{-1}$ is π^{-1} with its argument i written on the left. In a 1951 paper, Amitsur and Levitzki [1] show that $S_{2n}(x_1, \dots, x_{2n}) := \sum_{\pi \in \mathrm{Sym}(2n)} \mathrm{sgn}(\pi)[x_1, x_2, \dots, x_{2n}]_\pi$ is an identity of $M_n(\phi)$. This identity is known as the standard identity of $n \times n$ matrices.

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In 1956 Kaplansky [9] asked whether there exists a nonzero central identity for $M_n(\phi)$. Obviously, the constant polynomial or the constant polynomial added to the standard identity qualifies. Moreover, the well-known polynomial $(xy - yx)^2$ solves the problem for $n = 2$.

To avoid the above-mentioned trivial cases, in 1970 Kaplansky [10] rephrased the problem and asked whether there exists a homogeneous multilinear central identity of $M_n(\phi)$ of positive degree for $n \geqslant 3$. The same problem was also brought up in the 10th All-Union Algebra Colloquium, which took place in September 1969 in Novosibirsk, Russia. In 1972, Formanek [8] proved the existence of a central identity for each algebra $M_n(\phi)$. In the same year, Razmyslov [12] found a finite generating set for the identities of $M_2(\phi)$, where ϕ has characteristic zero. In 1973, Razmyslov [13] constructed a new central identity; its degree was $3n^2 - 1$. In the same year, U. Leron [11] proved that every multilinear identity of $M_n(\phi)$ with degree $2n + 1$ is a "consequence" of the standard identity S_{2n}.

In 1983, V. Drensky and A. Kasparian [6] showed that all the identities of degree < 9 of 3×3 matrices are "consequences" of the standard identity of degree 6. Furthermore, they found a central identity of degree 8 and showed that there are no central identities of lower degrees (see [5]).

Recently, V. Drensky and G. Piacentini [7] found a central identity of degree 13 for $M_4(\phi)$ with ϕ of characteristic zero which agrees with the conjecture [7, p. 1] that the minimal degree of a central identity of $M_n(\phi)$ is $(n^2 + 3n - 2)/2$.

Our procedure produces an independent generating set that implies all the identities and all the central identities of degree $n < 9$ of $M_3(\phi)$, where ϕ is a field of characteristic zero or $p > n$.

The method uses group representation theory and relies heavily on computational techniques. We first tested it by finding a set of independent multilinear identities for $M_2(Q)$, with Q the rationals, and then comparing the identities with Razmyslov's identities [12] (also see [4]).

Our results for 3×3 matrices are consistent with the literature and yield a new central identity in degree 8.

2. BASIC DEFINITIONS AND CONCEPTS

Let $A = \{f_1, \ldots, f_k\}$ be a nonempty set of polynomials of $\phi\langle X \rangle$ with f_i involving n_i variables. The so-called *T-ideal generated by A* is the ideal of $\phi\langle X \rangle$ generated by $B = \{f_i(y_1, \ldots, y_{n_i}) | i = 1, \ldots, k$ and $y_j \in \phi\langle X \rangle$ for $j = 1, \ldots, n_i\}$. The elements in the T-ideal generated by A are called the *identities implied by A*. An identity of a ϕ-algebra R is said to be *minimal* if it is not implied by a set of identities of R of lower degrees. Two identities f and g are said to be *equivalent* if f implies g and vice versa. Two central identities f and g of the algebra R are called equivalent if f or g, together with the polynomial identities of R, generate the same T-ideal.

Since concepts related to representations theory of the symmetric group of n objects, Sym(n), play a role in our procedure, we need some definitions due to Young and his students (also see [2]).

The *group ring* or the *group algebra* over the symmetric group Sym(n) consists of all sums of the form

$$a = \sum_{s \in \mathrm{Sym}(n)} \alpha(s)s \tag{1}$$

with arbitrary coefficients $\alpha(s) \in \phi$. This group ring is denoted by O_G.

A *frame* of degree n consists of n boxes arranged in rows in such a way that the leftmost boxes are located one under another and $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_r$, where m_i denotes the length of the ith row for $i = 1, \dots, r$. We shall refer to such a frame by $(m_1, \dots, m_r)$. Suppose two frames F and F' of the same degree are respectively given by $(m_1, \dots, m_r)$ and $(m'_1, \dots, m'_k)$. Then we write $F > F'$ if the first nonzero $m_i - m'_i$ is positive. The frames of a given degree are always listed in decreasing order. Here are all the frames of degree 4:

```
                                             [ ]
                                  [ ][ ]     [ ]
[ ][ ][ ][ ],  [ ][ ][ ] ,  [ ][ ]  [ ] ,    [ ].
               [ ]          [ ][ ], [ ]      [ ]
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A *tableau* of degree n is created by putting the numbers 1 to n into the n boxes of a frame. A *standard tableau* is a tableau in which the numbers are increasing in every row from left to right and in every column from top to bottom. We shall enumerate the standard tableaux of a given frame in the "systematic" or "dictionary" order $T_1, \dots, T_f$. The "nonstandard" tableaux are assumed to be ordered in some arbitrary way $T_{f+1}, \dots, T_{n!}$. As an example, we list all the standard tableaux $T_1, \dots, T_5$ of the (3, 2) frame:

```
[1][2][3]    [1][2][4]    [1][2][5]    [1][3][4]    [1][3][5]
[4][5]   ,   [3][5]   ,   [3][4]   ,   [2][5]   ,   [2][4]   .
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A permutation $\pi \in \mathrm{Sym}(n)$ applied to a tableau T is simply a renumbering of the contents of the boxes in T and is denoted by πT. For instance, if we apply the permutation (1 3 2) to the tableau [2][1][4][3][5], we get [1][3][4][2][5]. If T_i and T_j are two tableaux belonging to a fixed frame F, then the permutation that takes T_j to T_i is denoted by S_{ij} or $S_{i,j}$.

