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Mergelyan Sets and the Modulus of Continuity of Analytic Functions*

L. A. RUBEL

Department of Mathematics, University of Illinois, Urbana, Illinois 61801

A. L. SHIELDS

Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48104

AND

B. A. TAYLOR

*Institute for Advanced Study, Princeton, New Jersey 08540**Communicated by G. G. Lorentz*

1. INTRODUCTION

For f a function continuous on the closed unit disk and analytic in the interior, let

$$\omega(\delta, f) = \sup\{|f(z_1) - f(z_2)| : |z_1 - z_2| \leq \delta, |z_1| \leq 1, |z_2| \leq 1\}$$

and

$$\tilde{\omega}(\delta, f) = \sup\{|f(z_1) - f(z_2)| : |z_1 - z_2| \leq \delta, |z_1| = |z_2| = 1\}$$

denote, respectively, the modulus of continuity of f on the closed unit disk and the modulus of continuity of the restriction of f to the boundary $\{|z| = 1\}$. We consider here the question of determining the relationship of $\omega(\delta, f)$ and $\tilde{\omega}(\delta, f)$. Clearly, one has $\tilde{\omega}(\delta, f) \leq \omega(\delta, f)$, and we are concerned here with the extent to which the reverse inequality holds. For certain measures of growth, $\tilde{\omega}(\delta, f)$ and $\omega(\delta, f)$ are the same. For example, if α is given ($0 < \alpha \leq 1$) and if $\tilde{\omega}(\delta, f) \leq \delta^\alpha$, then $\omega(\delta, f) \leq \delta^\alpha$ (see Theorem 2.2). However, it is not true in general that $\omega(\delta, f) = \tilde{\omega}(\delta, f)$ (see Section 4 for an example). Nevertheless, we do have the following result.

The disk algebra A denotes the class of functions f that are continuous on $|z| \leq 1$ and analytic in $|z| < 1$.

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THEOREM 1.1. *There exists a constant $C > 0$ such that*

$$\omega(\delta, f) \leq C\tilde{\omega}(\delta, f)$$

for all $\delta > 0$ and all functions $f \in A$.

The example of Section 4 shows that $C > 1$, while the proof of Theorem 1.1 (see Section 2) shows that we may take $C \leq 3$. For contrast we state the following known result for harmonic functions.

THEOREM 1.1'. *There exists a constant C such that*

$$\omega(\delta, f) \leq C[\log(1/\delta)] \tilde{\omega}(\delta, f)$$

for $0 < \delta \leq \frac{1}{2}$, and all (complex-valued) functions f that are harmonic for $|z| < 1$ and continuous for $|z| \leq 1$.

At the end of Section 2 we say a few words about the proof of this result; in Section 4 we point out that the factor $\log(1/\delta)$ is *best possible*.

As noted above, the constant C in Theorem 1.1 is larger than 1. It is interesting to note that even in the small, the constant C is larger than 1; that is, there exists a function $f \in A$ with

$$\limsup_{\delta \rightarrow 0} \frac{\omega(\delta, f)}{\tilde{\omega}(\delta, f)} > 1.$$

We give an example of such a function in Section 4.

The problem of determining the relationship between $\omega(\delta, f)$ and $\tilde{\omega}(\delta, f)$ arises naturally in approximation theory. In particular, Theorem 1.1 answers a question posed by Sewell [7, p. 32]. The authors were led to the same question in the study of Mergelyan sets (see Section 3).

DEFINITION. A subset F of the open unit disk is called a Mergelyan set if and only if every function g that is analytic on the open unit disk and uniformly continuous on F can be uniformly approximated by polynomials on all sets of the form $F \cup \{|z| \leq r\}$ for each $0 < r < 1$.

In Section 3, we give some basic facts about Mergelyan sets. In particular, Theorem 1.1 allows us to prove that each set F that contains a *bullseye*, i.e., a set of the form $\bigcup\{|z| = r_n\}$, where r_n increases to 1, is a Mergelyan set (Proposition 3.5). An example is given that shows that the union of two Mergelyan sets need not be a Mergelyan set; another example shows that the intersection of two Mergelyan sets need not be a Mergelyan set. Since an earlier version of this paper was prepared, Stray [8] has given a characterization of Mergelyan sets.

It is of interest to study the relationship of $\omega(\delta, f)$ and $\tilde{\omega}(\delta, f)$ for domains other than the unit disk. If G is an open set and f is a continuous bounded function on G^- , the closure of G in the finite plane, then we can define

$$\omega(\delta, f; G) = \sup\{|f(z_1) - f(z_2)| : z_1, z_2 \in G^-, |z_1 - z_2| \leq \delta\}$$

and

$$\tilde{\omega}(\delta, f; G) = \sup\{|f(z_1) - f(z_2)| : z_1, z_2 \in \partial G, |z_1 - z_2| \leq \delta\}.$$

As before, we clearly have $\omega \leq \tilde{\omega}$. The analogue of Theorem 1.1 holds to the following extent. Let $\phi(\delta)$ be a continuous increasing function with $\phi(0) = 0$ and $\phi(\delta_1 + \delta_2) \leq \phi(\delta_1) + \phi(\delta_2)$.

THEOREM 1.2. *If G is simply connected, and if f is continuous and bounded on G^- and analytic in G , then*

$$\tilde{\omega}(\delta, f; G) \leq \phi(\delta) \quad \text{implies} \quad \omega(\delta, f; G) \leq C\phi(\delta)$$

where C is an absolute constant independent of G and f .

For arbitrary domains in the plane an extensive study, of when results like Theorem 1.2 hold has been made by Tamrazov [9], who gives conditions in terms of the capacity of the complement of G near boundary points. As far as we can determine the discoveries of theorems relating ω and $\tilde{\omega}$ by ourselves and Tamrazov occurred almost simultaneously [5]. However, the methods are different. While the results presented here are not as complete as those of [9], the proofs are quite straightforward and, for a large class of domains, depend only on simple versions of the maximum principle. In particular, in Section 2, we give an elegant proof of Theorem 1.1 due to Robert Kaufman. We thank Prof. Kaufman for permission to reproduce his argument here.

