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Weakly distributive categories

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Abstract

There are many situations in logic, theoretical computer science, and category theory where two binary operations – one thought of as a (tensor) "product", the other a "sum" – play a key role. In distributive and $*$-autonomous categories these operations can be regarded as, respectively, the AND/OR of traditional logic and the TIMES/PAR of (multiplicative) linear logic. In the latter logic, however, the distributivity of product over sum is conspicuously absent: this paper studies a "linearization" of that distributivity which *is* present in both case. Furthermore, we show that this weak distributivity is precisely what is needed to model Gentzen's cut rule (in the absence of other structural rules) and can be strengthened in two natural ways to generate full distributivity and $*$-autonomous categories.

0. Introduction

There are many situations in logic, theoretical computer science, and category theory where two binary operations, "tensor products" (though one may be a "sum"), play a key role. The multiplicative fragment of linear logic is a particularly interesting example as it is a Gentzen style sequent calculus in which the structural rules of contraction, thinning, and (sometimes) exchange are dropped. The fact that these rules are omitted considerably simplifies the derivation of the cut elimination theorem.

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Furthermore, the proof theory of this fragment is interesting and known [13] to correspond to $*$-autonomous categories as introduced by Barr in [2].

In the study of categories with two tensor products one usually assumes a distributivity condition, particularly in the case when one of these is either the product or sum. The multiplicative fragment of linear logic (viz. $*$-autonomous categories) is a significant exception to this situation; here the two tensors "times" ($\otimes$) and "par" ($⅋$ which we denote by $\oplus$ – note that this conflicts with the notation introduced by Girard) do not distribute one over the other.

However, $*$-autonomous categories are known to satisfy a weak notion of distributivity, in effect a "linearization" of the usual distributivity. This weak distributivity is given by maps of the form:

$$A \otimes (B \oplus C) \longrightarrow (A \otimes B) \oplus C$$
$$A \otimes (B \oplus C) \longrightarrow B \oplus (A \otimes C)$$

which amount to the requirement that one tensor is strong with respect to the other. To see what this means, we recall that a functor F is strong (with respect to a tensor product $\otimes$) if there is a natural transformation $X \otimes F(Y) \longrightarrow F(X \otimes Y)$ satisfying certain coherence conditions. If we take F to be the functor $F(Y) = Y \oplus C$, then this gives the weak distributive law $X \otimes (Y \oplus C) \longrightarrow (X \otimes Y) \oplus C$.

These maps, interpreted as entailments, are also valid in what might be considered the minimal logic of two such tensors, namely the classical Gentzen sequent calculus with the left and right introduction rules for conjunction and disjunction and with cut as the only structure rule. This Gentzen style proof theory has a categorical presentation already in the literature, viz. the polycategories of Lambek and Szabo [14]. It should therefore be possible to link $*$-autonomous categories and polycategories. However, this begs a wider question of precisely what properties a category must satisfy to be linked in this manner to the logical superstructure provided by a polycategory.

It turns out that these weak distributivity maps, when present coherently, are precisely the necessary structure required to construct a polycategory superstructure, and whence a Gentzen style calculus, over a category with two tensors. The weak distributivity maps allow the expression of the Gentzen cut rule in terms of ordinary (categorical) composition.

We call categories with two tensors linked by coherent weak distribution *weakly distributive categories*. They can be built up to be the proof theory of the full multiplicative fragment of classical linear logic [1] by coherently adding maps

$$\top \longrightarrow A \oplus A^{\perp}$$
$$A \otimes A^{\perp} \longrightarrow \perp$$

[1] The system FILL (full intuitionistic linear logic) of de Paiva [7] amounts to having just the second of these (families of) maps. From the autonomous category viewpoint, these are the more natural maps, as they correspond to evaluations. The symmetry of the $*$-autonomous viewpoint then suggests the first (family of) maps.

or to the proof theory of the $\wedge, \vee$ fragment of intuitionistic propositional logic by coherently adding contraction, thinning, and exchange. The former corresponds to $*$-autonomous categories and the latter to distributive categories.

In fact, weakly distributive categories lie at the base of a rich logical hierarchy, unifying several hitherto separate developments in the logics of theoretical computer science. In this paper we shall see some of these connections, in particular relating weakly distributive categories to $*$-autonomous categories, to distributive categories, and to braided monoidal categories. Furthermore, the duality involved in the definition of weakly distributive categories means that the opposite of a weakly distributive category is weakly distributive – so for example codistributive categories are weakly distributive. One has frequently been struck by the strangeness of the distributivity in such codistributive categories as the category of commutative rings, or the category of distributive lattices, and so on: they may now be seen as weakly distributive in the standard manner. Other famous examples of non-distributivity can be accommodated in this framework – the category of pointed sets is weakly distributive in the obvious way, with product and sum as the two tensors.

One point must be made about the connection with linear logic. A novel feature of our presentation is that we have considered the two tensor structure separately from the structure given by linear negation $(-)^{\perp}$. We show how to obtain the logic of $*$-autonomous categories from that of weakly distributive categories, giving, in effect, another presentation of $*$-autonomous categories. It sometimes happens that it is easier to verify $*$-autonomy this way; for example, verifying that a lattice with appropriate structure is $*$-autonomous becomes almost trivial if one checks the weak distributivity first (see [3]).

It is clear that weakly distributive categories constitute a very weak fragment of linear logic – one motivation behind studying them is indeed driven by this very weakness. The usual proof theoretic studies of classical linear logic introduce negation at a very early stage, making it difficult to see the interaction between the two multiplicative tensors TIMES and PAR. Weakly distributive categories isolate this interaction allowing a finer modularity of the construction of the proof theories.

We have been very brief about coherence questions here; these matters are more fully handled in a sequel [4], which contains as well an answer to the question of the conservativity of the extension to $*$-autonomous categories. In that paper, coherence is completely settled not only for weakly distributive categories but also for $*$-autonomous categories.

A note to end on: this paper is the "journal version" of [6]: it answers some of the problem we had to leave open at that time and provides more details where necessary. However, as time passes, one's view of things often alters, and so we now see a few matters differently. In particular, we are less convinced of the naturality of the original notion of weakly distributive category, and lean more to two extremes: viz. the symmetric case and the planar case. In the former, each tensor is symmetric and so we only need one of the weak distributivities (the others being derived via symmetry), and in the planar case, the tensors are not assumed symmetric, but only the non-permuting weak distributivities (those where the order of the variables is not changed) are assumed. This latter case is, we now agree, a better syntax for non-commutativity. Along with this goes the view that one ought to have two negations in the non-commutative case when passing to $*$-autonomy. The original notion is now relegated to the status of marginalia, at least until some compelling reason is found to reinstate it. The original notation, however, is useful as it allows a discussion of the two notions simultaneously – as we shall see below.

It turns out that there is rather more history to these notions than we had originally realized. The notion of weakly distributive category (in one flavour or another, often with variations) has been considered by a number of people, albeit usually only in the posetal case – the coherence conditions do not appear to have been considered before. A particularly good survey of the development of these notions appears in [11], and we refer the reader to Lambek's article for the details.

Note added in proof

An error in Proposition 3.1, where we claimed that distributive categories are weakly distributive, was found in proof. The result is totally incorrect: a distributive category is a cartesian weakly distributive category if and only if it is a preorder.

In particular, any distributive category which satisfies Eq. (13) for the choice of weak distributions described in the paper is immediately a preorder. This because in that diagram if $A = D = 1$ and $B = C = 0$ then, up to equivalence, we obtain for the two sides of the diagram the coproduct embeddings of 1 into $1 + 1$. This suffices to cause collapse. The argument can be modified to show that in any distributive category which is simultaneously weakly distributive (no matter how the weak distributions are defined) Boolean negation must have a fixed point. This also suffices to cause collapse.

Essentially, the reason why distributive categories are not weakly distributive is because the former cannot be a satisfactory semantics for any "reasonable" AND/OR logic. It is easy to verify that Eq. (13) is necessary if the semantics is to satisfy categorical cut elimination for the modest sequent calculus in the paper. (See Fig. 1 for example.) In particular, distributive categories cannot provide a sound semantics for any AND/OR logic, such as standard intuitionist or classical logic, which includes this sequent calculus as a fragment.

It is interesting to note, however, that by carefully choosing the weak distributions one can construct a cartesian weakly distributive category from an elementary distributive category by simply passing to the Kleisli category of the "exception monad" $E(X) = X + 1$. So, for example, although **Sets** is not weakly distributive itself, **Pointed Sets** is.

The error means, of course, that all discussion in the paper of non-posetal distributive categories as examples of weakly distributive categories must be discounted. This affects the Introduction, where in the seventh paragraph especially ("In fact, weakly distributive categories lie at the base of ...") must be read to exclude distributive categories, and Section 3, where Proposition 3.1 must be restated as indicated above, and the surrounding text must take this restatement into account. In particular, Theorem 3.3, although still correct, ought to be stengthened to state that a cartesian weakly distributive category is *a preorder* if and only if it has a strict initial object.

A version of this paper which contains a rewritten Introduction and Section 3 may be found on the WWW at this URL: <http://www.math.mcgill.ca/rags>.

1. Polycategories

We shall begin with a review of Szabo's notion of a polycategory:

Definition 1.1. A polycategory $\mathbf{C}$ consists of a set $\mathrm{Ob}(\mathbf{C})$ of objects and a set $\mathrm{M}\varphi(\mathbf{C})$ of morphisms, (also called arrows, polymorphisms, etc.) just like a category, except that the source and target of a morphism are finite sequences of objects

$$\textit{source}: \mathrm{M}\varphi(\mathbf{C}) \longrightarrow \mathrm{Ob}(\mathbf{C})^*$$

$$\textit{target}: \mathrm{M}\varphi(\mathbf{C}) \longrightarrow \mathrm{Ob}(\mathbf{C})^*$$

(where X^* = the free monoid generated by X).

There are identity morphisms $i_A: A \to A$ between singleton sequences (only) and a notion of composition given by the *cut rule*:

$$\frac{\Gamma_1, A, \Gamma_2 \xrightarrow{g} \Gamma_3 \quad \Delta_1 \xrightarrow{f} \Delta_2, A, \Delta_3}{\Gamma_1, \Delta_1, \Gamma_2 \xrightarrow{g\,{}_i\circ_j\, f} \Delta_2, \Gamma_3, \Delta_3}$$

where the length of Γ_1 is i and the length of Δ_2 is j. When the subscripts are clear from the context they shall be dropped. We shall place the following *restriction* on the cut rule: we allow cuts only if either Γ_1 or $\Delta_2 = \phi$, and either Γ_2 or $\Delta_3 = \phi$.