Given a tableau T_i, set

$$\bar{e}_i := \sum_{pq} \mathrm{sgn}(q)\, pq \in O_G, \tag{2}$$

the sum being taken over all products pq of permutations, where p is a horizontal and q is a vertical permutation for T_i. For instance, for the tableau

$$T_2 = \begin{matrix}[1][3]\\ [2][4]\end{matrix},$$

we have

$$\begin{aligned}\bar{e}_2 = \{&\mathrm{I} + (1\ 3) + (2\ 4) + (1\ 3)(2\ 4)\}\\ &\times\{\mathrm{I} - (1\ 2) - (3\ 4) + (1\ 2)(3\ 4)\}.\end{aligned}$$

Let $e_i := (d/n!)\bar{e}_i$, where d is the dimension of the left ideal in O_G generated by $\bar{e}_i$. Sometimes we use a superscript to distinguish the $e_i S_{ij}$ belonging to the tableaux of different frames. For example, $e_i^k S_{ij}^k$ belongs to the kth frame. If we are working with the tableaux of one particular frame, we may omit the superscript.

It is known that e_i is an idempotent element of O_G. Furthermore, all the $e_i S_{ij}$ of the standard tableaux of all the frames of a given degree n are linearly independent, and we will show how to express the multilinear identities and the multilinear central identities in terms of these elements.

3. PROCEDURE

We give a procedure that finds all the multilinear identities and all the multilinear central identities of $M_n(\phi)$, where ϕ is of characteristic zero or a large enough prime. We thus find all the identities and all the central identities of degree < 9 of $M_3(Q)$. In this section, "ϕ" denotes Q or Z_p. An identity or a central identity of degree n is called *multilinear* if it is of the form

$$\sum_{\pi \in \mathrm{Sym}(n)} \alpha_\pi [x_1, x_2, \ldots, x_n]_\pi \qquad \text{for some} \quad \alpha_\pi \in \phi. \tag{3}$$

This identity or central identity can be represented by

$$\sum_{\pi \in \mathrm{Sym}(n)} \alpha_\pi \pi \in O_G. \tag{4}$$

If $I(x_1, \dots, x_n)$ is an identity, then $I\pi(x_1, \dots, x_n) := I(x_{(1)\pi^{-1}}, \dots, x_{(n)\pi^{-1}})$ is an identity for any $\pi \in \mathrm{Sym}(n)$. Moreover, if $g = \sum_{\pi \in \mathrm{Sym}(n)} \alpha_\pi \pi \in O_G$, then $Ig(x_1, \dots, x_n) := \sum_{\pi \in \mathrm{Sym}(n)} \alpha_\pi I\pi(x_1, \dots, x_n)$ is also an identity. Similarly, if $I(x_1, \dots, x_n)$ is a central identity, then $Ig(x_1, \dots, x_n)$ is a central identity or an actual identity.

We say that a finite set of multilinear identities and central identities $I_1(x_1, \dots, x_n), \dots, I_k(x_1, \dots, x_n)$ is *independent under substitution* if for any choice of $g_1, \dots, g_k \in O_G$, $I_1 g_1(x_1, \dots, x_n) + \cdots + I_k g_k(x_1, \dots, x_n) = 0$ implies that $I_1 g_1(x_1, \dots, x_n) = \cdots = I_k g_k(x_1, \dots, x_n) = 0$. In the rest of the paper, *independent* means independent under substitution.

By the *identities of a given frame*, we mean all the multilinear identities of the form (4) which can be written as a linear combination of the $e_i S_{ij}$ of all the standard tableaux of that frame. The procedure described in this section is a general method of finding all the multilinear identities of a fixed frame. In Section 5, we will show that the identities found by our procedure form an independent generating set that implies all the multilinear identities of degree n.