We also thank Prof. L. Carleson for his comments on an earlier version of this manuscript. Most of the arguments in Section 2 are based on an idea he suggested, and are much simpler than our original arguments.

In Section 2, the main positive results relating $\omega(\delta, f)$ and $\tilde{\omega}(\delta, f)$ are presented. In Section 3, we discuss Mergelyan sets and some of their properties, and in Section 4 some examples showing that $C > 1$ are given.

2. THE MODULUS OF CONTINUITY

We begin with Kaufman's proof of Theorem 1.1.

Proof of Theorem 1.1. Let f be continuous for $|z| \leq 1$ and analytic for $|z| < 1$. For $0 < \delta \leq \pi/2$, set

$$g(z) = g(z, \delta) = \frac{1}{2\delta} \int_{-s}^{\delta} f(ze^{it}) dt.$$

Then g, g' are analytic for $|z| < 1$, continuous for $|z| \leq 1$, and have the following properties.

$$|g(z) - f(z)| \leq \tilde{\omega}(\delta, f), \quad |z| \leq 1, \quad (2.1)$$

$$|g'(z)| \leq \frac{1}{2\delta} \tilde{\omega}(2 \sin \delta, f), \quad |z| \leq 1. \quad (2.2)$$

For, from the maximum principle it suffices to prove (2.1) and (2.2) when $|z| = 1$. Further, $|g(e^{i\theta}) - f(e^{i\theta})| = |(1/(2\delta)) \int_{-\delta}^{\delta} \{f(e^{i(\theta+t)}) - f(e^{i\theta})\} dt| \leq \tilde{\omega}(|e^{i\delta} - 1|, f) = \tilde{\omega}(2\sin(\delta/2), f) \leq \tilde{\omega}(\delta, f)$, which proves (2.1). Similarly, $|g'(e^{i\theta})| = |(\partial/\partial\theta) g(e^{i\theta})| = (1/(2\delta)) |f(e^{i(\theta+\delta)}) - f(e^{i(\theta-\delta)})| \leq (1/(2\delta)) \tilde{\omega}(|e^{i\delta} - e^{-i\delta}|, f) = (1/(2\delta)) \tilde{\omega}(2 \sin \delta, f)$.

Then writing $f = (f - g) + g$, we have from (2.1) and (2.2) that $\omega(\delta, f) \leq \omega(\delta, f - g) + \omega(\delta, g) \leq 2 \sup\{|f(z) - g(z)| : |z| \leq 1\} + \delta \sup\{|g'(z)| : |z| \leq 1\} \leq 2\tilde{\omega}(\delta, f) + \frac{1}{2}\tilde{\omega}(2 \sin \delta, f)$. However, it is easy to check that $\tilde{\omega}(2 \sin \delta, f) \leq 2\tilde{\omega}(\delta, f)$, so we have

$$\omega(\delta, f) \leq 3\tilde{\omega}(\delta, f) \quad \text{for } 0 \leq \delta \leq \pi/2,$$

and Theorem 1.1 follows.

To begin the study of $\omega(\delta, f)$ on more general domains, we give a simple lemma on analytic functions.

LEMMA 2.1. *Let G be an open set in the plane and let u be a bounded, continuous function in G^- which is analytic in G . Then*

$$\begin{aligned} & \sup\{|u(z_1) - u(z_2)| : z_1, z_2 \in G^-, |z_1 - z_2| \leq \delta\} \\ &= \sup\{|u(z_1) - u(z_2)| : z_1 \in \partial G, z_2 \in G^-, |z_1 - z_2| \leq \delta\}. \end{aligned} \quad (2.3)$$

If G is bounded, the supremum in (2.3) can be replaced by maximum.

Proof. Let A, B denote, respectively, the left and right-hand sides of (2.3). Clearly $A \geq B$. To prove the other inequality, let $\epsilon > 0$ and choose $z_1, z_2 \in G^-$ with $|u(z_1) - u(z_2)| > A - \epsilon$. If z_1 or $z_2 \in \partial G$, then $B > A - \epsilon$. Thus, assume $z_1, z_2 \in G$. Let $b = z_2 - z_1$. Then the function $F(z) = u(z + b) - u(z)$ is bounded and continuous in G_b^- and analytic in G_b , where $G_b = \{z : z \in G \text{ and } z + b \in G\} = G \cap (G - b)$. Since F is bounded, we have

$$|F(z)| \leq \sup\{|F(w)| : w \in \partial G_b, w \neq \infty\}$$

for all $z \in G_b$. But $|F(z_1)| > A - \epsilon$, and the result follows.

Note that the supremum in (2.3) can sometimes be attained for pairs of points z_1, z_2 both of which are inside of G . This happens, for example,

for the analytic function $u(z) = z$. The lemma only asserts that among the pairs for which the supremum is attained, there must exist one that meets the boundary.

Hardy and Littlewood [4, p. 427] proved the following result. If $f \in A$ (the disk algebra) and if $\tilde{\omega}(\delta, f) \leq \delta^\alpha$, then $\omega(\delta, f) \leq c\delta^\alpha$.

This result was improved by Sewell [7, Theorem 1.2.7, p. 17], who showed that we may take $c = 1$. We give a short proof here based on a different idea. Sewell also extended the result to arbitrary Jordan domains.

THEOREM 2.2. *Let $f \in A$. Then for $0 < \alpha \leq 1$,*

$$\sup_{\delta > \gamma} \frac{\omega(\delta, f)}{\delta^\alpha} = \sup_{\delta > \gamma} \frac{\tilde{\omega}(\delta, f)}{\delta^\alpha}. \quad (2.4)$$

Proof. Fix α . The result is trivial if the right side of (2.4) is infinite, so without loss of generality we may assume that it is 1. That is, we have

$$|f(z) - f(w)| \leq |z - w|^\alpha \quad (|z| = 1, |w| = 1). \quad (2.5)$$

Now fix δ . By Lemma 2.1 there are points z_1, z_2 , with at least one on the boundary (we assume that $|z_2| = 1$) for which $\omega(\delta, f) = |f(z_1) - f(z_2)|$. We must show that $|f(z_1) - f(z_2)| \leq |z_1 - z_2|^\alpha$. This follows from (2.3) if $|z_1| = 1$, so we assume that $|z_1| < 1$.