We have the following equations[2]:

$$(1) \qquad \Gamma_1 \xrightarrow{f} \Gamma_2, A, \Gamma_3 \quad = \quad \frac{A \xrightarrow{i_A} A \quad \Gamma_1 \xrightarrow{f} \Gamma_2, A, \Gamma_3}{\Gamma_1 \xrightarrow{i_A\, {}_0\circ_j\, f} \Gamma_2, A, \Gamma_3}$$

$$(2) \qquad \Gamma_1, A, \Gamma_2 \xrightarrow{f} \Gamma_3 \quad = \quad \frac{\Gamma_1, A, \Gamma_2 \xrightarrow{f} \Gamma_3 \quad A \xrightarrow{i_A} A}{\Gamma_1, A, \Gamma_2 \xrightarrow{f\, {}_i\circ_0\, i_A} \Gamma_3}$$

[2] Below "$\Delta|\Gamma$" represents the trivial concatenation of a sequence and an empty sequence – as forced by the restriction on the cut rule.

$$(3)\qquad \dfrac{\Phi_1, B, \Phi_2 \xrightarrow{h} \Phi_3 \quad \dfrac{\Delta_1, A, \Delta_2 \xrightarrow{g} \Delta_3, B, \Delta_4 \quad \Gamma_1 \xrightarrow{f} \Gamma_2, A, \Gamma_3}{\Delta_1, \Gamma_1, \Delta_2 \xrightarrow{g\ {}_i\circ_j\ f} \Gamma_2, \Delta_3, B, \Delta_4, \Gamma_3}}{\Phi_1, \Delta_1, \Gamma_1, \Delta_2, \Phi_2 \xrightarrow{h\ {}_l\circ_{j+m}\ (g\ {}_i\circ_j\ f)} \Gamma_2, \Delta_3, \Phi_3, \Delta_4, \Gamma_3}$$

$$= \quad \dfrac{\dfrac{\Phi_1, B, \Phi_2 \xrightarrow{h} \Phi_3 \quad \Delta_1, A, \Delta_2 \xrightarrow{g} \Delta_3, B, \Delta_4}{\Phi_1, \Delta_1, A, \Delta_2, \Phi_2 \xrightarrow{h\ {}_l\circ_m\ g} \Delta_3, \Phi_3, \Delta_4} \quad \Gamma_1 \xrightarrow{f} \Gamma_2, A, \Gamma_3}{\Phi_1, \Delta_1, \Gamma_1, \Delta_2, \Phi_2 \xrightarrow{(h\ {}_l\circ_m\ g)\ {}_{l+i}\circ_j\ f} \Gamma_2, \Delta_3, \Phi_3, \Delta_4, \Gamma_3}$$

$$(4)\qquad \dfrac{\dfrac{\Phi_1, A, \Phi_2, B, \Phi_3 \xrightarrow{h} \Phi_4 \quad \Gamma_1 \xrightarrow{f} \Gamma_2, A, \Gamma_3}{\Phi_1, \Gamma_1, \Phi_2, B, \Phi_3 \xrightarrow{h\ {}_i\circ_j\ f} \Gamma_2, \Phi_4, \Gamma_3} \quad \Delta_1 \xrightarrow{g} \Delta_2, B, \Delta_3}{\Phi_1, \Gamma_1, \Phi_2, \Delta_1, \Phi_3 \xrightarrow{(h\ {}_i\circ_j\ f)\ {}_{i+k+l}\circ_m\ g} \Delta_2 | \Gamma_2, \Phi_4, \Delta_3 | \Gamma_3}$$

$$= \quad \dfrac{\dfrac{\Phi_1, A, \Phi_2, B, \Phi_3 \xrightarrow{h} \Phi_4 \quad \Delta_1 \xrightarrow{g} \Delta_2, B, \Delta_3}{\Phi_1, A, \Phi_2, \Delta_1, \Phi_3 \xrightarrow{h\ {}_{i+1+l}\circ_m\ g} \Delta_2, \Phi_4, \Delta_3} \quad \Gamma_1 \xrightarrow{f} \Gamma_2, A, \Gamma_3}{\Phi_1, \Gamma_1, \Phi_2, \Delta_1, \Phi_3 \xrightarrow{(h\ {}_{i+1+l}\circ_m\ g)\ {}_i\circ_j\ f} \Delta_2 | \Gamma_2, \Phi_4, \Delta_3 | \Gamma_3}$$

$$(5)\qquad \dfrac{\Phi_1, B, \Phi_2 \xrightarrow{h} \Phi_3 \quad \dfrac{\Delta_1, A, \Delta_2 \xrightarrow{g} \Delta_3 \quad \Gamma_1 \xrightarrow{f} \Gamma_2, A, \Gamma_3, B, \Gamma_4}{\Delta_1, \Gamma_1, \Delta_2 \xrightarrow{g\ {}_i\circ_j\ f} \Gamma_2, \Delta_3, \Gamma_3, B, \Gamma_4}}{\Phi_1 | \Delta_1, \Gamma_1, \Phi_2 | \Delta_2 \xrightarrow{h\ {}_m\circ_{j+k+l}\ (g\ {}_i\circ_j\ f)} \Gamma_2, \Delta_3, \Gamma_3, \Phi_3, \Gamma_4}$$

$$= \quad \dfrac{\Delta_1, A, \Delta_2 \xrightarrow{g} \Delta_3 \quad \dfrac{\Phi_1, B, \Phi_2 \xrightarrow{h} \Phi_3 \quad \Gamma_1 \xrightarrow{f} \Gamma_2, A, \Gamma_3, B, \Gamma_4}{\Phi_1, \Gamma_1, \Phi_2 \xrightarrow{h\ {}_m\circ_{j+1+l}\ f} \Gamma_2, A, \Gamma_3, \Phi_3, \Gamma_4}}{\Phi_1 | \Delta_1, \Gamma_1, \Phi_2 | \Delta_2 \xrightarrow{g\ {}_i\circ_j\ (h\ {}_m\circ_{j+1+l}\ f)} \Gamma_2, \Delta_3, \Gamma_3, \Phi_3, \Gamma_4}$$

Remark 1.2 (Other varieties of polycategories). The original definition of polycategory in [14], which we used in the preliminary version of this paper [6], differs from the definition above in that we have here placed a restriction on the cut rule. This amounts to imposing "planarity" on the proof theory of polycategories. This matter is discussed at some length by Lambek [11]: he gives three variant systems in which the cut rule has varying degrees of restriction.

In the weakest system, **BL1**, cut is restricted to instances where either $\Gamma_1 = \Gamma_2 = \phi$ or $\Delta_2 = \Delta_3 = \phi$. This corresponds to having no weak distributivities. A stronger system, **BL2**, allows cuts where either $\Gamma_1 = \Delta_3 = \phi$ or $\Gamma_2 = \Delta_2 = \phi$. This is the system we have adopted and is also the system of Abrusci in [1]. It is the basis of planar noncommutative linear logic. To motivate these restrictions, in the case when the tensors coincide (e.g. in compact closed categories) the cut is a composition of arrows of the form

$$\Gamma_1, \Delta_1, \Gamma_2 \longrightarrow \Gamma_1, \Delta_2, A, \Delta_3, \Gamma_2, \qquad \Delta_2, \Gamma_1, A, \Gamma_2, \Delta_3 \longrightarrow \Delta_2, \Gamma_3, \Delta_3.$$

This makes sense if and only if $\Gamma_1, \Delta_2 = \Delta_2, \Gamma_1$ and $\Gamma_2, \Delta_3 = \Delta_3, \Gamma_2$. In the non-symmetric theory, this can happen if and only if one of Δ_2 or Γ_1 is empty and also one of Γ_2 and Δ_3 is empty. We leave it to the reader to draw the "wiring diagrams" that make this clear.

On the other hand, if an unrestricted cut is allowed in our setup, which amounts to having two "permuting" weak distributivities, δ_R^L and δ_L^R, a certain amount of symmetry (or exchange) will be introduced. Thus, with unrestricted cut we are not dealing with a strictly non-commutative logic. On the other hand, nor would we be dealing with a commutative logic: if the weak distribution rules are inverted, the permuting ones give a braiding on the tensors. We shall return to this point later. We shall refer to models of the system with unrestricted cut as "non-planar polycategories".

We are now of the opinion that the two natural systems in this context are the planar system as given above, and the symmetric system which in addition contains the exchange rule:

$$\frac{\Gamma \xrightarrow{g} \Delta}{\sigma\Gamma \xrightarrow{g^\sigma_\tau} \tau\Delta}$$

for permutations σ, τ. We shall call models of the latter system "symmetric polycategories". They are, of course, also non-planar.

Next, we define a polycategory with two tensors: this amounts to having two binary operations $\otimes, \oplus$ on objects, extended to morphisms according to the following inference rules:

$$(\otimes L)\frac{\Gamma_1, A, B, \Gamma_2 \xrightarrow{f} \Gamma_3}{\Gamma_1, A \otimes B, \Gamma_2 \xrightarrow{f^{\otimes_i}} \Gamma_3}, \qquad (\otimes R)\frac{\Gamma_1 \xrightarrow{f} \Gamma_2, A, \Gamma_3 \qquad \Delta_1 \xrightarrow{g} \Delta_2, B, \Delta_3}{\Gamma_1, \Delta_1 \xrightarrow{f\ {}_i\otimes_j\ g} \Delta_2, \Gamma_2, A \otimes B, \Delta_3, \Gamma_3}$$

provided (in $(\otimes R)$) that $\Gamma_3 = \Delta_2 = \phi$ or $\Gamma_1 = \Gamma_3 = \phi$ or $\Delta_1 = \Delta_2 = \phi$. In $(\otimes L)$, $i =$ length of Γ_1; in $(\otimes R)$, $i =$ length of Γ_2, $j =$ length of Δ_2.

$$(\oplus L)\frac{\Gamma_1, A, \Gamma_2 \xrightarrow{f} \Gamma_3 \qquad \Delta_1, B, \Delta_2 \xrightarrow{g} \Delta_3}{\Delta_1, \Gamma_1, A \oplus B, \Delta_2, \Gamma_2 \xrightarrow{f\ {}_i\oplus_j\ g} \Gamma_3, \Delta_3}, \qquad (\oplus R)\frac{\Gamma_1 \xrightarrow{f} \Gamma_2, A, B, \Gamma_3}{\Gamma_1 \xrightarrow{f^{\otimes_i}} \Gamma_2, A \oplus B, \Gamma_3}$$

provided (in $(\oplus L)$) that $\Gamma_2 = \Delta_1 = \phi$ or $\Delta_1 = \Delta_3 = \phi$ or $\Gamma_2 = \Gamma_3 = \phi$. In $(\oplus L)$, $i =$ length of Γ_1, and $j =$ length of Δ_1; in $(\oplus R)$, $i =$ length of Γ_2.

(Note that we have indexed the labels as we did with cut; when clear from the context, we shall drop these subscripts.)

There are many further equivalences of derivations as in Definition 1.1. These can be considerably simplified if we give the following equivalent formulation of the tensor rules:

Definition 1.3. A two-tensor-polycategory is a polycategory with two binary operations $\otimes, \oplus$ on objects, with morphisms

$$m_{AB} : A, B \longrightarrow A \otimes B$$

$$w_{AB} : A \oplus B \longrightarrow A, B$$

and the rules of inference $(\otimes L)$ and $(\oplus R)$ above. These rules are to represent bijections stable under cut, so the following equations must hold:

- $m^{\otimes} = i$.
- $g \circ f^{\otimes} = (g \circ f)^{\otimes}$ for $g : \Delta_1, C, \Delta_2 \longrightarrow \Delta_3$ and $f : \Gamma_1 \longrightarrow \Gamma_2, C, \Gamma_3$, and where Γ_1 contains the sequence A, B.
- $f^{\otimes} \circ g = (f \circ g)^{\otimes}$ for $g : \Delta_1 \longrightarrow \Delta_2, C, \Delta_3$ and $f : \Gamma_1, C, \Gamma_2 \longrightarrow \Gamma_3$, and where one of Γ_1, Γ_2 contains the sequence A, B.
- $f = f^{\otimes_i} \circ m$. for $f : \Gamma_1, A, B, \Gamma_2 \longrightarrow \Gamma_3$.
- $\left(f^{\otimes_i}\right)^{\otimes_j} = \left(f^{\otimes_j}\right)^{\otimes_i}$. for $f : \Gamma_1, A, B, \Gamma_2, C, D, \Gamma_3 \longrightarrow \Gamma_4$.
- $w^{\oplus} = i$.
- $g \circ f^{\oplus} = (g \circ f)^{\oplus}$. for $g : \Delta_1, C, \Delta_2 \longrightarrow \Delta_3$ and $f : \Gamma_1 \longrightarrow \Gamma_2, C, \Gamma_3$, and where one of Γ_2, Γ_3 contains the sequence A, B.
- $f^{\oplus} \circ g = (f \circ g)^{\oplus}$. for $g : \Delta_1 \longrightarrow \Delta_2, C, \Delta_3$ and $f : \Gamma_1, C, \Gamma_2 \longrightarrow \Gamma_3$, and where Γ_3 contains the sequence A, B.
- $f = w \circ f^{\oplus_i}$ for $f : \Gamma_1 \longrightarrow \Gamma_2, A, B, \Gamma_3$.
- $\left(f^{\oplus_i}\right)^{\oplus_j} = \left(f^{\oplus_j}\right)^{\oplus_i}$ for $f : \Gamma_1 \longrightarrow \Gamma_2, A, B, \Gamma_3, C, D, \Gamma_4$.
- $\left(f^{\otimes_i}\right)^{\oplus_j} = \left(f^{\oplus_j}\right)^{\otimes_i}$ for $f : \Gamma_1, A, B, \Gamma_2 \longrightarrow \Gamma_3, C, D, \Gamma_4$.