Let F be a fixed frame of degree n with f standard tableaux $T_1, \dots, T_f$. Define

$$E_k := e_k S_{k1}, \qquad k = 1, \dots, f. \tag{5}$$

Notice that $e_k S_{k1} = S_{k1} e_1$ (see [3, p. 248]). A crucial point, which we prove later, is that every multilinear identity or multilinear central identity of a frame F may be expressed as a linear combination of E_k's of that frame. If $g = \sum_{\pi \in \mathrm{Sym}(n)} \alpha_\pi \pi$, then by $[x_1, \dots, x_n]_g$ we mean $\sum_{\pi \in \mathrm{Sym}(n)} \alpha_\pi [x_1, \dots, x_n]_\pi$. Then a multilinear identity of the given frame is of the form

$$I(x_1, \dots, x_n) = \sum_{k=1}^{f} \alpha_k [x_1, \dots, x_n]_{E_k} = \sum_{k=1}^{f} \alpha_k E_k. \tag{6}$$

Thus we want to find coefficients $\alpha_1, \dots, \alpha_n \in \phi$, not all zero, such that the following relation holds for any choice of matrices $M_1, \dots, M_n \in M_m(\phi)$:

$$\sum_{k=1}^{f} \alpha_k [M_1, \dots, M_n]_{E_k} = 0_{m \times m}. \tag{7}$$

Our technique requires several choices of sets of matrices $\{M_1^{(1)}, \ldots, M_n^{(1)}\}, \{M_1^{(2)}, \ldots, M_n^{(2)}\}, \ldots$. For brevity, we will refer to $[M_1^{(i)}, \ldots, M_n^{(i)}]_{E_k}$ by h_{ik}.

We note that (7) gives us a system of m^2 equations, that is, one equation for each entry of the $m \times m$ matrix. Because of this, it is reasonable to consider the $m \times m$ matrices h_{ik} as $m^2 \times 1$ column vectors. Returning to (7), our problem simplifies to finding solutions $[\alpha_1, \ldots, \alpha_f]^T$ of

$$\begin{bmatrix} h_{11} & \cdots & h_{1f} \\ \vdots & & \vdots \\ h_{r1} & \cdots & h_{rf} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_f \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{8}$$

i.e., finding the nullspace of (h_{ij}). Each row of the block matrix (h_{ij}) represents m^2 equations. It may be regarded as an $(m^2 r) \times f$ matrix. The number r is the number of trials, and it ought to be chosen large enough to make sure that (h_{ij}) reaches its maximum rank.

Below, we have given a more detailed outline of the procedure.

Step 1. Set $t = 0$.

Step 2. Set $i = 1$. Create in dictionary order all the standard tableaux $T_1, \ldots, T_f$ of F_{t+1}.

Step 3. Create $E_k = e_k S_{k1}$ for each $k = 1, \ldots, f$.

Step 4. Create n $m \times m$ random matrices $M_1^{(i)}, \ldots, M_n^{(i)}$, each expressed as a column vector.

Step 5. Compute $h_{i1}, \ldots, h_{if}$. We need a program which multiplies the n random matrices in any order, i.e., a program which evaluates $[M_{(1)\pi^{-1}} \cdots M_{(n)\pi^{-1}}]$ for any $\pi \in \mathrm{Sym}(n)$.

Step 6. Find the row canonical form of the matrix (h_{ij}). The first time that the procedure is executed for a fixed frame, the matrix whose row canonical form is to be computed will be the same as the one that is found in step 5. The row canonical form is stored in some matrix. After that, every time that the procedure is executed, the new $h_{i1}, \ldots, h_{if}$ of step 5 are placed at the bottom of the stored matrix, and the row canonical form of the resulting matrix is computed.

Step 7. Augment i and return to step 4. Steps 4, 5, and 6 are repeated until we believe that the row canonical matrix of step 6 has reached its maximum rank (see step 8). The decision to go to step 8 is made after the rank is unchanged through several iterations. If we go to step 8 before reaching the maximum rank, we will discover it later.

Step 8. Find the nullspace of the final matrix. This will give us the set of all possible coefficients $\alpha_1, \ldots, \alpha_f$ of $E_1, \ldots, E_f$, which is an independent generating set of multilinear identities for the given frame.

Step 9. Augment t and go to step 2.

Step 10. Check all the potential identities found in step 8. It is sufficient to check an identity of degree n on all possible sets of n $m \times m$ matrix units, that is, for $(m^2)^n$ sets.

4. RESULTS FOR 3×3 MATRICES

Similarly we can obtain an independent generating set that implies all the multilinear central identities of 3×3 matrices. Computations show that all the identities of degree < 9 of the 3×3 matrices are consequences of the standard identity $S_6(x_1, \ldots, x_6)$, as expected (see [6]). We do not include the identities of the frames which did not yield any central identities.

The multilinear central identities appear for the first time in degree 8. The first two multilinear central identities, which are given by (9) and (10), belong to the fourteenth frame F_{14}, which is (3, 3, 1, 1) or