Let $\phi(z) = 1/(z - z_2)^\alpha$ (any branch). Then ϕ is analytic for $|z| < 1$ and $\phi \in H^p$ ($p < 1/\alpha$) (see [2, Section 4.6, Lemma, p. 65]). Hence, the function $g(z) = [f(z) - f(z_2)]\phi(z)$ is also in H^p . Further, g is continuous onto the boundary except perhaps at $z = z_2$. On the boundary we have by (2.5):

$$|g(z)| = \frac{|f(z) - f(z_2)|}{|z - z_2|^\alpha} \leq 1 \quad (|z| = 1, z \neq z_2).$$

Since $g \in H^p$ and $|g| \leq 1$ almost everywhere on the boundary it follows that $|g(z)| \leq 1$ for $|z| < 1$ (see [2, Theorem 2.11, p. 28]). In particular, $|g(z_1)| \leq 1$, which completes the proof.

Since the constant C of Theorem 1.1 is greater than 1, it seems unlikely that Theorem 1.1 can be deduced from the maximum principle.

One could prove Theorem 2.2 without recourse to H^p theory by invoking a Phragmén–Lindelöf theorem. The function $g(z)$ is analytic in the unit disk, bounded by 1 on the boundary except at z_2 , and does not grow too fast in the interior as we approach z_2 . Hence, it is bounded by 1 in the whole disk. (See [10, Section 5.61] for the corresponding Phragmén–Lindelöf theorem in a half-plane.)

We now indicate an alternative approach to this theorem. By the open polydisk in \mathbb{C}^2 we mean the set $\{(z, w) : |z| < 1, |w| < 1\}$. By the distinguished boundary we mean the set $\{|z| = 1, |w| = 1\}$. Assuming the hypotheses of Theorem 2.2 let

$$\begin{aligned} F(z, w) &= \log \frac{|f(z) - f(w)|}{|z - w|^\alpha} \\ &= \log \left| \frac{f(z) - f(w)}{z - w} \right| + (1 - \alpha) \log |z - w|. \end{aligned}$$

Then F is plurisubharmonic in the open polydisk, continuous on the closed polydisk except for the subset of the distinguished boundary where $z = w$. Further, $F \leq \alpha \log c/|z - w|$ in the open polydisk, and $F \leq 0$ on the distinguished boundary, except where $z = w$. Consequently, the family of functions $F_r^+(e^{i\theta}, e^{i\phi}) = F^+(re^{i\theta}, re^{i\phi})$, where $F^+ = \max(F, 0)$, is a uniformly integrable family, and $F \leq 0$ in the open polydisk follows from an extended version of the maximum principle [6, Theorem 3.2.4 (vi), p. 42].

In the remainder of this section we give the proof of theorems analogous to Theorem 1.1 but for more general domains. For G a domain in the plane, let $A(G^-)$ denote the algebra of bounded continuous functions on G^- , the closure of G in the finite plane, that are continuous in G^- and analytic in G . For $f \in A(G^-)$, let

$$\omega(\delta, f; G) = \sup\{|f(z_1) - f(z_2)| : z_1, z_2 \in G^-, |z_1 - z_2| \leq \delta\} \quad (2.6)$$

$$\tilde{\omega}(\delta, f; G) = \sup\{|f(z_1) - f(z_2)| : z_1, z_2 \in \partial G, |z_1 - z_2| \leq \delta\}. \quad (2.7)$$

Further, if $z \in G$, then let $\zeta(z)$ denote a point of the boundary of G with

$$|z - \zeta(z)| = \min\{|z - \zeta| : \zeta \in \partial G\}.$$

LEMMA 2.3. *If $f \in A(G^-)$, then*

$$\omega(\delta, f; G) \leq \tilde{\omega}(2\delta, f; G) + \sup\{|f(z) - f(\zeta(z))| : z \in G, |z - \zeta(z)| \leq \delta\}.$$

Proof. Let $\epsilon > 0$. By Lemma 2.1 we can find $z_1 \in \partial G$, $z \in G$ with $|f(z_1) - f(z)| \geq \omega(\delta, f; G) - \epsilon$, and $|z_1 - z| \leq \delta$. Then $|\zeta(z) - z| \leq \delta$ so $|z_1 - \zeta(z)| \leq 2\delta$ and

$$\begin{aligned} \omega(\delta, f; G) &\leq |f(z_1) - f(z)| + \epsilon \leq |f(z_1) - f(\zeta(z))| + |f(z) - f(\zeta(z))| + \epsilon \\ &\leq \tilde{\omega}(2\delta, f) + \epsilon + |f(z) - f(\zeta(z))|. \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, the lemma follows.

From Lemma 2.3, we see that the only problem in relating $\omega(\delta, f)$ and $\tilde{\omega}(\delta, f)$ is to estimate the term $|f(z) - f(\zeta(z))|$ when $z \in G$. For nice domains, this is not difficult to do. To see this, fix a point $\zeta \in \partial G$. Then define $u(z) = \log |f(z) - f(\zeta)|$.

LEMMA 2.4. For $\zeta \in \partial G$, $f \in A(G)$, u is a subharmonic function on G and we have the inequalities

$$\begin{aligned} u(z) &\leq \log \tilde{\omega}(2\delta, f; G), & z \in \partial G, & \quad |z - \zeta| \leq 2\delta. \\ u(z) &\leq \log \tilde{\omega}(\lambda\delta, f; G), & z \in \partial G, & \quad |z - \zeta| = \lambda\delta, \quad \lambda \geq 1. \end{aligned} \tag{2.8}$$

Proof. These inequalities are clear.