We shall leave it as an exercise to show that this is equivalent to the other presentation. However, we must stress that cut elimination does *not* hold for the second presentation of two-tensor-polycategories; the amount of cut built into the rules $(\otimes R)$ and $(\oplus L)$ is necessary to prove cut elimination.

It is straightforward to define the category of polycategories (just keep in mind that we interpret sequents $\Gamma \longrightarrow \Delta$ as maps $\bigotimes \Gamma \longrightarrow \bigoplus \Delta$, and functors should preserve the tensors). So a functor $F : \mathbf{C} \longrightarrow \mathbf{D}$ is a map $\mathrm{Ob}(\mathbf{C}) \longrightarrow \mathrm{Ob}(\mathbf{D})$ and a map $\mathrm{M}\varphi(\mathbf{C}) \longrightarrow \mathrm{M}\varphi(\mathbf{D})$ so that this and the induced map $\mathrm{Ob}(\mathbf{C})^* \longrightarrow \mathrm{Ob}(\mathbf{D})^*$ commute with *source* and with *target*. A functor between two-tensor-polycategories must preserve the two tensors.

A natural transformation $\alpha : F \longrightarrow G$ assigns a $\mathbf{D}$ morphism $\alpha_A : F(A) \longrightarrow G(A)$ to each singleton sequence A from $\mathbf{C}$, satisfying the usual naturality condition.

We shall denote the 2-category of polycategories by **PolyCat**, and the 2-category of two-tensor-polycategories by $\mathbf{PolyCat}_{\otimes\oplus}$. We then note that the latter is a conservative extension of the former:

Proposition 1.4. *There is a 2-adjunction $F \dashv U$*

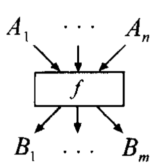
$$\mathbf{PolyCat} \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{PolyCat}_{\otimes\oplus}$$

whose unit $\mathbf{C} \longrightarrow UF(\mathbf{C})$ is full and faithful for each polycategory $\mathbf{C}$.

Proof. Given a polycategory $\mathbf{C}$, $F(\mathbf{C})$ is the free two-tensor-polycategory generated by $\mathbf{C}$. That is, close the set $\mathrm{Ob}(\mathbf{C})$ under the tensors $\otimes, \oplus$ to obtain the objects of $F(\mathbf{C})$, and take the sequents of $\mathbf{C}$ as non-logical axioms, closing under the inference rules and quotienting by the equivalence to obtain the morphisms of $F(\mathbf{C})$. For a two-tensor-polycategory, U just forgets the two tensor structure.

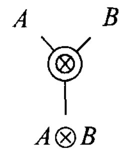
For a two-tensor-polycategory $\mathbf{D}$, the counit $FU(\mathbf{D}) \longrightarrow \mathbf{D}$ collapses the new tensor structure onto the old. For a polycategory $\mathbf{C}$, the unit $\mathbf{C} \longrightarrow UF(\mathbf{C})$ is the usual inclusion into the free structure. To see that this map is full, we use the cut elimination theorem for two-tensor-polycategories. $F(\mathbf{C})$ has only the sequents of $\mathbf{C}$ as its non-logical axioms, so by cut elimination any derivation in $F(\mathbf{C})$ is equivalent to one with cuts restricted to sequents from $\mathbf{C}$. If $\Gamma \longrightarrow \Delta$ is a tensor-free sequent of $F(\mathbf{C})$ (i.e. is in the image of the unit) then any derivation of $\Gamma \longrightarrow \Delta$ is equivalent to a derivation in $\mathbf{C}$. This because with the cuts restricted to the tensor-free part of $F(\mathbf{C})$, none of the left or right introduction rules could be used in the derivation (they introduce tensors that could never be eliminated).

To verify that the unit is faithful is a bit more involved – the proof we prefer uses proof nets and the techniques developed in [4]. We sketch the details here: we do not wish to enter a long digression on proof nets, so we refer the reader to that paper for more details. For polycategories (without tensors) the notion of (two-sided) proof net is particularly simple: in the free case, we have just identity wires – not a rich structure, admittedly. For arbitrary polycategories, we add the possibility of starting with "non-logical axioms" corresponding to primitive morphisms of the polycategory. Such a non-logical axiom, corresponding to a morphism $A_1, \ldots, A_n \longrightarrow B_1, \ldots, B_m$, takes the from of a component

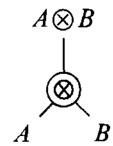


To pass to the two-tensor-polycategory context, we just add the usual (two-sided) links for $\otimes, \oplus$:

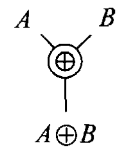
$(\otimes I)$



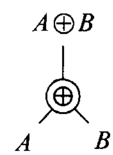
$(\otimes E)$



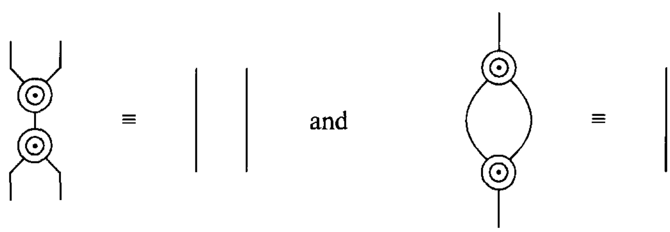
$(\oplus I)$



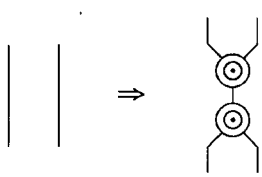
$(\oplus E)$



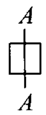
We have the following rewrites in the two-tensor case. (We use $\odot$ to represent either $\otimes$ or $\oplus$.)



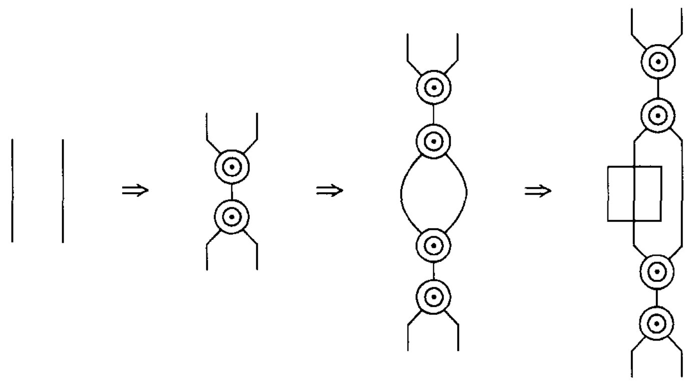
Back to the proof of faithfulness: the basic idea is to consider a chain of "rewrites" in the two-tensor-polycategory $UF(\mathbf{C})$ which display the equality of two morphisms from $\mathbf{C}$; since the morphisms (rather their representing nets) originate in $\mathbf{C}$, the first rewrite which is not tensor free must involve an "expansion" (technically, a backwards reduction) step introducing a tensor (or cotensor).



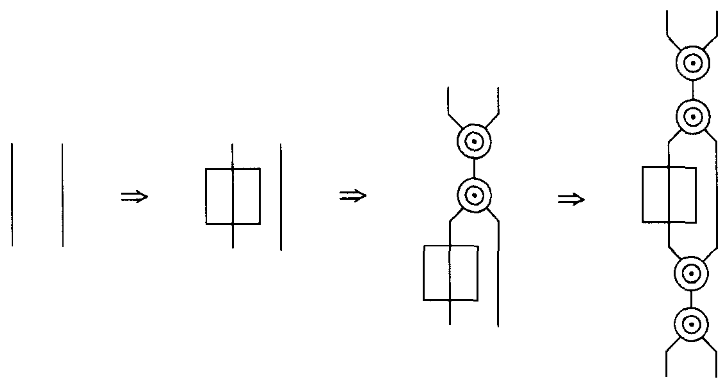
We then consider the next step in the chain: if it is a rewrite from $\mathbf{C}$ then we can in fact switch the order of these rewrites, pushing the two-tensor-rewrite (the expansion) later in the chain. For example, if a rewrite in $\mathbf{C}$ replaces an identity on A with another link, say



(where the box represents some suitable subgraph). Then we would replace a subsequence of rewrites of this form:



with one of this form:



If the next step were not a **C** rewrite, but instead a tensor rewrite, we would just proceed by induction to delay all the tensor steps. In this way we can push all the **C** steps to the beginning of the chain and all the tensor steps to the end of the chain – since these latter steps eventually end up with a net from **C**, they must cancel out (i.e. the chain may be terminated before the two-tensor rewrites begin) to give a chain without tensors. Hence the two morphisms must be equal in **C**. □

Remark. While we have not considered the question of the units $\top, \bot$ for the tensors $\otimes, \oplus$ these can be added (see [13]) and the above proof can be extended for this case. We shall feel free to consider $\mathbf{PolyCat}_{\otimes\oplus}$ with these units when this makes matters technically simpler. The full treatment of the units is quite technical and forms the main subject of [4].

2. Weakly distributive categories

2.1. Definitions

A *weakly distributive category* $\mathbf{C}$ is a category with two tensors and two weak distribution natural transformations. The two tensors will be denoted by $\otimes$ and $\oplus$ and we shall call $\otimes$ the *tensor* and $\oplus$ the *cotensor*. Each tensor comes equipped with a unit object, an associativity natural isomorphism, and a left and right unit natural isomorphism:

$$(\otimes, \top, a_\otimes, u_\otimes^L, u_\otimes^R)$$

$$a_\otimes : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$$

$$u_\otimes^R : A \otimes \top \longrightarrow A$$

$$u_\otimes^L : \top \otimes A \longrightarrow A$$

$$(\oplus, \bot, a_\oplus, u_\oplus^L, u_\oplus^R)$$

$$a_\oplus : (A \oplus B) \oplus C \longrightarrow A \oplus (B \oplus C)$$

$$u_\oplus^R : A \oplus \bot \longrightarrow A$$

$$u_\oplus^L : \bot \oplus A \longrightarrow A$$

The two weak distribution transformations shall be denoted by

$$\delta_L^L : A \otimes (B \oplus C) \longrightarrow (A \otimes B) \oplus C$$

$$\delta_R^R : (B \oplus C) \otimes A \longrightarrow B \oplus (C \otimes A).$$

If the tensors are symmetric, the following "permuting" weak distributivities are induced:

$$\delta_R^L : A \otimes (B \oplus C) \longrightarrow B \oplus (A \otimes C)$$

$$\delta_L^R : (B \oplus C) \otimes A \longrightarrow (B \otimes A) \oplus C.$$

We shall call $\mathbf{C}$ a *non-planar weakly distributive category* if it has these permuting weak distributivities as well. A special case is when the two tensors are symmetric; see the next section for the definition of a symmetric weakly distributive category.