$$\begin{matrix} [\ ][\ ][\ ] \\ [\ ][\ ][\ ] \\ [\ ] \\ [\ ] \end{matrix} .$$

Then

$$\begin{aligned} I_{c1} = [& -407S_{11} - 518S_{21} - 814S_{41} + 444S_{61} + 259S_{71} - 222S_{81} \\ & -259S_{91} + 444S_{10,1} + 407S_{11,1} + 259S_{12,1} + 333S_{13,1} - 481S_{14,1} \\ & -222S_{15,1} + 111S_{16,1} + 629S_{17,1} + 1110S_{18,1} + 37S_{19,1} \\ & -999S_{20,1} - 555S_{21,1} - 148S_{22,1} + 370S_{23,1} - 518S_{24,1} - 259S_{25,1} \\ & -814S_{26,1} - 592S_{27,1} - 481S_{28,1} + 259S_{29,1} - 148S_{30,1} \\ & -296S_{31,1} - 185S_{32,1} - 481S_{33,1} + 185S_{34,1} - 629S_{35,1} - 148S_{36,1} \\ & +407S_{37,1} - 1147S_{38,1} - 925S_{39,1} + 222S_{40,1} - 555S_{41,1} + 333S_{42,1} \\ & +259S_{43,1} - 148S_{44,1} - 1073S_{45,1} - 185S_{46,1} - 666S_{47,1} \\ & -962S_{48,1} + 333S_{49,1} - 444S_{50,1} + 777S_{51,1} + 518S_{52,1} - 296S_{53,1} \\ & -925S_{54,1} - 370S_{55,1}]e_1, \end{aligned} \tag{9}$$

$$I_{c2} = [-124S_{11} - 31S_{21} - 125S_{31} + 67S_{41} - 85S_{51} - 262S_{61} - 232S_{71}$$
$$+31S_{81} - 23S_{91} - 172S_{10,1} - 246S_{11,1} - 222S_{12,1} - 149S_{13,1}$$
$$-42S_{14,1} + 31S_{15,1} + 87S_{16,1} - 222S_{17,1} - 355S_{18,1} - 51S_{19,1}$$
$$+272S_{20,1} + 255S_{21,1} - 11S_{22,1} - 145S_{23,1} + 94S_{24,1} + 12S_{25,1}$$
$$+152S_{26,1} + 101S_{27,1} + 13S_{28,1} - 67S_{29,1} - 136S_{30,1} + 73S_{31,1}$$
$$+25S_{32,1} + 13S_{33,1} - 95S_{34,1} + 17S_{35,1} - 31S_{36,1} - 46S_{37,1}$$
$$+196S_{38,1} + 160S_{39,1} - 91S_{40,1} - 40S_{41,1} - 104S_{42,1} - 112S_{43,1}$$
$$-11S_{44,1} + 209S_{45,1} - 22S_{47,1} + 111S_{48,1} - 164S_{49,1} + 12S_{50,1}$$
$$-241S_{51,1} - 209S_{52,1} - 22S_{53,1} + 220S_{54,1} - 185S_{56,1}]e_1. \qquad (10)$$

The multilinear central identity (11) and the multilinear identities (12) and (13) belong to the frame F_{15}, i.e., the frame

[ ][ ][ ].
[ ][ ]
[ ][ ]
[ ]

Then

$$I_{c3} = [322S_{11} + 340S_{21} + 438S_{31} + 612S_{41} + 169S_{51} - 380S_{61} - 16S_{71}$$
$$-3S_{81} + 426S_{91} - 17S_{10,1} - 229S_{11,1} + 6S_{12,1} + 293S_{13,1} + 66S_{14,1}$$
$$+191S_{15,1} + 49S_{16,1} - 140S_{17,1} + 339S_{18,1} - 122S_{19,1} + 211S_{20,1}$$
$$+11S_{21,1} + 143S_{22,1} + 277S_{23,1} + 440S_{24,1} + 55S_{25,1} - 669S_{26,1}$$
$$-252S_{27,1} - 84S_{28,1} + 351S_{29,1} + 182S_{30,1} - 37S_{31,1} - 97S_{32,1}$$
$$-375S_{33,1} - 471S_{34,1} - 136S_{35,1} + 47S_{36,1} + 259S_{37,1} - 53S_{38,1}$$
$$-133S_{39,1} + 454S_{40,1} - 469S_{41,1} + 22S_{42,1} + 356S_{43,1} + 31S_{44,1}$$
$$-314S_{45,1} - 30S_{46,1} + 84S_{47,1} + 93S_{48,1} + 400S_{49,1} + 197S_{50,1}$$
$$-154S_{51,1} - 308S_{52,1} + 43S_{53,1} - 368S_{54,1} + 86S_{56,1} - 98S_{57,1}$$
$$+15S_{58,1} + 46S_{59,1} + 285S_{60,1} + 227S_{61,1} + 453S_{62,1} + 710S_{63,1}$$
$$+119S_{64,1} - 410S_{65,1} - 238S_{66,1} - 43S_{67,1} - 634S_{68,1}]e_1, \qquad (11)$$

$$\begin{aligned} I_1 = [& -224S_{11} - 422S_{21} + 132S_{31} - 252S_{41} + 55S_{51} - 440S_{61} - 280S_{71} \\ & -201S_{81} - 24S_{91} + 13S_{10,1} + 29S_{11,1} - 246S_{12,1} - 313S_{13,1} \\ & -60S_{14,1} + 143S_{15,1} + 223S_{16,1} - 20S_{17,1} + 249S_{18,1} + 52S_{19,1} \\ & +115S_{20,1} - 37S_{21,1} - 49S_{22,1} - 215S_{23,1} + 194S_{24,1} - 77S_{25,1} \\ & -219S_{26,1} + 72S_{27,1} + 96S_{28,1} + 81S_{29,1} + 188S_{30,1} - 67S_{31,1} \\ & +17S_{32,1} - 15S_{33,1} - 75S_{34,1} + 50S_{35,1} + 107S_{36,1} + 37S_{37,1} \\ & +301S_{38,1} + 413S_{39,1} + 34_{40,1} + 203S_{41,1} - 74S_{42,1} - 88S_{43,1} \\ & +43S_{44,1} + 40S_{45,1} + 42S_{46,1} + 120S_{47,1} + 291S_{48,1} + 304S_{49,1} \\ & +329S_{50,1} - 76S_{51,1} - 314S_{52,1} + 91S_{53,1} - 14S_{54,1} + 54S_{55,1} \\ & +290S_{56,1} - 14S_{57,1} - 75S_{58,1} + 76S_{59,1} + 87S_{60,1} + 71S_{61,1} \\ & +3S_{62,1} + 32S_{63,1} - 37S_{64,1} + 34S_{65,1} + 20S_{66,1} - 91S_{67,1} \\ & -106S_{68,1} - 54S_{69,1}]e_1, \end{aligned} \tag{12}$$