Thus, to estimate the subharmonic function u in the interior of G , we should find a harmonic function that dominates u on ∂G . Then, by the maximum principle, it will also dominate u inside of G . For nice domains G , we can write down such a function explicitly. One such class of domains is the following.

DEFINITION 2.5. We say that the complement of G is *fat* if there exist constants $C \geq 1$, $\delta_0 > 0$ such that for all $\zeta \in \partial D$ and all $0 < \delta \leq \delta_0$, the ball $\{z : |z - \zeta| \leq \delta\}$ contains a point $\zeta' \notin G^-$ such that $|z - \zeta'| \geq (1/C)\delta$ for all $z \in G^-$.

For example, if G has a smooth boundary, the condition of Definition 2.5 holds. Note also that if the complement of G is fat and $0 < \delta \leq \delta_0$, $\zeta \in \partial G$, and $\zeta' \notin G^-$ is a point related to ζ as above, then the function

$$h(z) = h(z, \delta, \zeta) = \log |C(z - \zeta')/\delta| \tag{2.9}$$

has the following properties.

- $h(z)$ is harmonic in a neighborhood of G^- ;
- $h(z) \geq 0$ if $z \in G^-$;
- $h(z) \geq \log(|z - \zeta|/\delta) + \log(C/2)$ if $z \in \partial G$ and $\lambda = |z - \zeta|/\delta \geq 2$.

All the assertions are clear, except for the last one. But, if $z \in G^-$ and $|z - \zeta| \geq 2\delta$, then $|z - \zeta'| \geq \frac{1}{2}|z - \zeta|$, so

$$h(z) \geq \log \left(\frac{C|z - \zeta|}{2\delta} \right).$$

With this function h , we can now prove the following Theorem.

THEOREM 2.6. *Suppose the complement of G is fat, in the sense of Definition 2.5, and that ϕ is a continuous, increasing subadditive function. Then for all $f \in A(G^-)$, and $0 < \delta \leq \delta_0$,*

$$\tilde{\omega}(\delta, f; G) \leq \phi(\delta) \Rightarrow \omega(\delta, f; G) \leq (2 + 8c) \phi(\delta).$$

Proof. From Lemma 2.4, we have for $\zeta \in \partial G$, $u(z) = \log |f(z) - f(\zeta)| \leq \log \phi(2\delta)$ if $z \in \partial G$ and $|z - \zeta| < 2\delta$, while $u(z) \leq \log \phi(\lambda\delta)$ if $z \in \partial G$ and $|z - \zeta| = \lambda\delta$, $\lambda \geq 2$. However, because ϕ is subadditive, $\phi(2\delta) \leq 2\phi(\delta)$, and, in general, for $\lambda > 1$, we have $\phi(\lambda\delta) \leq 2\lambda\phi(\delta)$. In particular, if h is the function of (2.9), then for $z \in \partial G$ and $|z - \zeta| \leq 2\delta$, we have

$$u(z) \leq \log \phi(\delta) + \log 2 \leq h(z) + \log \phi(\delta) + \log 2$$

while if $|z - \zeta| \geq 2\delta$ and $z \in \partial G$,

$$\begin{aligned} u(z) &\leq h(z) + \log 2(|z - \zeta|/\delta) + \log \phi(\delta) \\ &\leq h(z) + \log \phi(\delta) + \log 2 - \log(C/2) \\ &\leq h(z) + \log \phi(\delta) + \log 4. \end{aligned}$$

Thus, in any case,

$$u(z) \leq h(z) + \log \phi(\delta) + \log 4, \quad z \in \partial G.$$

If we replace u by $\tilde{u} = \max(u, \log \phi(\delta) + \log 4)$, then the same inequality still holds, and \tilde{u} is a bounded continuous function on G^- with \tilde{u} subharmonic on G . Since $h(z) + \log \phi(\delta) + \log 4$ is harmonic and dominates \tilde{u} on ∂G , it also dominates \tilde{u} inside G . Thus, if $z \in G$ and $|z - \zeta| \leq \delta$, then

$$\begin{aligned} u(z) &\leq h(z) + \log \phi(\delta) + \log 4 = \log(C|z - \zeta|/\delta) + \log \phi(\delta) + \log 4 \\ &\leq \log 2C + \log \phi(\delta) + \log 4 = \log 8C\phi(\delta). \end{aligned}$$

Hence, $|f(z) - f(\zeta)| \leq 8C\phi(\delta)$. Combining this estimate with that of Lemma 2.3, and using the fact that $\phi(2\delta) \leq 2\phi(\delta)$ again, we have

$$\omega(\delta, f; G) \leq (2 + 8C) \phi(\delta).$$

Finally, we give a theorem for arbitrary simply connected regions. For this we will use an estimate of A. Beurling [1, p. 55] for harmonic measure.

Let G be a simply connected domain in the plane, let $\gamma \subset \partial G$, and let $z \in G$. Let $r(z, G) =$ distance from z to ∂G , and $r(z, \gamma) =$ distance from z to γ . Further, let $\omega(z, \gamma, G)$ denote the harmonic measure of γ for the domain G with respect to the point z .

THEOREM (Beurling, [1, p. 55]).

$$\omega(z, \gamma, G) \leq \frac{4}{\pi} \arctan \left[\frac{r(z, G)}{r(z, \gamma)} \right]^{1/2}.$$

THEOREM 2.7. *Let G be simply connected and ϕ a continuous increasing subadditive function for $\delta \geq 0$ with $\phi(\delta) \geq 0$. Then for $f \in A(G^-)$,*

$$\bar{\omega}(\delta, f) \leq \phi(\delta) \Rightarrow \omega(\delta, f) \leq C\phi(\delta)$$

where C is an absolute constant, independent of G .