This data must satisfy certain coherence conditions which we shall discuss shortly. Before doing so we remark that there are some symmetries which arise from this data: [op′] reverse the arrows and swap both the $\otimes$ and $\oplus$ and $\top$ and $\bot$; this gives the following assignment of the generating maps:

$$\begin{array}{lll} \delta_L^L \leftrightarrow \delta_R^R & a_\otimes \mapsto a_\oplus^{-1} & a_\oplus \mapsto a_\otimes^{-1} \\ \delta_R^L \mapsto \delta_R^L & u_\otimes^R \mapsto (u_\oplus^R)^{-1} & u_\oplus^R \mapsto (u_\otimes^R)^{-1} \\ \delta_L^R \mapsto \delta_L^R & u_\otimes^L \mapsto (u_\oplus^L)^{-1} & u_\oplus^L \mapsto (u_\otimes^L)^{-1} \end{array}$$

[$\odot'$] simultaneously reverse the tensor $\otimes$ and the cotensor $\oplus$; this assigns

$$\begin{array}{lll} \delta_L^L \leftrightarrow \delta_R^R & a_\otimes \mapsto a_\otimes^{-1} & a_\oplus \mapsto a_\oplus^{-1} \\ \delta_R^L \leftrightarrow \delta_L^R & u_\otimes^R \leftrightarrow u_\otimes^L & u_\oplus^R \leftrightarrow u_\oplus^L \end{array}$$

In the non-planar case, the $\odot'$ symmetry can be decomposed into its obvious components, as follows. Note then that the following are true only in case the "permuting" weak distributivities are valid:
[$\otimes'$] reverse the tensor $\otimes$; this assigns

$$\begin{array}{lll} \delta_L^L \leftrightarrow \delta_L^R & a_\otimes \mapsto a_\otimes^{-1} & a_\oplus \mapsto a_\oplus \\ \delta_R^L \leftrightarrow \delta_R^R & u_\otimes^R \leftrightarrow u_\otimes^L & u_\oplus^R \mapsto u_\oplus^R \\ & & u_\oplus^L \mapsto u_\oplus^L \end{array}$$

[$\oplus'$] reverse the cotensor $\oplus$; this assigns

$$\begin{array}{lll} \delta_L^L \leftrightarrow \delta_R^L & a_\otimes \mapsto a_\otimes & a_\oplus \mapsto a_\oplus^{-1} \\ \delta_L^R \leftrightarrow \delta_R^R & u_\otimes^R \mapsto u_\otimes^R & u_\oplus^R \leftrightarrow u_\oplus^L \\ & u_\otimes^L \mapsto u_\otimes^L & \end{array}$$

The notion of a weakly distributive category is preserved by these symmetries (the latter two only for non-planar weakly distributive categories) and we shall use this fact to give an economical statement of the required commuting conditions, which are as follows[3]:

2.1.1. Tensors

The two tensor products must satisfy the usual conditions of a tensor product. Explicitly, the data $(\otimes, \top, a_\otimes, u_\otimes^R, u_\otimes^L)$, where $a_\otimes$, $u_\otimes^R$, and $u_\otimes^L$ are natural isomorphisms,

[3] We shall give the equations for non-planar weakly distributive categories – of course, the equations involving the permuting weak distributivities are only intended for the non-planar case. We only number the planar equations.

must satisfy:

$$\begin{tikzcd}
(A\otimes\top)\otimes B \arrow[rr, "a_\otimes"] \arrow[dr, "u^R_\otimes\otimes i_B"'] & & A\otimes(\top\otimes B) \arrow[dl, "i_A\otimes u^L_\otimes"] \\
& A\otimes B &
\end{tikzcd}$$

and the well-known pentagon diagram

$$\begin{tikzcd}
& ((A\otimes B)\otimes C)\otimes D \arrow[dl, "a_\otimes\otimes i_D"'] \arrow[ddr, "a_\otimes"] & \\
(A\otimes(B\otimes C))\otimes D \arrow[ddd, "a_\otimes"'] & & \\
& & (A\otimes B)\otimes(C\otimes D) \arrow[ddl, "a_\otimes"] \\
& & \\
A\otimes((B\otimes C)\otimes D) \arrow[r, "i_A\otimes a_\otimes"'] & A\otimes(B\otimes(C\otimes D)) &
\end{tikzcd}$$

This gives the following equations[4]:

$$a_\otimes; i\otimes u^L_\otimes = u^R_\otimes \otimes i \tag{1}$$

$$a_\otimes; a_\otimes = a_\otimes \otimes i; a_\otimes; i\otimes a_\otimes \tag{2}$$

Similarly, the data $(\oplus, \top, a_\oplus, u^R_\oplus, u^L_\oplus)$ must satisfy the diagrams obtained by applying the symmetry op′ to these. This, of course, makes $\oplus$ a tensor product and gives the following equations:

$$a_\oplus; i\oplus u^L_\oplus = u^R_\oplus \oplus i \tag{3}$$

$$a_\oplus; a_\oplus = a_\oplus \oplus i; a_\oplus; i\oplus a_\oplus \tag{4}$$

[4] We use two notations for composition: $f;g$ and $g\circ f$ both denote "first f then g".

2.1.2. Unit and distribution

The following commutativity linking the unit and distribution must hold:

$$\begin{tikzcd}
\top \otimes (A \oplus B) \arrow[d, "\delta_L^L"'] \arrow[dr, "u_\otimes^L"] & \\
(\top \otimes A) \oplus B \arrow[r, "u_\otimes^L \oplus i_B"'] & A \oplus B
\end{tikzcd}$$

Furthermore all the forms of this diagram under the symmetries must hold. This gives the following set of identities:

$$u_\otimes^L = \delta_L^L ; u_\otimes^L \oplus i \tag{5}$$

$$u_\otimes^R = \delta_R^R ; i \oplus u_\otimes^R \tag{6}$$

$$u_\oplus^L \otimes i = \delta_R^R ; u_\oplus^L \tag{7}$$

$$i \otimes u_\oplus^R = \delta_L^L ; u_\oplus^R \tag{8}$$

$$u_\otimes^L = \delta_R^L ; i \oplus u_\otimes^L$$

$$u_\otimes^R = \delta_L^R ; u_\otimes^R \oplus i$$

$$u_\oplus^R \otimes i = \delta_L^R ; u_\oplus^R$$

$$i \otimes u_\oplus^L = \delta_R^L ; u_\oplus^L$$

For example, the diagram obtained by reversing arrows (after writing isomorphisms in the "positive" direction) is

$$\begin{tikzcd}
(\bot \oplus A) \otimes B \arrow[r, "\delta_R^R"] \arrow[dr, "u_\oplus^L \otimes i_B"'] & \bot \oplus (A \otimes B) \arrow[d, "u_\oplus^L"] \\
& A \otimes B
\end{tikzcd}$$

2.1.3. Associativity and distribution

All the remaining diagrams actually result from the interaction of these maps. However, they also arise as an interaction between other maps and to emphasize these interactions we shall have a representative diagram from each interaction chosen so as to provide a complete axiomatization. Here therefore is the representative diagram for this interaction:

$$\begin{tikzcd}
(A\otimes B)\otimes(C\oplus D) \arrow[r, "a_\otimes"] \arrow[dd, "\delta_L^L"'] & A\otimes(B\otimes(C\oplus D)) \arrow[d, "i_A\otimes\delta_L^L"] \\
= & A\otimes((B\otimes C)\oplus D) \arrow[d, "\delta_L^L"] \\
((A\otimes B)\otimes C)\oplus D \arrow[r, "a_\otimes\oplus i_D"'] & (A\otimes(B\otimes C))\oplus D
\end{tikzcd}$$

Notice that this diagram brings outside tensors on one side into an argument of the cotensor. The symmetries now generate the following set of equations:

$$\delta_L^L; a_\otimes \oplus i = a_\otimes; i \otimes \delta_L^L; \delta_L^L \tag{9}$$

$$a_\otimes; \delta_R^R = \delta_R^R \otimes i; \delta_R^R; i \oplus a_\otimes \tag{10}$$

$$\delta_R^R; a_\oplus = a_\oplus \otimes i; \delta_R^R; i \oplus \delta_R^R \tag{11}$$

$$i \otimes a_\oplus; \delta_L^L = \delta_L^L; i \oplus \delta_L^L; a_\oplus \tag{12}$$

$$\delta_R^L; i \oplus a_\otimes = a_\otimes; i \otimes \delta_R^L; \delta_R^L$$

$$a_\otimes; \delta_L^R = \delta_L^R \otimes i; \delta_L^R; a_\otimes \oplus i$$

$$a_\oplus \otimes i; \delta_L^R = \delta_L^R; \delta_L^R \oplus i; a_\oplus$$

$$\delta_R^L; a_\oplus = i \otimes a_\oplus; \delta_R^L; i \oplus \delta_R^L$$

Again, for example:

$$\begin{tikzcd}
((A\oplus B)\oplus C)\otimes D \arrow[r, "a_\oplus\otimes i_D"] \arrow[dd, "\delta_R^R"'] & (A\oplus(B\oplus C))\otimes D \arrow[d, "\delta_R^R"] \\
= & A\oplus((B\oplus C)\otimes D) \arrow[d, "i_A\otimes\delta_R^R"] \\
(A\oplus B)\oplus(C\otimes D) \arrow[r, "a_\oplus"'] & A\oplus(B\oplus(C\otimes D))
\end{tikzcd}$$

which is an interaction of coassociativity and distribution and brings an outside tensor onto an inside cotensor.

2.1.4. Distribution and distribution

$$\begin{tikzcd}
& (A\oplus B)\otimes(C\oplus D) \arrow[dl, "\delta_L^L"'] \arrow[dr, "\delta_R^R"] & \\
((A\oplus B)\otimes C)\oplus D \arrow[d, "\delta_R^R\oplus i_D"'] & = & A\oplus(B\otimes(C\oplus D)) \arrow[d, "i_A\oplus\delta_L^L"] \\
(A\oplus(B\otimes C))\oplus D \arrow[rr, "a_\oplus"'] & & A\oplus((B\otimes C)\oplus D)
\end{tikzcd}$$

The symmetries then generate the following equations:

$$\delta_L^L;\delta_R^R\oplus i;a_\oplus=\delta_R^R;i\oplus\delta_L^L \tag{13}$$

$$a_\otimes;i\otimes\delta_R^R;\delta_L^L=\delta_L^L\otimes i;\delta_R^R \tag{14}$$

$$\delta_L^R;\delta_R^L\oplus i;a_\oplus=\delta_R^L;i\oplus\delta_L^R$$

$$a_\otimes;i\otimes\delta_L^R;\delta_R^L=\delta_R^L\otimes i;\delta_L^R$$

Notice that the symmetries $\otimes'$ and $\oplus'$ generate the same equation this time. When the symmetry op$'$ is applied the diagram becomes

$$\begin{tikzcd}
A\otimes((B\oplus C)\otimes D) \arrow[rr, "a_\otimes"] \arrow[d, "i_A\otimes\delta_R^R"'] & & (A\otimes(B\oplus C))\otimes D \arrow[d, "\delta_L^L\otimes i_D"] \\
A\otimes(B\oplus(C\otimes D)) \arrow[dr, "\delta_L^L"'] & = & ((A\otimes B)\oplus C)\otimes D \arrow[dl, "\delta_R^R"] \\
& (A\otimes B)\oplus(C\otimes D) &
\end{tikzcd}$$

which is the case when tensors on either side are moved into different arguments of the cotensor.

2.1.5. Coassociativity and distribution

The last basic diagram applies only to the non-planar case, and concerns a further interaction of coassociativity and distributivity:

$$\begin{tikzcd}
A\otimes((B\oplus C)\oplus D) \arrow[rr, "i_A\otimes a_\oplus"] \arrow[d, "\delta^L_L"'] & & A\otimes(B\oplus(C\oplus D)) \arrow[d, "\delta^L_R"] \\
(A\otimes(B\oplus C))\oplus D \arrow[d, "\delta^L_R\oplus i_D"'] & = & B\oplus(A\otimes(C\oplus D)) \arrow[d, "i_B\oplus\delta^L_L"] \\
(B\oplus(A\otimes C))\oplus D \arrow[rr, "a_\oplus"'] & & B\oplus((A\otimes C)\oplus D)
\end{tikzcd}$$

which brings an outside tensor onto the middle cotensor.

The symmetries then generate the following equations:

$$\delta^L_L;\delta^L_R\oplus i;a_\oplus = i\otimes a_\oplus;\delta^L_R;i\oplus\delta^L_L$$

$$\delta^R_L;\delta^R_R\oplus i;a_\oplus = a_\oplus\otimes i;\delta^R_R;i\oplus\delta^R_L$$

$$a_\otimes;i\otimes\delta^R_L;\delta^L_L = \delta^L_L\otimes i;\delta^R_L;a_\otimes\oplus i$$

$$a_\otimes;i\otimes\delta^R_R;\delta^L_R = \delta^L_R\otimes i;\delta^R_R;i\oplus a_\otimes$$

Notice that the original diagram is already "symmetric" in $\oplus$ so that this symmetry generates nothing new.