$$\begin{aligned} I_2 = [& -172S_{11} - 136S_{21} - 132S_{31} - 126S_{41} - 19S_{51} - 118S_{61} - 80S_{71} \\ & -69S_{81} - 30S_{91} + 77S_{10,1} + 43S_{11,1} - 24S_{12,1} - 47S_{13,1} + 6S_{14,1} \\ & +37S_{15,1} + 29S_{16,1} + 56S_{17,1} + 75S_{18,1} + 38S_{19,1} - 25S_{20,1} \\ & -53S_{21,1} - 95S_{22,1} - 73S_{23,1} - 68S_{24,1} - 49S_{25,1} - 51S_{26,1} \\ & -18S_{27,1} + 12S_{28,1} + 27S_{29,1} + 46S_{30,1} + 31S_{31,1} + S_{32,1} + 15S_{33,1} \\ & +21S_{34,1} + 22S_{35,1} + 19S_{36,1} - S_{37,1} + 59S_{38,1} + 91S_{39,1} \\ & +56S_{40,1} + 85S_{41,1} + 2S_{42,1} - 56S_{43,1} - 7S_{44,1} - 4S_{45,1} + 12S_{46,1} \\ & +42S_{47,1} + 33S_{48,1} + 56S_{49,1} + 13S_{50,1} - 14S_{51,1} - 28S_{52,1} - S_{53,1} \\ & -4S_{54,1} - 2S_{56,1} - 4S_{57,1} - 33S_{58,1} - 40S_{59,1} - 33S_{60,1} + S_{61,1} \\ & -57S_{62,1} - 68S_{63,1} - 53S_{64,1} + 2S_{65,1} + 52S_{66,1} + S_{67,1} + 16S_{68,1} \\ & -54S_{70,1}]e_1. \end{aligned} \tag{13}$$

The last independent central identity belongs to the frame F_{18}, or

$$\begin{matrix} [\ ][\ ] \\ [\ ][\ ] \\ [\ ][\ ] \\ [\ ][\ ] \end{matrix}$$

and is given by

$$I_{c4} = [4S_{11} + 21S_{21} + 12S_{31} + 21S_{41} + 7S_{51} + 21S_{61} + 8S_{71} + 21S_{81}$$
$$+22S_{91} + 8S_{10,1} + 7S_{11,1} + 8S_{12,1} + 12S_{13,1} + 38S_{14,1}]e_1. \quad (14)$$

As we shall prove in the next section, the multilinear identities and central identities of the frames of degree n found by our procedure form an independent generating set which implies all the multilinear central identities of degree n. Therefore, the central identities (9), (10), (11), and (14) together with the identities implied by S_6 found by our procedure form an independent generating set which implies all the multilinear central identities of degree 8.

5. PROOFS

In this section, we will prove that the identities of degree n computed by our procedure form an independent generating set for all the multilinear identities of degree n. Similar proofs can be used to show that the identities and the central identities of degree n computed by our procedure form an independent generating set for all the multilinear identities and all the multilinear central identities of degree n. Furthermore, through complete linearization one can prove that every identity of degree n of $M_m(\phi)$ is implied by a set of multilinear identities of degrees $\leqslant n$, where ϕ has characteristic zero or $p > n$ [15, pp. 14–17]. Thus we only need to state the proofs for multilinear identities. In the rest of this section, an identity means a degree n identity of $M_m(\phi)$. Since we work only with multilinear identities of degree n, for brevity we shall refer to an identity $I(x_1, \dots, x_n)$ by I.

The following theorem is an important result in group representation theory which implies that the product of the $e_i S_{ij}$ of different frames is zero. For a proof see [14].