Proof. Exactly as in the proof of Theorem 2.6, we must estimate the function $u(z) = \log |f(z) - f(\zeta)|$ for fixed $\zeta \in \partial G$. Let $h(z)$ be the harmonic function on G such that on ∂G ,

$$\begin{aligned} h(\xi) &= \log \phi(2\delta), \quad \xi \in \partial G, \quad |\xi - \zeta| \leq 2\delta \\ h(\xi) &= \log \phi(\lambda\delta), \quad \xi \in \partial G, \quad |\xi - \zeta| \geq 2\delta, \quad \lambda = |\xi - \zeta|/\delta. \end{aligned}$$

That is, $h(z) = \int_{\partial G} h(\xi) \omega(z, d\xi; G)$. If $z \in G$ and $|z - \zeta| \leq \delta$, then

$$h(z) = \sum_{n=1}^{\infty} \int_{E_n} h(\xi) \omega(z, d\xi; G)$$

where $E_1 = \{\xi \in \partial G : |\xi - \zeta| \leq 2\delta\}$ and for $n \geq 2$, $E_n = \{\xi \in \partial G : 2^n\delta < |\xi - \zeta| \leq 2^{n+1}\delta\}$. On the set E_1 , $h(\xi) \leq \log \phi(2\delta) \leq \log 2\phi(\delta)$ while on E_n , $h(\xi) \leq \log \phi(\lambda\delta) \leq \log 2\lambda\phi(\delta) = \log \phi(\delta) + \log 2 + \log |\xi - \zeta|/\delta \leq \log 2\phi(\delta) + (n+1)$. Thus, $h(z) \leq \log 2\phi(\delta) + \sum_{n=2}^{+\infty} (n+1) \omega(z, E_n; G)$. But, $|z - \zeta| \leq \delta$ implies $r(z, G) \leq \delta$ and $r(z, E_n) \geq 2^n\delta - |z - \xi| \geq 2^{n-1}\delta$. Thus, by Beurling's theorem, $\omega(z, E_n; G) \leq (4/\pi) \tan^{-1}[2^{-(1/2)(n-1)}]$. Since $\tan^{-1} x \leq x$, $x \geq 0$, we have

$$\omega(z, E_n; G) \leq \frac{4}{\pi} 2^{-(n-1)/2},$$

so if

$$A = \frac{4}{\pi} \sum_{n=2}^{+\infty} (n+1) 2^{-(n-1)/2},$$

we have

$$h(z) \leq A + \log 2\phi(\delta), \quad z \in G, \quad |z - \zeta| \leq \delta. \tag{2.10}$$

The remainder of the proof is exactly the same as in the proof of Theorem 2.6. The constant C is $2 + 2e^A$.

3. MERGELYAN SETS

DEFINITION 3.1. Let F be a subset of the open unit disk D . Then $U(F)$ is the space of all functions that are analytic in D and uniformly continuous on F , in the topology of uniform convergence on every set of the form $K \cup F$, as K ranges over the compact subsets of D .

Remark. By a standard device, $U(F)$ is seen to be a metric space with metric

$$\rho(f, g) = \sum_{n=1}^{\infty} 2^{-n} \frac{\|f - g\|_{K_n \cup F}}{1 + \|f - g\|_{K_n \cup F}},$$

where K_n is a sequence of compact subsets of D such that every compact set in D is contained in a finite union of the K_n , and

$$\|f\|_E = \sup\{|f(z)| : z \in E\}.$$

DEFINITION 3.2. A relatively closed subset F of D is said to be a Mergelyan set if the polynomials are dense in $U(F)$.

PROPOSITION 3.3. If F is a Mergelyan set and if $\phi : D \rightarrow D$ is a one-to-one conformal map, then $\phi(F)$ is a Mergelyan set.

Proof. The proof is an easy consequence of the fact that ϕ must be analytic on the closure of D , and we omit it.

DEFINITION 3.4. A set F in D is said to be radial if there is a sequence $r_n \rightarrow 1^-$ such that $F_n \cap D \subset F$ where $F_n = \{z/r_n : z \in F\}$.

PROPOSITION 3.5. Every radial set is a Mergelyan set.

The proof uses the familiar mapping $f \rightarrow f_r$, where $f_r(z) = f(rz)$, and we omit it.

DEFINITION 3.6. The polynomial hull of G , $H_p(G)$, is the set of points z for which

$$|p(z)| \leq \sup\{|p(w)| : w \in G\}$$

for all polynomials p .

DEFINITION 3.7. The uniformly continuous analytic hull of $G \subset D$ with respect to F , $H_{U,F}(G)$, is the set of all points $z \in D$ for which

$$|f(z)| \leq \sup\{|f(w)| : w \in G\}$$

for all $f \in U(F)$.

PROPOSITION 3.8. If F is a Mergelyan set, then

$$H_{U,F}(K \cup F) = H_p(K \cup F) \cap D$$

for every compact subset K of D .

Proof. It is clear that $H_{U,F}(G) \subset H_p(G) \cap D$ for all subsets G of D . In the other direction, take $z \in H_p(K \cup F) \cap D$, and let $f \in U(F)$. Let P_n be a sequence of polynomials which converge to f in $U(F)$. Then

$$\begin{aligned} |f(z)| &= \lim_{n \rightarrow \infty} |P_n(z)| \leq \limsup_{n \rightarrow \infty} \{|P_n(w)| : w \in K \cup F\} \\ &= \sup\{|f(w)| : w \in K \cup F\}. \end{aligned}$$

and the result follows.

Stray [8] has recently established the converse to this proposition.

PROPOSITION 3.9. Let $F = \{z_n\}$ be a Blaschke sequence (i.e., $\sum 1 - |z_n| < +\infty$, $|z_n| < 1$) such that every point of ∂D is a limit point of the z_n . Then F is not a Mergelyan set.

Proof. It is clear that the Blaschke product over the z_n cannot be approximated in $U(F)$ by polynomials.

DEFINITION 3.10. A bullseye is a closed subset of D that contains circles $\{z : |z| = r\}$ for values of r arbitrarily close to 1.

PROPOSITION 3.11. Every bullseye is a Mergelyan set.

Proof. By Theorem 1.1, if $f \in U(F)$ then f must be uniformly continuous on $|z| < 1$ and is, consequently, the uniform limit on D of a sequence of polynomials.