When the symmetry which reverses arrows is applied the diagram becomes

$$\begin{tikzcd}
(A\otimes(B\oplus C))\otimes D \arrow[r, "a_\otimes"] \arrow[d, "\delta^L_L\otimes i"'] & A\otimes((B\oplus C)\otimes D) \arrow[d, "i\otimes\delta^R_L"] \\
((A\otimes B)\oplus C)\otimes D \arrow[d, "\delta^R_L"'] & A\otimes((B\otimes D)\oplus C) \arrow[d, "\delta^L_L"] \\
((A\otimes B)\otimes D)\oplus C \arrow[r, "a_\otimes\oplus i"'] & (A\otimes(B\otimes D))\oplus C
\end{tikzcd}$$

which brings the tensors on each side onto an argument of the cotensor.

2.2. *Weakly distributive categories and polycategories*

In this section we show that weakly distributive categories and two-tensor-polycategories are essentially the same thing. With Proposition 1.4 this justifies our claiming that weakly distributive categories constitute the essential content of polycategories. We shall denote by **WkDistCat** the category of weakly distributive categories with functors which preserve the tensor, cotensor, units, and weak distributivities. We shall assume that the two-tensor-polycategories have units, to correspond to the units in the weakly distributive categories. In this section, any of the versions of polycategories and of weakly distributive categories may be used, as long as one "moves in parallel" – i.e. both non-planar, both planar, or both symmetric.

Theorem 2.1. *There is an equivalence of 2-categories*

$$\mathbf{PolyCat}_{\otimes\oplus} \underset{P}{\overset{W}{\rightleftarrows}} \mathbf{WkDistCat}.$$

Proof. Given a weakly distributive category $\mathbf{W}$, $P(\mathbf{W})$ is the polycategory with the same set of objects as $\mathbf{W}$, and with morphisms given by: $\Gamma \longrightarrow \Delta$ is a morphism if and only if $\bigotimes \Gamma \longrightarrow \bigoplus \Delta$ is a morphism of $\mathbf{W}$. (We observe the usual convention that an empty tensor is the unit of the tensor.) To check that the cut rule, and the left and right introduction rules, are valid, we use the weak distributivities; for example, we shall illustrate the following instance of cut. (To see how all the distributivities work, we use an unrestricted instance of cut, which needs the permuting distributivities as well as the planar ones.) Given maps (in $\mathbf{W}$) $C_1 \otimes A \otimes C_2 \xrightarrow{g} C_3$ and $D_1 \xrightarrow{f} D_2 \oplus A \oplus D_3$, we can construct $C_1 \otimes D_1 \otimes C_2 \xrightarrow{g\ {}_1{\circ}_1\ f} D_2 \oplus C_3 \oplus D_3$ as follows (ignoring some instances of associativity for simplicity):

$$\begin{aligned}
C_1 \otimes D_1 \otimes C_2 &\xrightarrow{i\otimes f\otimes i} C_1 \otimes (D_2 \oplus A \oplus D_3) \otimes C_2 \\
&\xrightarrow{\delta_R^L\otimes i} (D_2 \oplus (C_1 \otimes (A \oplus D_3))) \otimes C_2 \\
&\xrightarrow{\delta_R^R} D_2 \oplus ((C_1 \otimes (A \oplus D_3)) \otimes C_2) \\
&\xrightarrow{i\oplus(\delta_L^L\otimes i)} D_2 \oplus (((C_1 \otimes A) \oplus D_3) \otimes C_2) \\
&\xrightarrow{i\oplus\delta_L^R} D_2 \oplus ((C_1 \otimes A \otimes C_2) \oplus D_3) \\
&\xrightarrow{i\oplus g\oplus i} D_2 \oplus C_3 \oplus D_3
\end{aligned}$$

(Other ways of introducing the distributivities to move the C's next to A are equivalent, by the coherence conditions on the interaction of distributivity with itself and with associativity.)

Similarly, as an example of $(\oplus L)$, consider maps $C_1 \otimes A \xrightarrow{f} C_3$ and $B \otimes D_2 \xrightarrow{g} D_3$; construct $f_1 \oplus_0 g$ as (again, ignoring associativity):

$$\begin{aligned}
C_1 \otimes (A \oplus B) \otimes D_2 &\xrightarrow{\delta_L^L\otimes i} ((C_1 \otimes A) \oplus B) \otimes D_2 \\
&\xrightarrow{(f\oplus i)\otimes i} (C_3 \oplus B) \otimes D_2 \\
&\xrightarrow{\delta_R^R} C_3 \oplus (B \otimes D_2) \\
&\xrightarrow{i\oplus g} C_3 \oplus D_3
\end{aligned}$$

(The other cases are similar – note that $(\otimes L)$ and $(\oplus R)$ are trivial.)

$$
\begin{tikzcd}
E_1 \otimes C_1 \otimes D_1 \arrow[r, "i \otimes g"] \arrow[d, "i \otimes f \otimes i"] & E_1 \otimes C_1 \otimes (B \oplus D_3) \arrow[r, "\delta_L^L"] \arrow[d, dotted, "i \otimes f \otimes i"'] & (E_1 \otimes C_1 \otimes B) \oplus D_3 \arrow[d, "i \otimes f \otimes i"'] \\
(E_1 \otimes (C_2 \oplus A)) \otimes D_1 \arrow[r, dotted, "i \otimes g"] \arrow[d, "\delta_R^L \otimes i"] & E_1 \otimes (C_2 \oplus A) \otimes (B \oplus D_3) \arrow[r, dotted, "\delta_L^L"] \arrow[d, dotted, "\delta_R^L \otimes i"'] & (E_1 \otimes (C_2 \oplus A) \otimes B) \oplus D_3 \arrow[d, "\delta_R^L \otimes i"'] \\
(C_2 \oplus (E_1 \otimes A)) \otimes D_1 \arrow[r, dotted, "i \otimes g"] \arrow[d, "\delta_R^R"] & (C_2 \oplus (E_1 \otimes A)) \otimes (B \oplus D_3) \arrow[r, dotted, "\delta_L^L"] \arrow[d, dotted, "\delta_R^R"'] & ((C_2 \oplus (E_1 \otimes A)) \otimes B) \oplus D_3 \arrow[d, "\delta_R^R \oplus i"'] \\
C_2 \oplus (E_1 \otimes A \otimes D_1) \arrow[r, "i \otimes g"] & C_2 \oplus (E_1 \otimes A \otimes (B \oplus D_3)) \arrow[r, phantom, "(13)"] \arrow[d, "i \oplus \delta_L^L"'] & (C_2 \oplus (E_1 \otimes A \otimes B)) \oplus D_3 \arrow[d, "a_\oplus"'] \\
 & C_2 \oplus ((E_1 \otimes A \otimes B) \oplus D_3) \arrow[r, dotted, "i"] \arrow[dr, "i \oplus h \oplus i"'] & C_2 \oplus ((E_1 \otimes A \otimes B) \oplus D_3) \arrow[d, "i \oplus h \oplus i"'] \\
 & & C_2 \oplus E_4 \oplus D_3
\end{tikzcd}
$$

Fig. 1. $(h \circ f) \circ g = (h \circ g) \circ f$.

We must then check that all the equivalences of two-tensor-polycategories follow from the coherence diagrams of weakly distributive categories. This is a frightful exercise; we shall just illustrate one case. (But note that the extra structure due to the two tensors is easy since $(\otimes L)$ and $(\oplus R)$ are identities. So we really only need check the five equivalences of Definition 1.1.)

Consider the following instance of the fourth equivalence scheme: we have maps $E_1 \otimes A \otimes B \xrightarrow{h} E_4$, $C_1 \xrightarrow{f} C_2 \oplus A$, and $D_1 \xrightarrow{g} B \oplus D_3$. To show $(h \;{}_1\circ_1\; f) \;{}_2\circ_0\; g = (h \;{}_2\circ_0\; g) \;{}_1\circ_1\; f$ amounts to showing (the outer part of) Fig. 1 commutes. (Again, we omit the associativities.) The inner diagrams all commute because of naturality, except for the hexagon marked (13), which is an instance of the equation with that number above. The other equivalences are shown in a like manner. Thus $P(\mathbf{W})$ is a two-tensor-polycategory.

Next, given a two-tensor-polycategory $\mathbf{P}$, the weakly distributive category $W(\mathbf{P})$ is just the category part of $\mathbf{P}$, viz. those morphisms whose source and target are singletons. The distributivities are given by the cut rule and the axioms $(\otimes R)$ and

($\oplus L$). For instance, δ_L^L is given as $m\ {}_1\circ_0\ w$:

$$\frac{A,B \longrightarrow A\otimes B \quad B\oplus C \longrightarrow B,C}{A,B\oplus C \longrightarrow A\otimes B,C}$$

Likewise, in the non-planar case, δ_R^L is given as (note how the unrestricted cut induces an "exchange" – this is where the non-planar version picks up its degree of symmetry):

$$\frac{A,C \longrightarrow A\otimes C \quad B\oplus C \longrightarrow B,C}{A,B\oplus C \longrightarrow B,A\otimes C}$$

and the coherence conditions follow from the equivalences for polycategories.

It is clear from the constructions above that $WP(\mathbf{W})$ is isomorphic to $\mathbf{W}$; indeed they are the same category. And essentially for the same reason, $\mathbf{P}$ is isomorphic to $PW(\mathbf{P})$. (The reason referred to is the bijection

$$\frac{\Gamma \longrightarrow \Delta}{\bigotimes\Gamma \longrightarrow \bigoplus\Delta}$$

which means that the category part of a two-tensor-polycategory carries all the information of the polycategory.) □

3. Distributive categories

How are weakly distributive categories related to distributive categories? It turns out that they are very close indeed – if the tensors are the cartesian product and coproduct (nicely), then the two notions coincide. This reinforces the view that weak distributivity is the natural notion for general tensors.

A weakly distributive category is *symmetric* (resp. $\otimes$-symmetric, $\oplus$-symmetric) in case the tensors are symmetric (resp. the tensor is symmetric with $s_\otimes$, the cotensor with $s_\oplus$) and

$$\begin{tikzcd}
A\otimes(B\oplus C) \arrow[r, "i\otimes s_\oplus"'] \arrow[d, "\delta_L^L"'] & A\otimes(C\oplus B) \arrow[r, "s_\otimes"'] \arrow[d, "\delta_R^L"'] & (C\oplus B)\otimes A \arrow[r, "s_\oplus\otimes i"'] \arrow[d, "\delta_R^R"'] & (B\oplus C)\otimes A \arrow[d, "\delta_L^R"'] \\
(A\otimes B)\oplus C \arrow[r, "s_\oplus"'] & C\oplus(A\otimes B) \arrow[r, "i\oplus s_\otimes"'] & C\oplus(B\otimes A) \arrow[r, "s_\oplus"'] & (B\otimes A)\oplus C
\end{tikzcd}$$

commuting in all squares (resp. those squares which exist). (This just says that the permuting distributivities are canonically induced by the planar ones and symmetry.)

A weakly distributive category is *bicartesian* (resp. $\otimes$-cartesian or $\oplus$-cartesian) if the category is symmetric (resp. $\otimes$-symmetric, $\oplus$-symmetric) with the tensor a product (with $\top$ the final object) and the cotensor a coproduct (with $\perp$ the initial object).

A source of motivation for the study of weak distributivity is the fact that distributive categories are (bicartesian) examples. This means that the category of **Sets** (or any topos) is a bicartesian weakly distributive category.

We now verify that distributive categories are a source of examples. A *distributive category* [5, 15] has finite products and coproducts such that the comparison map from the coproduct

$$\langle i \times b_0 | i \times b_1 \rangle : A \times B + A \times C \longrightarrow A \times (B + C)$$

is an isomorphism. We shall denote the inverse of $\langle i \times b_0 | i \times b_1 \rangle$ by δ.