THEOREM 1. *Let ϕ be a field of characteristic zero or $> n$. Then the $e_i S_{ij}$ of the frames of degree n form a basis for the group ring O_G. Furthermore, we have $O_G \cong F^1 \oplus \cdots \oplus F^k$, where k is the number of frames of degree n, and F^λ is the subring generated by the $e_i^\lambda S_{ij}^\lambda$.*

Suppose the frame associated with F^λ has f standard tableaux. Associate with $A = (\alpha_{ij}) \in M_m(\phi)$ the element $\overline{A} := \sum_{i=1,\, j=1}^{f,f} \alpha_{ij} e_i S_{ij} \in F^\lambda$. Then, using $*$ for multiplication in O_G, Clifton [3, p. 249, line 4] tells us

$$\overline{A} * \overline{B} = \overline{A(\varepsilon_{ij})B}, \tag{15}$$

where the numbers ε_{ij} for $i, j = 1, 2, \ldots, f$ are defined as follows [3, p. 248]:

$$\varepsilon_{ij} := \begin{cases} \operatorname{sgn}(q) & \text{if} \quad S_{ji} = qp \ \ (\text{for } T_i), \\ 0 & \text{otherwise.} \end{cases}$$

Let $(\varepsilon_{ij}) = \varepsilon$. It is easy to verify that ε is an upper triangular matrix with 1's on the diagonal; hence invertible. Note that $Ig(x_1, \ldots, x_n) = (I * g)(x_1, \ldots, x_n)$.

LEMMA 2. *If $I = \sum_{i=1,\, j=1}^{f,f} \alpha_{ij} e_i S_{ij}$ is an identity of the frame F, then so is $I_j = \sum_{i=1}^{f} \alpha_{ij} e_i S_{ij}$ for each $j = 1, \ldots, f$.*

Proof. Let $I = \overline{A}$, and let E_{ij} be the matrix with 1 in position ij and zeros elsewhere. Then

$$\sum_{i=1}^{f} \alpha_{ij} e_i S_{ij} = \overline{AE_{jj}} = \overline{A\varepsilon\varepsilon^{-1}E_{jj}} = \overline{A} * \overline{\varepsilon^{-1}E_{jj}}$$

is also an identity. ∎

Similarly,

LEMMA 3. *If $I_l = \sum_{i=1}^{f} \alpha_i e_i S_{il}$ is an identity of the frame F, then so is $I_m = \sum_{i=1}^{f} \alpha_i e_i S_{im}$ for each $m = 1, \ldots, f$.*

THEOREM 4. *If I is an identity of the frame F, then for some $g_m \in O_G$, we have $I = \sum_{m=1}^{f} L_m * g_m$, where each L_m is an identity of the form $\sum_{i=1}^{f} \beta_{i1} e_i S_{i1}$ for some $\beta_{11}, \ldots, \beta_{f1} \in \phi$.*

Sketch of a proof. By Lemma 2, each column of $I = \overline{(\alpha_{ij})}$ is an identity of the frame F. Then by Lemma 3, the location of each individual column is immaterial. So we can represent each of them by putting the nonzero column first. In other words, I is equivalent to the identities represented by the matrices $(\alpha_{rs})E_{jj}$ for $j = 1, \dots, f$, which in turn are equivalent to the identities represented by the matrices $(\alpha_{rs})E_{j1}$ for $j = 1, \dots, f$. ∎

Hence every identity of the frame F is implied by the set of all the identities of the form $\sum_{i=1}^{f} \alpha_{i1} e_i S_{i1}$. Thus the identities computed by our procedure imply all the identities of the given frame. The following theorem shows that every multilinear identity is implied by the set of all the identities of all of the frames.

THEOREM 5. *Let I be a multilinear identity of degree n, and k be the number of frames of degree n. Then $I = I_1 + \cdots + I_k$, where I_m is an identity of the frame F_m for each $m = 1, \dots, k$.*

Proof. By Theorem 1, $I = \sum_{m=1}^{k} I_m$, where

$$I_m = \sum_{i=1,\ j=1}^{f_m,\ f_m} \alpha_{ij}^m e_i^m S_{ij}^m \qquad \text{for some} \quad \alpha_{ij}^m \in \phi.$$

We need to show that each I_m is an identity. Since I is an identity, then so is $I * g$ for any $g \in O_G$. In particular, let $g = \overline{\varepsilon^{-1}}$, where $\varepsilon = (\varepsilon_{ij})$ for the frame F_m. Also let A_m be the matrix such that $\overline{A_m} = I_m$, and let the identity $I * g$ be represented by $(\overline{A_1} + \cdots + \overline{A_k}) * \overline{\varepsilon^{-1}}$. Then

$$\begin{aligned} I * g &= \overline{A_m} * \overline{\varepsilon^{-1}} \\ &= \overline{A_m \varepsilon \varepsilon^{-1}} \\ &= \overline{A_m}. \end{aligned}$$

Therefore, I_m is also an identity for each $m = 1, \dots, k$. ∎

Next, we will prove that the independent identities for a frame F_λ found by our procedure form an independent generating set for all of the identities of that frame.