THEOREM 3.12. There exist two Mergelyan sets whose intersection is not a Mergelyan set.

Proof. It is clear that there are two bullseyes whose intersection is a Blaschke sequence that is dense on ∂D . Propositions 3.9 and 3.11 now apply.

PROPOSITION 3.13. Let J be a simple closed Jordan curve in \bar{D} that intersects ∂D only at $z = 1$, and let $J' = J \setminus \{1\}$. Then $H_{U,J'}(J') = J'$.

Proof. We need only prove that if $z \notin J'$ then there is an $f \in U(J')$ with $|f(z)| > 1$ but $|f(w)| \leq 1$ for $w \in J'$. Without loss of generality, we will take $z = 0$. Let $\phi(z) = (1+z)/(1-z)$ and let $A = \phi(J')$. By Arakelian's

theorem [2, Theorem 3.1, p. 37], there exists an entire function λ such that $|\lambda(1)| > 1$ but $|\lambda(w)| \leq 1$ for $w \in A$, and such that $\lambda(w) \rightarrow 0$ as $w \rightarrow \infty$, $w \in A$. (Just take a continuous function α on $A \cup \{1\}$ that is large at 1 and small on A , tending to 0 at ∞ , and approximate it within $\epsilon/(1 + |z|)$ on $A \cup \{1\}$, where $\epsilon = 1/10$, say.) Let $f(z) = \lambda(\phi(z))$ to complete the proof.

THEOREM 3.14. *There exist two Mergelyan sets E and F in D whose union is not a Mergelyan set.*

Proof. Take E to be the radius from 0 to 1 and let F be a short circular arc that touches ∂D at 1 at right angles to ∂D . By Propositions 3.3 and 3.5 both E and F are Mergelyan sets. But let K be the straight line segment that joins the endpoints of E and F that lie in D , and let $B = E \cup F$. Then $B \cup K$ is a set J' of the form just discussed and so $D \cap H_p(B \cup K) \neq H_{U,B}(B \cup K)$. Indeed, the left-hand side is the inside of J' while the right-hand side is just the curve J' . By Proposition 3.13, $E \cup F$ is not a Mergelyan set, and the result is proved.

4. EXAMPLES

We give some examples which indicate that, in some sense, Theorem 1.1 is best possible. It is well-known that Theorem 1.1 is false for harmonic functions. A simple example is given by the harmonic function on $|z| < 1$ with the boundary function $u(e^{it}) = |e^{it} - 1| \sim |t|$ for small t . Clearly, $\tilde{\omega}(\delta, u) \leq c\delta$. But, if $\delta = 1 - r$, then the Poisson kernel $P(r, t)$ satisfies $P(r, t) \geq (\delta/t^2)$ for $|t| \geq \delta$, so

$$u(r) \geq c\delta \int_{\delta}^{\pi} \frac{u(e^{it})}{t^2} dt \geq c'\delta \log \frac{1}{\delta}.$$

Thus, $\omega(\delta, u) \geq |u(r) - u(1)| \geq c'\delta \log(1/\delta)$. This example shows that the logarithmic factor in Theorem 1.1' is best possible.

An explicit example which shows that $C > 1$ in Theorem 1.1 is the function F which conformally maps the unit circle onto the region interior to the two circles which pass through $+1$, -1 , and $\pm((2)^{1/2} - 1)i$, respectively. Normalize F so that $F(0) = 0$, $F(1) = 1$. Then an explicit formula for F is

$$F(z) = \frac{1 - (1 - z^2)^{1/2}}{z}.$$

It can be verified that $\tilde{\omega}(1, F) = |F(e^{i\pi/3}) - F(1)| = 0.9332... < 1 = |F(1) - F(0)| \leq \omega(1, F)$. We will not give the details of this calculation here, however, since there are slightly less explicit examples which are easier to check.

A very simple function which is *almost* an example is the function

$$g(z) = z^{1/4} + z^{1/3}$$

in the half-plane $H = \{z : \operatorname{Re} z > 0\}$. Unfortunately, g is not bounded, so it is not really an example. However, it is easy to modify g to obtain such an example. We first give the main properties of g .

LEMMA 4.1. *Let $0 < a < b \leq 1/3$ and for $\operatorname{Re} z \geq 0$ let*

$$g(z) = z^a + z^b = r^a e^{ia\theta} + r^b e^{ib\theta}, \quad z = r e^{i\theta}, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2},$$

$$\tilde{\omega}(\delta) = \tilde{\omega}(\delta, g) = \sup\{|g(iy_1) - g(iy_2)| : |y_1 - y_2| \leq \delta\}$$

$$\omega(\delta) = \omega(\delta, g) = \sup\{|g(z_1) - g(z_2)| : |z_1 - z_2| \leq \delta, \operatorname{Re} z_i \geq 0\}.$$

Then

$$\tilde{\omega}(\delta) = |g(i\delta)| < g(\delta) \leq \omega(\delta), \quad \delta > 0.$$

Proof. Fix $t > 0$ and define $\psi(y) = |g(iy) - g(i(y-t))|^2$. We will show that

$$\sup\{\psi(y) : -\infty < y < +\infty\} = \psi(t) = \psi(0) = |g(it)|^2 \quad (4.1)$$

and that

$$\max\{|g(it)| : 0 \leq t \leq \delta\} = |g(i\delta)|. \quad (4.2)$$

These two facts imply the lemma. We first prove (4.2). Write

$$g(iy) = u(y) + iv(y)$$

so that

$$u(y) = y^a \cos \frac{a\pi}{2} + y^b \cos \frac{b\pi}{2} \quad y \geq 0$$

$$v(y) = y^a \sin \frac{a\pi}{2} + y^b \sin \frac{b\pi}{2} \quad y \geq 0$$

and

$$u(y) = u(-y), \quad v(y) = -v(y).$$

Clearly, u, v are non-negative increasing functions for $y \geq 0$ and therefore so is $|g(iy)|$, which proves (4.2).