Proposition 3.1. *Distributive categories are bicartesian weakly distributive categories.*

Proof. Let

$$\delta_L^L : A \times (B + C) \xrightarrow{\delta} A \times B + A \times C \xrightarrow{i+p_1} A \times B + C.$$

Then, as $+$ and $\times$ are symmetric the other weak distributions can be obtained from this. Due to the symmetry of product and coproduct it suffices to prove that the four basic diagrams hold together with their op′ duals. This gives eight diagrams to check. However, examining these diagrams, bearing in mind the use of the symmetric maps in defining the weak distributions, shows that there are only actually six distinct things to prove. These are that the different ways of expressing the following arrows are equal:

$$\top \times (A + B) \longrightarrow A + B$$

$$(\bot + A) \times B \longrightarrow A \times B$$

$$(A \times B) \times (C + D) \longrightarrow A \times (B \times C) + D$$

$$(A \times B) \times (C + D) \longrightarrow A \times C + B \times D$$

$$A \times ((C + D) + E) \longrightarrow A \times C + (D + E)$$

$$(A + B) \times (C + D) \longrightarrow A + (B \times C) + D$$

For the first of these consider:

$$\begin{tikzcd}
\top \times (A+B) \arrow[r, "p_1"] \arrow[d, "\delta"'] & A+B \\
\top \times A + \top \times B \arrow[r, "i+p_1"'] & \top \times A + B \arrow[u, "p_1+i"']
\end{tikzcd}$$

As $\delta = \langle i \times b_0 | i \times b_1 \rangle^{-1}$ to obtain commutativity it suffices to show

$$
\begin{aligned}
&\top \times A \xrightarrow{b_0} \top \times A + \top \times B \xrightarrow{\langle i \times b_0 | i \times b_1 \rangle} \top \times (A + B) \xrightarrow{p_1} A + B \\
&\quad = \top \times A \xrightarrow{p_1} A \xrightarrow{b_0} A + B \\
&\top \times B \xrightarrow{b_1} \top \times A + \top \times B \xrightarrow{\langle i \times b_0 | i \times b_1 \rangle} \top \times (A + B) \xrightarrow{p_1} A + B \\
&\quad = \top \times B \xrightarrow{p_1} B \xrightarrow{b_1} A + B
\end{aligned}
$$

which are clear.

For the op$'$ dual of this we have

$$
\begin{tikzcd}
(\perp + A) \times B) \arrow[r, "\delta"] \arrow[d, "b_1^{-1} \times i"'] & \perp \times B + A \times B \arrow[r, "p_0 + i"] & \perp + A \times B \arrow[d, "b_1^{-1}"] \\
A \times B \arrow[rr, equal] & & A \times B
\end{tikzcd}
$$

which commutes as $b_1 \times i; \delta; p_0 + i = b_1; p_0 + i = b_1$.

The next two equations are consequences of the fact that the following diagram commutes:

$$
\begin{tikzcd}
(A \times B) \times (C + D) \arrow[rr, "a_\times"] \arrow[dd, "\delta"'] & & A \times (B \times (C + D)) \arrow[d, "i \times \delta"] \\
& = & A \times (B \times C + B \times D) \arrow[d, "\delta"] \\
(A \times B) \times C + (A \times B) \times D \arrow[rr, "a_\otimes + a_\otimes"'] & & A \times (B \times C) + A \times (B \times D)
\end{tikzcd}
$$

which can be checked using the inverse of δ and checking the values on the components of the coproduct in the lower left corner. To obtain the weak distribution maps (and in particular the diagrams which we require) it suffices to project the components of the coproducts. This easily gives the desired commutativities.

The remaining equations are checked in the same manner. That is we use full distribution applied in two different ways, check that the diagram commute for the components of the coproducts using the inverses of these distributions, and finally project to obtain the weak distributions. $\square$

It is of some interest to wonder what conditions must be added to a bicartesian weakly distributive category to force it to be distributive. Demanding that it is bicartesian is not sufficient: this can be seen in several ways. For instcance, there are well-known examples of categories with products and coproducts that are weakly distributive but not distributive – one simple example is the category of pointed sets. (Notice, in view of the result below, that the initial object in this category is not strict.) There are two interesting general classes of examples worth mentioning.

First, an Abelian category is a bicartesian weakly distributive category as it is a symmetric monoidal category on the biproduct. In fact, any braided monoidal category is a non planar weakly distributive category by letting the non permuting weak distributions be the associativity of the tensor and the permuting weak distributions be given by the braiding. Thus, certainly any symmetric monoidal category is a symmetric weakly distributive category. However, while an Abelian category is a symmetric weakly distributive category it is certainly not distributive.

Second, the dual of a distributive category (a codistributive category) is clearly bicartesian weakly distributive as the latter is a self-dual notion. However, a codistributive category is not distributive. Indeed, a codistributive category which is simultaneously a distributive category must be a preorder (as the final object is costrict).

In order to obtain a distributive category there must, therefore, be some relationship required between the distribution, projection, and embedding maps. Our first attempt to pin this down is as follows:

Lemma 3.2. *A bicartesian weakly distributive category is distributive if and only if the following diagrams commute*:

$$\begin{tikzcd}
A\times B \arrow[r, "i\times b_0"] \arrow[dr, "b_0"'] & A\times(B+C) \arrow[d, "\delta_L^L"] \\
 & (A\times B)+C
\end{tikzcd}
\qquad
\begin{tikzcd}
A\times B \arrow[r, "i\times b_0"] \arrow[d, "p_1"'] & A\times(B+C) \arrow[d, "\delta_R^L"] \\
B \arrow[r, "b_0"'] & B+(A\times C)
\end{tikzcd}$$

Proof. It is easy to check that a distributive category satisfies the two diagrams. For the converse, we must construct the inverse δ of $\langle i \times b_0 | i \times b_1 \rangle$.

We set

$$\delta = A \times (B + C) \xrightarrow{A\times i} (A \times A) \times (B + C) \xrightarrow{a_\times} A \times (A \times (B + C))$$

$$\xrightarrow{i\times\delta_R^L} A \times (B + A \times C) \xrightarrow{\delta_L^L} A \times B + A \times C.$$

To show that this is the inverse of $\langle i \times b_0 | i \times b_1 \rangle$ we precompose with $i \times b_0$ (by symmetry the same thing will happen on precomposing with $i \times b_1$) and show the result is b_0:

$$\begin{tikzcd}
A\times B \arrow[r, "i\times b_0"] \arrow[d, "\Delta\times i;a_\times"'] & A\times(B+C) \arrow[d, "\Delta\times i;a_\times"] \\
A\times(A\times B) \arrow[r, "i\times(i\times b_0)"] \arrow[d, "p_1"'] \arrow[dr, "i\times b_0"] & A\times(A\times(B+C)) \arrow[d, "1\times\delta_L^L"] \\
A\times B \arrow[dr, "b_0"] & A\times(A\times B+C) \arrow[d, "\delta_R^L"] \\
 & A\times B+A\times C
\end{tikzcd}$$

where the triangle and parallelogram are the two conditions added. □

An initial object is *strict* in case every map to it is an isomorphism. Notice that, as an Abelian category has a zero, it cannot have a strict initial object without being trivial. The initial object of a distributive category, however, is necessarily strict (see [5]). This is a difference we now exploit:

Theorem 3.3. *A bicartesian weakly distributive category is a distributive category if and only if it has a strict initial object.*

Proof. It suffices to show that the two diagrams above commute in the presence of a strict initial object. To see this consider the two naturality diagrams

$$\begin{tikzcd}
A\times(B+\bot) \arrow[r, "i\times(i+\bot)"] \arrow[d, "\delta_L^L"'] & A\times(B+C) \arrow[d, "\delta_L^L"] \\
A\times B+\bot \arrow[r, "i+\bot"'] & (A\times B)+C
\end{tikzcd}
\qquad
\begin{tikzcd}
A\times(B+\bot) \arrow[r, "i\times(i+\bot)"] \arrow[d, "\delta_R^L"'] & A\times(B+C) \arrow[d, "\delta_R^L"] \\
B+(A\times\bot) \arrow[r, "i+(i\times\bot)"'] & B+(A\times C)
\end{tikzcd}$$

The first immediately yields the first condition of the lemma. The second due to strictness has the bottom left object isomorphic to B and the horizontal map is then the coproduct embedding. It suffices to prove that the vertical map is essentially a projection. For this consider

$$\begin{tikzcd}
A\times(B+\perp) \arrow[r, "!\times i"] \arrow[d, "\delta_R^L"'] & \top\times(B+\perp) \arrow[d, "\delta_R^L"] \\
B+A\times\perp \arrow[r, "i+(!\times i)"'] & B+\top\times\perp
\end{tikzcd}$$

The lower horizontal map is an isomorphism due to the strictness of the initial object. However, the map across the square is clearly equivalent to a projection. □

Note added in proof

As mentioned earlier, Proposition 3.1 should now read:

Proposition 3.1. *An elementary distributive category is a cartesian weakly distributive category if and only if it is a preorder.*

Thus, cartesian weakly distributive categories and distributive categories are almost orthogonal notions!

Proof. The following diagram always commutes in a symmetric weakly distributive category:

$$\begin{tikzcd}
(A\oplus B)\otimes(C\oplus D) \arrow[r, "c_\oplus\otimes c_\oplus"] \arrow[d, "\delta_R^R"'] & (B\oplus A)\otimes(D\oplus C) \arrow[r, "c_\otimes"] & (D\oplus C)\otimes(B\oplus A) \arrow[d, "\delta_L^L"] \\
A\oplus(B\otimes(C\oplus D)) \arrow[d, "i\oplus\delta_L^L"'] & & ((D\oplus C)\otimes B)\oplus A \arrow[d, "\delta_R^R\oplus i"] \\
A\oplus((B\otimes C)\oplus D) \arrow[r, "c_\oplus;c_\oplus\oplus i"] & (D\oplus(B\otimes C))\oplus A \arrow[r, "(i\oplus c_\otimes)\oplus i"] & (D\oplus(C\otimes B))\oplus A
\end{tikzcd}$$

Substituting $A = D = 1$ and $B = C = 0$, where 0 is the initial object of a distributive category and 1 is a final object, makes the top horizontal map the identity on $(1+0)\times(0+1)$. Similarly the bottom maps give, up to equivalence, the Boolean negation map. However, under this substitution, condition (13) yields the same diagram with, up to the same equivalence, the identity map across the bottom. This implies that, in any distributive category which is also weakly distributive, Boolean negation has a fixed point. This happens only when the distributive category is a preorder (see [5]). □

The subsequent discussion must take into account this observation. In particular, Theorem 3.3 can now be strengthened to say:

Theorem 3.3. *A cartesian weakly distributive category is posetal if and only if the initial object is strict.*

Given these results it all the more surprising to discover that "pointed sets" – and more generally the Kleisli category of a distributive category with respect to the exception monad – do give examples of cartesian weakly distributive categories. Here the weak distribution is given by annihilating the offending "lifted product component", in the following sense. $A \times B = A + A \otimes B + B$ is the product in this Kleisli category, where $\_ \otimes \_$ is the lifted (or amalgamated) product. The final object is also the initial object: so that this category has a zero. To define the weak distribution we use the fact that the lifted product is distributive so that there are a series of natural equivalences:

$$A \times (B + C) \equiv A + A \otimes (B + C) + B + C \equiv A + A \otimes B + A \otimes C + B + C.$$

Using the fact that we have a zero we may now annihilate the $A \otimes C$ component to obtain an object naturally equivalent to $(A \times B) + C$. This defines the weak distribution δ_L^L. Then it is routine to verify the following.

Proposition 3.4. *For any distributive category the Kleisli category for the exception monad is a cartesian weakly distributive category.*

Thus, we may faithfully include a distributive category (via the Kleisli left adjoint) into a cartesian weakly distributive category, and the lifted product is connected both monoidally and comonoidally to the product in the Kleisli category.