LEMMA 6. *A set $I_1, \ldots, I_\tau$ of linearly independent identities of the form $\sum_{i=1}^{f_\lambda} \alpha_{i1} e_i^\lambda S_{i1}^\lambda$ for a given frame F_λ is also independent under substitution.*

Proof. Let $I_m = \overline{A}_m$, where

$$A_m = \begin{bmatrix} \alpha_{11}^m & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{f_\lambda 1}^m & 0 & \cdots & 0 \end{bmatrix} \qquad \text{for} \quad m = 1, \ldots, \tau.$$

If $I_1, \ldots, I_\tau$ are not independent under substitution, then there exist group ring elements $g_1, \ldots, g_\tau \in O_G$ such that $I_m * g_m \neq 0$ for some m and

$$I_1 * g_1 + \cdots + I_\tau * g_\tau = 0.$$

Since $O_G \cong F^1 \oplus \cdots \oplus F^k$, each $g_m = (g_m)_1 + \cdots + (g_m)_k$ for some $(g_m)_i \in F^i$, and each

$$I_m * g_m = I_m * (g_m)_1 + \cdots + I_m * (g_m)_k.$$

Since the $e_i S_{ij}$ of different frames annihilate one another, we get

$$I_m * g_m = I_m * (g_m)_\lambda \qquad \text{for} \quad m = 1, \ldots, \tau.$$

Hence

$$I_1 * (g_1)_\lambda + \cdots + I_\tau * (g_\tau)_\lambda = 0.$$

Let $(g_m)_\lambda = \overline{(G_m)_\lambda}$, and let $\varepsilon(G_m)_\lambda$ be represented by the following matrix, where ε is the (ε_{ij}) matrix for the frame F_λ:

$$\varepsilon(G_m)_\lambda = \begin{bmatrix} \beta_{11}^m & \cdots & \beta_{1f_\lambda}^m \\ \vdots & \ddots & \vdots \\ \beta_{f_\lambda 1}^m & \cdots & \beta_{f_\lambda f_\lambda}^m \end{bmatrix} \qquad \text{for} \quad m = 1, \ldots, \tau.$$

Then

$$\begin{aligned} I_m * g_m &= I_m * (g_m)_\lambda \\ &= \overline{A}_m * \overline{(G_m)_\lambda} \\ &= \overline{A_m \varepsilon (G_m)_\lambda}, \end{aligned}$$

and

$$A_m \varepsilon(G_m)_\lambda = \begin{bmatrix} \beta_{11}^m \alpha_{11}^m & \cdots & \beta_{1f_\lambda}^m \alpha_{11}^m \\ \vdots & \ddots & \vdots \\ \beta_{11}^m \alpha_{f_\lambda 1}^m & \cdots & \beta_{1f_\lambda}^m \alpha_{f_\lambda 1}^m \end{bmatrix}.$$

We have assumed that $I_m * g_m \neq 0$ for some m; so let column l of $A_m \varepsilon(G_m)_\lambda$ be nonzero. Then $\beta_{1l}^m \neq 0$, and

$$\sum_{m=1}^{\tau} \overline{A_m \varepsilon(G_m)_\lambda E_{l1}} = \sum_{m=1}^{\tau} \overline{A_m \varepsilon(G_m)_\lambda} * \overline{\varepsilon^{-1} E_{l1}}$$

$$= \left(\sum_{m=1}^{\tau} I_m * g_m \right) * \overline{\varepsilon^{-1} E_{l1}} = 0,$$

since $\sum_{m=1}^{\tau} I_m * g_m = 0$. But

$$A_m \varepsilon(G_m)_\lambda E_{l1} = \begin{bmatrix} \beta_{11}^m \alpha_{11}^m & \cdots & \beta_{1f_\lambda}^m \alpha_{11}^m \\ \vdots & \ddots & \vdots \\ \beta_{11}^m \alpha_{f_\lambda 1}^m & \cdots & \beta_{1f_\lambda}^m \alpha_{f_\lambda 1}^m \end{bmatrix} E_{l1}$$

$$= \begin{bmatrix} \beta_{1l}^m \alpha_{11}^m & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{1l}^m \alpha_{f_\lambda 1}^m & 0 & \cdots & 0 \end{bmatrix}$$

$$= \beta_{1l}^m A_m.$$

Therefore, we have

$$\beta_{1l}^1 I_1 + \cdots + \beta_{1l}^\tau I_\tau = 0,$$

which implies that $I_1, \ldots, I_\tau$ are linearly dependent, a contradiction. ■

Now, we need to show that all of the identities of all of the frames given by the procedure form an independent generating set for all the multilinear identities of degree n.

THEOREM 7. *Let each* H_λ *be a set of linearly independent identities of the form* $\sum_{i=1}^{f_\lambda} \alpha_{i1}^\lambda e_i^\lambda S_{i1}^\lambda$ *for each* $\lambda = 1, \ldots, k$, *where* k *is the number of the frames of degree* n. *Then the set* $\bigcup_{\lambda=1}^{k} H_\lambda$ *is independent under substitution.*

Proof. Let $I_1^\lambda, \ldots, I_{\tau_\lambda}^\lambda$ be the identities of the set H_λ. Assume there exist nonzero elements g_m^λ of the group ring such that

$$\sum_{\lambda=1}^{k} \sum_{m=1}^{\tau_\lambda} I_m^\lambda * g_m^\lambda = 0.$$

The linear independence of all of the $e_i S_{ij}$'s of all of the frames implies that

$$\sum_{m=1}^{\tau_\lambda} I_m^\lambda * g_m^\lambda = 0 \qquad \text{for} \quad \lambda = 1, \ldots, k.$$

In other words, the linearly independent identities of each set H_λ are not independent under substitution, which contradicts Lemma 6. ∎

We now describe step 10, checking the identities. Suppose that $I(x_1, \ldots, x_n) = \sum_{i=1}^{f} \alpha_i e_i S_{i1}$ is a potential identity of degree n of a frame F_λ. To verify that I is an identity, it suffices to show that $I(x_1, \ldots, x_n) = 0$ for any choice of $x_1, \ldots, x_n$ of $\{E_{ij}\}$.