To prove (4.1), consider the function $\varphi(y) = \arg g(iy)$. Now $\tan \varphi(y) = v(y)/u(y) = (\tan(a\pi/2))\{(1 + \lambda y^c)/(1 + \mu y^c)\}$ where $c = b - a > 0$,

$$\lambda = \frac{\sin(b\pi/2)}{\sin(a\pi/2)} > 1 \quad \text{and} \quad \mu = \frac{\cos(b\pi/2)}{\cos(a\pi/2)} < 1.$$

It is then easy to check that $y \mapsto (1 + \lambda y^c)/(1 + \mu y^c)$ is increasing for $y \geq 0$. In particular, $\tan \varphi(y) \leq \lim_{y \rightarrow \infty} \tan \varphi(y) = \tan(b\pi/2)$. Thus, $0 \leq \varphi(y) \leq b\pi/2 \leq \pi/6$ for $y \geq 0$.

Now write $g(it) = \rho_0 e^{i\varphi_0}$. Since $|g(iy)|$ and $\arg g(iy)$ are increasing for $y \geq 0$, and since $g(iy) = \overline{g(-iy)}$, it follows that $g(iy)$ belongs to the sector

$$S = \{w = \rho e^{i\varphi} : 0 \leq \rho \leq \rho_0, \varphi \leq \varphi_0\}$$

whenever $|y| \leq t$. However, since $\varphi_0 \leq \pi/6$, it is clear that the diameter of the sector is

$$\rho_0 = |\rho_0 e^{i\varphi_0} - 0| = |g(it) - g(0)| = |g(it)|.$$

Therefore,

$$\max\{\psi(y) : 0 \leq y \leq t\} = |g(it)|^2. \quad (4.3)$$

Next consider $y \geq t$. A short calculation shows $\psi(y) = [h_a(y)]^2 + [h_b(y)]^2 + 2h_a(y)h_b(y)\cos(b-a)\pi/2$, $y \geq t$, where $h_a(y) = y^a - (y-t)^a$. The function ψ is thus the sum of three decreasing functions so

$$\sup\{\psi(y) : y \geq t\} = \psi(t) = |g(it)|^2. \quad (4.4)$$

Eq. (4.1) is a consequence of (4.3) and (4.4). This completes the proof.

Remark. The same argument will show that $g(z) = z^a + z^b$ with $0 < a < \frac{1}{3}$, $b \leq 1$ has $\tilde{\omega}(\delta) = g(i\delta)$ for small $\delta > 0$.

As a consequence of the lemma, we can also see that $\tilde{\omega}(\delta, g) \sim \omega(\delta, g)$ as $\delta \rightarrow 0$. In fact, we have, with

$$M(\delta) = \frac{|g(\delta) - g(0)|}{\tilde{\omega}(\delta, g)} = \left| \frac{g(\delta)}{g(i\delta)} \right|$$

that

$$[M(\delta)]^2 = \frac{(\delta^a + \delta^b)^2}{\delta^{2a} + \delta^{2b} + 2\delta^{a+b}\cos((b-a)\pi/2)} = \frac{(1+t)^2}{(1+t)^2 - 2t\eta}$$

where $t = \delta^{b-a}$ and $\eta = 1 - \cos(b-a)\pi/2 > 0$. Thus, as $\delta \rightarrow 0$, we have that

$$M(\delta) = 1 + \eta\delta^{b-a} + O(\delta^{2(b-a)}). \quad (4.5)$$

We want to modify g to obtain an example in the unit disc. The idea is to multiply g by $R/(z+R)$ where R is a large positive number. This new function is then bounded in $\operatorname{Re} z \geq 0$. If we then restrict it to a large disk $|z-N| < N$ in the right half-plane, it provides an example on this large disk that can then be transferred to the unit disk. We will do a little more work and obtain an example in the unit disk $D = \{|z| < 1\}$ with

$$\limsup_{\delta \rightarrow 0} \frac{\omega(\delta, f; D)}{\tilde{\omega}(\delta, f; D)} > 1.$$

LEMMA 4.2. Let $D = \{z : |z| < 1\}$. Then there exist numbers $\delta_1 > 0$, $\lambda > 1$, $c > 0$ such that: for every δ , $0 < \delta \leq \delta_1$, there is a function $k = k_\delta$ analytic for $|z| < 1$, continuous for $|z| \leq 1$, and such that

- (1) $|k(1 - \delta) - k(1)| \geq \lambda \tilde{\omega}(\delta, k; D)$,
- (2) $|k(z)| \leq 1$, and
- (3) $\tilde{\omega}(\delta, k; D) \geq c$.

Proof. For $R \geq 1$, set $h(z) = (R/(z + R))g(z)$ where g is as in Lemma 4.1. Then

$$h(z_1) - h(z_2) = \frac{R(z_2 - z_1)}{(z_1 + R)(z_2 + R)} g(z_1) + \frac{R}{z_2 + R} [g(z_1) - g(z_2)].$$

Therefore, if $A(R) = \max\{|g(z_1)/(z_1 + R)| : \operatorname{Re} z_1 \geq 0\}$ we have $\tilde{\omega}(\delta, h) \leq A(R)\delta + \tilde{\omega}(\delta, g)$ and then

$$\begin{aligned} \frac{|h(\delta) - h(0)|}{\tilde{\omega}(\delta, h)} &\geq \frac{(R/(\delta + R))g(\delta)}{\tilde{\omega}(\delta, g) + A(R)\delta} \\ &= \frac{1}{1 + \delta/R} \frac{g(\delta)}{|g(i\delta)| + A(R)\delta}. \end{aligned} \quad (4.6)$$

Now as $R \rightarrow \infty$, we have $A(R) \rightarrow 0$ so we can fix $R = R_0 \leq 1$ so large that $A(R) \leq 1$. Then, for small $\delta > 0$, we have $|g(\delta)| \sim \delta^a$, $|g(i\delta)| \sim \delta^a$, so the right-hand side of (4.6) is equal to

$$\frac{g(\delta)}{|g(i\delta)|} + O(\delta) \quad \text{as } \delta \rightarrow 0.$$