4. Adding negation

Definition 4.1. We define a weakly distributive category with negation to be a weakly distributive category with object functions ${}^{\perp}(-)$ and $(-)^{\perp}$, together with the following parametrized families of maps ("contradiction" and "tertium non datur"):

$$\begin{array}{ll} {}^{\perp}A \otimes A \xrightarrow{\gamma_A^L} \perp & \qquad A \otimes A^{\perp} \xrightarrow{\gamma_A^R} \perp \\ \top \xrightarrow{\tau_A^L} A^{\perp} \oplus A & \qquad \top \xrightarrow{\tau_A^R} A \oplus {}^{\perp}A \end{array}$$

which satisfy the following coherence conditions:

$$\begin{tikzcd}
A \otimes \top \arrow[r, "i \otimes \tau^L"] \arrow[dd, "u_\otimes^R"'] & A \otimes (A^{\perp} \oplus A) \arrow[d, "\delta_L^L"] \\
= & (A \otimes A^{\perp}) \oplus A \arrow[d, "\gamma^R \oplus i"] \\
A & \perp \oplus A \arrow[l, "u_\oplus^L"]
\end{tikzcd}$$

As before, this induces a further set of equations:

$$i \otimes \tau^L; \delta_L^L; \gamma^R \oplus i; u_\oplus^L = u_\otimes^R \tag{15}$$

$$\tau^R \otimes i; \delta_R^R; i \oplus \gamma^L; u_\oplus^R = u_\otimes^L \tag{16}$$

$$i \otimes \tau^R; \delta_L^L; \gamma^L \oplus i; u_\oplus^L = u_\otimes^R \tag{17}$$

$$\tau^L \otimes i; \delta_R^R; i \oplus \gamma^R; u_\oplus^R = u_\otimes^L \tag{18}$$

We illustrate the third of these, as it does not result from a symmetry:

$$\begin{tikzcd}
{}^\perp A \otimes \top \arrow[r, "i\otimes\tau^R"] \arrow[dd, "u_\otimes^R"'] & {}^\perp A \otimes (A \oplus {}^\perp A) \arrow[d, "\delta_L^L"] \\
= & ({}^\perp A \oplus A) \otimes {}^\perp A \arrow[d, "\gamma^L \oplus i"] \\
{}^\perp A & \perp \oplus {}^\perp A \arrow[l, "u_\oplus^L"]
\end{tikzcd}$$

Notice that we have not required that $(-)^\perp$ and ${}^\perp(-)$ be contravariant functors, but merely that they be defined on objects. Nor have we required that there be a natural isomorphism between A, ${}^\perp(A^\perp)$, and $({}^\perp A)^\perp$. Of course, $(-)^\perp$ and ${}^\perp(-)$ do extend to contravariant functors and these natural isomorphisms exist but these are consequence of the axioms as we shall see. Note also that the symmetries in the definition of a planar weakly distributive categories can be extended with the assumption that both the symmetries op′ and $\odot'$ also swap the functor $(-)^\perp$ with ${}^\perp(-)$.

Lemma 4.2. *In a weakly distributive category with negation we have the following adjunctions*:

$$A \otimes - \dashv A^\perp \oplus - \qquad {}^\perp A \otimes - \dashv A \oplus -$$

$$- \otimes B \dashv - \oplus {}^\perp B \qquad - \otimes B^\perp \dashv - \oplus B$$

corresponding to the following bijections:

$$\frac{A \otimes B \longrightarrow C}{B \longrightarrow A^\perp \oplus C} \qquad \frac{{}^\perp A \otimes B \longrightarrow C}{B \longrightarrow A \oplus C}$$

$$\frac{A \otimes B \longrightarrow C}{A \longrightarrow C \oplus {}^\perp B} \qquad \frac{A \otimes B^\perp \longrightarrow C}{A \longrightarrow C \oplus B}$$

Proof. We shall treat just the adjunction $- \otimes B \dashv - \oplus {}^\perp B$, the symmetries can be used to derive the rest. Given a map $A \otimes B \longrightarrow C$, we derive the corresponding map as $A \longrightarrow A \otimes \top \longrightarrow A \otimes (B \oplus {}^{\perp}B) \longrightarrow (A \otimes B) \oplus {}^{\perp}B \longrightarrow C \oplus {}^{\perp}B$. Conversely, given $A \longrightarrow C \oplus {}^{\perp}B$, we have $A \otimes B \longrightarrow (C \oplus {}^{\perp}B) \otimes B \longrightarrow C \oplus ({}^{\perp}B \otimes B) \longrightarrow C \oplus \perp \longrightarrow C$.

In particular, the unit $\eta_A : A \longrightarrow (A \otimes B) \oplus {}^{\perp}B$ is given by

$$\eta_A : A \xrightarrow{u_{\otimes}^{R-1}} A \otimes \top \xrightarrow{i \otimes \tau^R} A \otimes (B \oplus {}^{\perp}B) \xrightarrow{\delta_L^L} (A \otimes B) \oplus {}^{\perp}B$$

and the counit $\varepsilon_A : (A \oplus {}^{\perp}B) \otimes B \longrightarrow A$ is given (as the symmetries might suggest) by

$$\varepsilon_A : (A \oplus {}^{\perp}B) \otimes B \xrightarrow{\delta_R^R} A \oplus ({}^{\perp}B \otimes B) \xrightarrow{i \oplus \gamma^L} A \oplus \perp \xrightarrow{u_{\oplus}^R} A$$

To check the triangle identities, we must verify that the (outer part of the) diagram in Fig. 2 commutes. The marked regions commute by the corresponding coherence conditions; the others are by naturality. There is a similar diagram for the other triangle identity which we leave as an exercise for the reader. □

$$\begin{tikzcd}
 & (A \oplus {}^{\perp}B) \otimes \top \arrow[r, "i \otimes \tau^R"] \arrow[d, dotted, "\delta_R^R"'] & (A \oplus {}^{\perp}B) \otimes (B \oplus {}^{\perp}B) \arrow[r, "\delta_L^L"] \arrow[d, dotted, "\delta_R^R"] & ((A \oplus {}^{\perp}B) \otimes B) \oplus {}^{\perp}B \arrow[dd, "\delta_R^R \otimes i"] \\
A \oplus {}^{\perp}B \arrow[ur, "u_{\otimes}^{R-1}"] \arrow[r, phantom, "(6)"] & A \oplus ({}^{\perp}B \otimes \top) \arrow[l, dotted, "i \oplus u_{\otimes}^R"'] \arrow[r, dotted, "i \oplus i \otimes \tau^R"] \arrow[d, phantom, "(A \oplus (17))"] & A \oplus ({}^{\perp}B \otimes (B \oplus {}^{\perp}B)) \arrow[d, dotted, "i \oplus \delta_L^L"] \arrow[r, phantom, "(13)"] & \\
 & A \oplus (\perp \oplus {}^{\perp}B) \arrow[ul, dotted, "i \oplus u_{\oplus}^L"] & A \oplus (({}^{\perp}B \otimes B) \oplus {}^{\perp}B) \arrow[l, dotted, "i \oplus \gamma^L \oplus i"'] & (A \oplus ({}^{\perp}B \otimes B)) \oplus {}^{\perp}B \arrow[l, dotted, "a_{\oplus}"'] \arrow[dll, "i \oplus \gamma^L \oplus i"] \\
 & (A \oplus \perp) \oplus {}^{\perp}B \arrow[uul, "u_{\oplus}^R \oplus i"] \arrow[u, dotted, "a_{\oplus}"] \arrow[u, phantom, "(3)", near end] & &
\end{tikzcd}$$

Fig. 2. Triangle identity.

We may now use the adjunctions to define the effect of $(-)^\perp$ on maps:

$$\frac{\dfrac{\dfrac{\dfrac{A \longrightarrow B}{A \otimes \top \longrightarrow B}}{\top \longrightarrow A^\perp \oplus B}}{\top \otimes B^\perp \longrightarrow A^\perp}}{B^\perp \longrightarrow A^\perp}$$

and similarly for ${}^\perp(-)$:

$$\frac{\dfrac{\dfrac{\dfrac{A \longrightarrow B}{\top \otimes A \longrightarrow B}}{\top \longrightarrow B \oplus {}^\perp A}}{{}^\perp B \otimes \top \longrightarrow A^\perp}}{{}^\perp B \longrightarrow {}^\perp A}$$

It is then a matter of verifying that this is functorial, by explicitly giving the "formulas" for $B^\perp \longrightarrow A^\perp$, and for ${}^\perp B \longrightarrow {}^\perp A$, in terms of $A \longrightarrow B$, and verifying that the appropriate diagrams commute when this is done for i_A and for $f;g$.

For example, then given $A \xrightarrow{f} B$, we get

$$\begin{aligned}&{}^\perp B \xrightarrow{{}^\perp f} {}^\perp A \text{ as } {}^\perp B \xrightarrow{u^{-1}} {}^\perp B \otimes \top \xrightarrow{i\otimes\tau} {}^\perp B \otimes (A \oplus {}^\perp A) \xrightarrow{i\otimes(f\oplus i)} {}^\perp B \otimes (B \oplus {}^\perp A)\\ &\xrightarrow{\delta^L_L} ({}^\perp B \otimes B) \oplus {}^\perp A \xrightarrow{\gamma\oplus i} \bot \oplus {}^\perp A \xrightarrow{u} {}^\perp A\end{aligned}$$

(To get this from the definition of the adjunction, we use the fact that the map

$$\top \xrightarrow{u^{-1}} \top \otimes \top \xrightarrow{i\otimes\tau} \top \otimes (A \oplus {}^\perp A) \xrightarrow{\delta^L_L} (\top \otimes A) \oplus {}^\perp A \xrightarrow{u\oplus i} A \oplus {}^\perp A$$

is equal to τ. This follows from Eq. (5) and naturality.)

The case $f = i$ gives an instance of coherence condition (17), so that is no problem. The other case is a bit more complicated: to verify that ${}^\perp(-)$ preserves composition, one must show that ${}^\perp(f;g) = {}^\perp g; {}^\perp f$ for $A \xrightarrow{f} B \xrightarrow{g} C$. This translates to showing that the following diagram commutes:

$$\begin{tikzcd}
 & {}^\perp C \otimes \top \arrow[r, "i\otimes\tau^R"] \arrow[dd, equal] & {}^\perp C \otimes (B \oplus {}^\perp B) \arrow[r, "i\otimes(g\oplus i);\,\delta^L_L;\,\gamma^L\oplus i;\,u^L"] \arrow[dd, dotted, "i\otimes(i\oplus{}^\perp f)"] & {}^\perp B \arrow[dd, "{}^\perp f"] \\
{}^\perp C \arrow[ur, "u^{-1}"] \arrow[dr, "u^{-1}"'] & & & \\
 & {}^\perp C \otimes \top \arrow[r, "i\otimes\tau^R;\, i\otimes(f\oplus i)"'] & {}^\perp C \otimes (B \oplus {}^\perp A) \arrow[r, "i\otimes(g\oplus i);\,\delta^L_L;\,\gamma^L\oplus i;\,u^L"'] & {}^\perp A
\end{tikzcd}$$

The right-hand square (involving g) commutes by naturality, and the left square (involving f) reduces to the diagram:

$$\begin{tikzcd}
\top \arrow[r, "\tau^R"] \arrow[d, "\tau^R"'] & B \oplus {}^{\perp}B \arrow[d, "i \oplus {}^{\perp}f"] \\
A \oplus {}^{\perp}A \arrow[r, "f \oplus i"'] & B \oplus {}^{\perp}A
\end{tikzcd}$$

To see this commutes we construct the decomposition of the diagram shown in Fig. 3. Furthermore, notice that $(-)^{\perp}$ is full and faithful as there is a bijection $\mathrm{Hom}(A,B) \simeq \mathrm{Hom}(B^{\perp},A^{\perp})$.