There are $(m^2)^n$ choices to check. This checking was done in the following way. We ordered the matrix units $\{E_{ij}\}$ in some way. For each choice with $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$, we wanted to check for all permutations π that $I\pi(x_1, \ldots, x_n) = 0$. Normally each choice would require checking $n!$ arrangements of these arguments.

In the matrix representation of I, only the first column of I is nonzero. For $j = 1, \ldots, f$ set $I_j = \sum_{i=1}^{f} \alpha_i e_i S_{ij}$, which has the nonzero column moved to the jth column. We checked that $I_j(x_1, \ldots, x_n) = 0$ for $j = 1, \ldots, f$. Instead of checking one polynomial expression $n!$ times, we checked f polynomial expressions one time. This suffices, since by representations theory, $I\pi(x_1, \ldots, x_n) = \sum_{j=1}^{f} \beta(\pi, j) I_j$ for some coefficients $\beta(\pi, j)$.

We can check whether an expression I is a central identity as well. We only need to check that $I_j(x_1, \ldots, x_n)$ is in the center when $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$ and $j = 1, \ldots, f$.

If we tried to check an identity $I(x_1, \ldots, x_8)$ of F_{15} for the 3×3 matrices directly, we would have to check $9^8 = 43{,}046{,}721$ substitutions. Using the method explained above, there are 12,870 arrangements where $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_8$ and $f = 70$, which gives (70)(12,870) = 900,900 substitutions which have to be checked.

6. REMARKS

Our further computations showed that the central identity of degree 8 found by V. Drensky and A. Kasparian [5] gives a central identity in F_{14}, a central identity in F_{18}, an identity in each of the frames F_{16}, F_{19}, F_{21}, F_{22}, and two identities in F_{20}. The central identity in F_{14} is a consequence of the central identities (9) and (10), while the central identity in F_{18} is the same as the central identity (14), and all of the identities are consequences of the standard identity of degree 6. Therefore, the central identity (11) which belongs to the frame F_{15} is not a consequence of the one found by Drensky and Kasparian.

The procedure was originally designed to find an independent generating set for all the multilinear identities of $M_3(Q)$. But one may encounter roundoff error or some type of numerical problem when the degree $n > 7$. However, since the procedure is valid for any field Z_p where p is a large enough prime, one may execute it for several primes. Once all the multilinear identities of several $M_3(Z_p)$'s are known, one can find all the multilinear identities of $M_3(Q)$ among them.

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REFERENCES

1 S. A. Amitsur and J. Levitzki, Remarks on minimal identities for algebras, *Proc. Amer. Math. Soc.* 2:320–327 (1951).

2 H. Boerner, *Representation of Groups with Special Considerations for the Needs of Modern Physics*, North-Holland, Amsterdam, 1963.

3 J. Clifton, A simplification of the computation of the natural representation of the symmetric group, *Proc. Amer. Math. Soc.* 83:248–250 (1981).

4 V. Drensky, A minimal basis of identities for a second-order matrix algebra over a field of characteristic 0 (in Russian), *Algebra i Logika* 20(3):282–290 (1981); transl., *Algebra and Logic* 20:188–194 (1981).

5 V. Drensky and A. Kasparian, A new central polynomial for 3 × 3 matrices, *Comm. Algebra* 13(3):745–752 (1985).

6 V. Drensky and A. Kasparian, Polynomial identities of eight degree for 3 × 3 matrices, *Annuaire Univ. Sofia* 77:175–194 (1983).

7 V. Drensky and G. M. Piacentini, A central polynomial of low degree for 4 × 4 matrices, *J. Algebra* 168:469–478 (1994).

8 E. Formanek, Central polynomials for matrix rings, *J. Algebra* 23:129–132 (1972).

9 I. Kaplansky, Problems in the Theory of Rings, Nat. Acad. of Sci. Nat. Res. Counc. Publ. 502, 1957.
10 I. Kaplansky, Problems in the theory of rings revisited, *Amer. Math. Monthly* 77:445–454 (1970).
11 U. Leron, Multilinear identities of the matrix ring, *Trans. Amer. Math. Soc.* 183:175–202 (1973).
12 Yu. Razmyslov, Finite basing of the identities of a matrix algebra of second order over a field of characteristic 0 (in Russian), *Algebra i Logika* 12:83–113 (1973); transl., *Algebra and Logic* 12:47–63 (1973).
13 Yu. Razmyslov, On the Kaplansky problem (in Russian), *Izv. Akad. Nauk SSSR Ser. Mat.* 37:483–501 (1973); transl., *Math. USSR Izv.* 7:479–496 (1973).
14 D. Rutherford, *Substitutional Analysis*, Edinburgh U.P., Edinburgh, 1948.
15 I. P. Shestakov, A. I. Shirshov, A. M. Slin'ko, and K. A. Zhevlakov, *Rings That Are Nearly Associative*, Academic, New York, 1982.

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