Combined with (4.5), this yields

$$\frac{|h(\delta) - h(0)|}{\tilde{\omega}(\delta, h)} = 1 + 2\eta\delta^{b-a} + o(\delta^{b-a}) \quad \text{as } \delta \rightarrow 0.$$

Thus, there is a constant $\delta_0 > 0$ such that

$$\frac{|h(\delta) - h(0)|}{\tilde{\omega}(\delta, h)} \geq 1 + \frac{\eta\delta^{b-a}}{2} \quad 0 < \delta < \delta_0. \quad (4.7)$$

For $N \geq 1$ and $|\zeta| \leq 1$, let $z = N(1 - \zeta)$ and $k(\zeta) = k(\zeta, N) = h(z)$. We claim that for large N , the functions $k_\delta = k(\cdot, N)$, where $\delta = \delta_0/N$, essentially satisfy the conditions of the lemma. For, since $h(z) \rightarrow 0$ as $|z| \rightarrow \infty$, it follows that the modulus of continuity of the restriction of h to the large circles $|z - N| = N$ tends to the modulus of continuity of the restriction of

h to the imaginary axis. Thus, for every $\tau > 0$, there is a number $N_0 \geq 1$ so large that

$$\tilde{\omega}(\delta_0/N, k(\cdot, N); D) \leq (1 + \tau) \tilde{\omega}(\delta_0, h),$$

and

$$\tilde{\omega}(\delta_0/N, k(\cdot, N); D) \geq (1 - \tau) \tilde{\omega}(\delta_0, h).$$

But then

$$\frac{|k(1 - \delta_0/N, N) - k(1, N)|}{\tilde{\omega}(\delta_0/N, k(\cdot, N); D)} \geq \frac{|h(\delta_0)|}{(1 + \tau) \tilde{\omega}(\delta_0, h)} \geq \frac{1 + (\eta/2) \delta_0^{b-a}}{1 + \tau}.$$

If τ is small, say $\tau < \frac{1}{4} \eta \delta_0^{b-a}$, then $\lambda = [1 + (\eta/2) \delta_0^{b-a}]/(1 + \tau) > 1$ and (1) of the lemma holds with $\delta_1 = \delta_0/N_0$, $\delta = \delta_0/N$, and $k_\delta = k(\cdot, \delta_0/N)$. Since $\tilde{\omega}(\delta_0/N, k(\cdot, N); D) \geq (1 - \tau) \tilde{\omega}(\delta_0, h)$, the condition (3) holds. Condition (2) may not hold, but since h is bounded we can divide each of the $k(\cdot, N)$ by $\sup |h(z)|$ to make it hold. This will not destroy (1) or (3). This completes the proof of the lemma.

PROPOSITION 4.3. *There exists a function F analytic for $|z| < 1$ and continuous for $|z| \leq 1$ such that, with $D = \{|z| < 1\}$,*

$$\limsup_{\delta \rightarrow 0} \frac{|F(1 - \delta) - F(1)|}{\tilde{\omega}(\delta, F; D)} > 1.$$

Proof. The function F may be written explicitly as follows. For suitable sequences of positive numbers ϵ_j, δ_j ,

$$F(z) = \sum_{j=1}^{\infty} \epsilon_j k_j(z)$$

where k_j is a function as in Lemma 4.2 associated to the number δ_j . To carry out the construction, choose $\eta > 0$ so small that $(1 + 2\eta)/(1 - 2\eta) < \lambda$, where λ is as in Lemma 4.2. Set $\epsilon_1 = 1$, $\delta_1 =$ the number δ_1 of Lemma 4.2, and k_1 a function satisfying the conditions (1)–(3) of Lemma 4.2. Then inductively choose positive numbers ϵ_n, δ_n , and functions k_n so that

$$\sum_{j=n+1}^{\infty} 2\epsilon_j < \eta \epsilon_n \tilde{\omega}(\delta_n, k_n) \tag{4.8}$$

$$\sum_{j=1}^{n-1} \epsilon_j \omega(\delta_n, k_j) < \eta \epsilon_n \tilde{\omega}(\delta_n, k_n). \tag{4.9}$$

It is possible to do this since $\tilde{\omega}(\delta_n, k_n) \geq c > 0$ while $\omega(\delta, k_j) \rightarrow 0$ as $\delta \rightarrow 0$ for $j \leq n$. We can also assume $\epsilon_j \leq 2^{-j}$.

Now, we claim that with

$$F(z) = \sum_{j=1}^{\infty} \epsilon_j k_j(z)$$

we have that $F(1 - \delta_n) - F(1)$ is about equal to $\epsilon_n[k_n(1 - \delta_n) - k_n(1)]$ and $\tilde{\omega}(\delta_n, F)$ is about equal to $\epsilon_n \tilde{\omega}(\delta_n, k_n)$. For

$$\begin{aligned} \tilde{\omega}(\delta_n, F) &\leq \sum_{j=1}^{n-1} \epsilon_j \tilde{\omega}(\delta_n, k_j) + \epsilon_n \tilde{\omega}(\delta_n, k_n) + \sum_{j=n+1}^{\infty} 2\epsilon_j \\ &\leq (1 + 2\eta) \epsilon_n \tilde{\omega}(\delta_n, k_n) \end{aligned}$$

by (4.8) and (4.9). Similarly, we have

$$|F(1 - \delta_n) - F(1)| \geq (1 - 2\eta)\epsilon_n |k_n(1 - \delta_n) - k_n(1)|.$$

Therefore,

$$\begin{aligned} \limsup_n \frac{|F(1 - \delta_n) - F(1)|}{\tilde{\omega}(\delta_n, F)} &\geq \limsup_n \frac{1 - 2\eta}{1 + 2\eta} \frac{|k_n(1 - \delta_n) - k_n(1)|}{\tilde{\omega}(\delta_n, k_n)} \\ &\geq \frac{1 - 2\eta}{1 + 2\eta} \quad \lambda > 1. \end{aligned}$$

This completes the proof.

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