In the symmetric case, it is appropriate to identify the two negation operators:

Definition 4.3. A symmetric weakly distributive category with negation is a symmetric weakly distributive category with an object function $(-)^{\perp}$, together with the following parametrized families of maps.

$$A \otimes A^{\perp} \xrightarrow{\gamma_A^R} \perp$$

$$\top \xrightarrow{\tau_A^R} A \oplus A^{\perp}$$

These induce the following families (by composing with symmetry maps):

$$A^{\perp} \otimes A \xrightarrow{\gamma_A^L} \perp$$

$$\top \xrightarrow{\tau_A^L} A^{\perp} \oplus A$$

which together satisfy the following coherence conditions:

$$i \otimes \tau^L; \delta_L^L; \gamma^R \oplus i; u_{\oplus}^L = u_{\otimes}^R$$

$$i \otimes \tau^R; \delta_L^L; \gamma^L \oplus i; u_{\oplus}^L = u_{\otimes}^R$$

We leave to the reader the exercise of showing that the other equations generated by the standard symmetries are consequences of the above. For the record, they are the following:

$$\tau^R \otimes i; \delta_R^R; i \oplus \gamma^L; u_{\oplus}^R = u_{\otimes}^L$$

$$\tau^L \otimes i; \delta_R^R; i \oplus \gamma^R; u_{\oplus}^R = u_{\otimes}^L$$

$$i \otimes \tau^R; \delta_R^L; i \oplus \gamma^R; u_{\oplus}^R = u_{\otimes}^R$$

$$\tau^L \otimes i; \delta_L^R; \gamma^L \oplus i; u_{\oplus}^L = u_{\otimes}^L$$

$$i \otimes \tau^L; \delta_R^L; i \oplus \gamma^L; u_{\oplus}^R = u_{\otimes}^R$$

$$\tau^R \otimes i; \delta_L^R; \gamma^R \oplus i; u_{\oplus}^L = u_{\otimes}^L$$

$$
\begin{tikzcd}[column sep=small]
 & & & B \oplus {}^{\bot}B \arrow[dll, "i \oplus u^{R-1}"'] \arrow[dl, "u^{R-1}"] \arrow[dl, phantom, bend right, "(6)"] & \\
 & B \oplus ({}^{\bot}B \otimes \top) \arrow[d, "i \otimes \tau^R"'] & (B \oplus {}^{\bot}B) \otimes \top \arrow[l, "\delta^R_R"'] \arrow[d, "i \otimes \tau^R"'] & \top \otimes \top \arrow[l, "\tau^R \oplus i"'] \arrow[d, "i \otimes \tau^R"'] & \top \arrow[ul, "\tau^R"'] \arrow[l, shift right, "u^{R-1}"'] \arrow[l, shift left, "u^{L-1}"] \arrow[dd, "\tau^R"'] \\
 & B \oplus ({}^{\bot}B \otimes (A \oplus {}^{\bot}A)) \arrow[dl, "i \oplus \delta^L_L"'] & (B \oplus {}^{\bot}B) \otimes (A \oplus {}^{\bot}A) \arrow[l, "\delta^R_R"'] \arrow[dd, "\delta^L_L"'] & \top \otimes (A \oplus {}^{\bot}A) \arrow[l, "\tau^R \oplus i"'] \arrow[dd, "\delta^L_L"'] & \\
B \oplus (({}^{\bot}B \otimes A) \oplus {}^{\bot}A) \arrow[dr, "a^{-1}"] \arrow[dd, "i \oplus ((i \otimes f) \oplus i)"'] & (13) & & (5) & A \oplus {}^{\bot}A \arrow[ul, "u^{L-1}"'] \arrow[dl, "u^{L-1} \oplus i"] \arrow[ddd, "f \oplus i"] \\
 & (B \oplus ({}^{\bot}B \otimes A)) \oplus {}^{\bot}A \arrow[d, "(i \oplus (i \otimes f)) \oplus i"'] & ((B \oplus {}^{\bot}B) \otimes A) \oplus {}^{\bot}A \arrow[l, "\delta^R_R"'] \arrow[d, "(i \otimes f) \oplus i"'] & (\top \otimes A) \oplus {}^{\bot}A \arrow[l, "(\tau^R \otimes i) \oplus i"'] \arrow[d, "(i \otimes f) \oplus i"'] & \\
B \oplus (({}^{\bot}B \otimes B) \oplus {}^{\bot}A) \arrow[r, "a^{-1}"] \arrow[d, "i \oplus (\gamma^L \oplus i)"'] & (B \oplus ({}^{\bot}B \otimes B)) \oplus {}^{\bot}A \arrow[d, "(i \oplus \gamma^L) \oplus i"'] & ((B \oplus {}^{\bot}B) \otimes B) \oplus {}^{\bot}A \arrow[l, "\delta^R_R \oplus i"'] \arrow[d, phantom, "((16) \oplus {}^{\bot}A)"] & (\top \otimes B) \oplus {}^{\bot}A \arrow[l, "(\tau^R \otimes i) \oplus i"'] \arrow[dr, "u^L \oplus i"] & \\
B \oplus (\bot \oplus {}^{\bot}A) \arrow[r, "a^{-1}"] \arrow[rrrr, bend right, "i \oplus u^L"'] & (B \oplus \bot) \oplus {}^{\bot}A \arrow[rrr, "u^R \oplus i"'] \arrow[rrr, phantom, shift right=3, "(1)"] & & & B \oplus {}^{\bot}A
\end{tikzcd}
$$

Fig. 3. ${}^{\bot}(-)$ preserves composition.

The corresponding lemma is immediate:

Lemma 4.4. *In a symmetric weakly distributive category with negation we have the following adjunctions*:

$$A \otimes - \dashv A^{\perp} \oplus - \qquad A^{\perp} \otimes - \dashv A \oplus -$$
$$- \otimes B \dashv - \oplus B^{\perp} \qquad - \otimes B^{\perp} \dashv - \oplus B$$

corresponding to the following bijections:

$$\frac{A \otimes B \longrightarrow C}{B \longrightarrow A^{\perp} \oplus C} \qquad \frac{A^{\perp} \otimes B \longrightarrow C}{B \longrightarrow A \oplus C}$$

$$\frac{A \otimes B \longrightarrow C}{A \longrightarrow C \oplus B^{\perp}} \qquad \frac{A \otimes B^{\perp} \longrightarrow C}{A \longrightarrow C \oplus B}$$

Of course, the point of all this is the following.

Theorem 4.5. *The notions of symmetric weakly distributive categories with negation and $*$-autonomous categories coincide.*

Proof. One direction is more or less automatic from the lemma, in view of Barr's characterization of $*$-autonomous categories in [2]. That is to say, symmetric weakly distributive categories with negation are $*$-autonomous. Of course, to make the translation to Barr's framework, we must make the following (standard) definition: $A \multimap B = A^{\perp} \oplus B$.

The involutive nature of $(-)^{\perp}$ follows from the lemma straightforwardly: viz. the iso $A = A^{\perp\perp}$ is induced by the adjunctions:

$$\begin{array}{c} A \longrightarrow B \\ \hline \top \otimes A \longrightarrow B \\ \hline \top \longrightarrow B \oplus A^{\perp} \\ \hline \top \otimes A^{\perp\perp} \longrightarrow B \\ \hline A^{\perp\perp} \longrightarrow B \end{array}$$

Then we can conclude that $A \multimap B = (A \otimes B^{\perp})^{\perp}$ also.

In either case, it is now easy to verify the essential bijection:

$$\begin{array}{c} A \longrightarrow (B \multimap C^{\perp}) \\ \hline A \longrightarrow B^{\perp} \oplus C^{\perp} \\ \hline A \otimes C \longrightarrow B^{\perp} \\ \hline C \otimes A \longrightarrow B^{\perp} \\ \hline C \longrightarrow B^{\perp} \oplus A^{\perp} \\ \hline C \longrightarrow (B \multimap A^{\perp}) \end{array}$$

Next the other half of the proof: here we give just a brief sketch. It is a straightforward verification to check that $*$-autonomous categories are weakly distributive, though the diagrams can be pretty horrid. We shall just indicate how the weak distribution δ_L^L is obtained, leaving the rest to the faith of the reader.

Defining $A \oplus B = A^\perp \multimap B$, we need $\delta_L^L : A \otimes (B^\perp \multimap C) \longrightarrow (A \otimes B)^\perp \multimap C$. While it is possible to give a formula for this morphism, it is perhaps more instructive to give its derivation:

First note that under the functor $(-)^\perp$, the internal hom bijection becomes

$$\frac{C^\perp \longrightarrow (A \otimes B)^\perp}{B \otimes C^\perp \longrightarrow A^\perp}$$

From this it is easy to derive maps $A \otimes (A \otimes B)^\perp \longrightarrow B^\perp \longrightarrow (B^\perp \multimap C) \multimap C$. Then we can use the bijection

$$\frac{A \otimes X \longrightarrow Y \multimap C}{A \otimes Y \longrightarrow X \multimap C}$$

to derive the map $A \otimes (B^\perp \multimap C) \longrightarrow (A \otimes B)^\perp \multimap C$ as needed. □

Remark 4.6 (*Planar non-commutativity*). The above suggests that (non-symmetric) weakly distributive categories with negation provide a natural notion of non-symmetric $*$-autonomous categories, and hence of non-commutative linear logic (rather, the multiplicative fragment thereof). The planar version we have outlined at the beginning of this section has become widely accepted – an account of this syntax (in a posetal context) appears in [1]. Note that in this context, there are (natural) isomorphisms $({}^\perp A)^\perp \simeq A$, ${}^\perp(A^\perp) \simeq A$. Furthermore, there are two internal homs: $A \multimap B = A^\perp \oplus B \simeq (A \otimes {}^\perp B)^\perp$ and $B \circ\!- A = B \oplus {}^\perp A \simeq {}^\perp(B^\perp \otimes A)$.

In [6] we also presented a hybrid definition, with just one negation operator with non-symmetric tensors. Our original presentation arose in an attempt to describe commutative linear logic: it displayed some of the features of the planar non-commutative form as well as the commutative form. At this time we feel it is premature to pronounce definitively on the "best" degree of non commutativity in linear logic, and so we offer only these comments: First, our main observation is that the core of the multiplicative fragment of linear logic may be found in the two tensors, connected by weak distributivity. (We do not believe that the central role played by the weak distributivities, permutative or not, had been sufficiently observed before.[5]) Second, to include negation and internal hom, one need only add negation in the most simple minded manner (the internal hom structure follows naturally). For example, Barr has pointed out that in some contexts the simplest way to show that a category is $*$-autonomous is to show that it is weakly distributive with negation [3]. Third, the various versions of this fragment may be classified by the degree of the weak distributivity assumed and the nature of the negation added. □

[5] An exception is recent work of de Paiva and Hyland, which has among other things pointed out some of the aspects of the distributivities we have in mind here.

5. Some examples

To conclude, we shall briefly consider some examples of weakly distributive categories, beginning with preorders and working up to more substantial examples.

5.1. Posetal weakly distributive categories

The beauty of the posetal weakly distributive categories is that one need not check the coherence conditions as all diagrams commute! Thus, it suffices to have the weak distributions present. Notice first that, when such a category is bicartesian, the initial object is necessarily strict, giving:

Lemma 5.1. *All bicartesian weakly distributive categories which are preorders are equivalent to distributive lattices.*

Thus, the interesting posetal examples occur when one or both tensors are non-cartesian. There are plenty of examples of these. Here are two sources:

- (Droste) Let L be a lattice ordered monoid (that is a set having a commutative, associative, and idempotent operation $x \wedge y$, and an associative operation $x \cdot y$ with unit 1 such that $z \cdot (x \wedge y) = (z \cdot x) \wedge (z \cdot y)$ and $(x \wedge y) \cdot z = (x \cdot z) \wedge (y \cdot z)$) in which every element is less than 1 (so this is the unit of $\wedge$ too) then L is a posetal weakly distributive category. This because

  $$x \cdot (y \wedge z) = (x \cdot y) \wedge (x \cdot z) \leq (x \cdot y) \wedge (1 \cdot z) = (x \cdot y) \wedge z$$

  and similarly for the other weak distributions.
  An example of such an L is the negative numbers. In general one may take the negative portion of any lattice ordered group (free groups can be lattice ordered so that the multiplication need not be commutative).
- A shift monoid is a commutative monoid $(M, 0, +)$ with a designated invertible element a. This allows one to define a second "shifted" multiplication $x \cdot y = x + y - a$ with unit a for which we have the following identity:

  $$x \cdot (y + z) = (x \cdot y) + z$$

  which clearly is a weak distribution. In this manner a shift monoid becomes a discrete weakly distributive category. Furthermore, it is not hard to show that every discrete symmetric weakly distributive category must be a shift monoid.
  This example is also of interest as it suggests that when one inverts the weak distributions (which produces braidings on the tensors), the tensors, which need not be equivalent, are related by a $\oplus$ invertible object. This is, in fact, what happens in general, as we shall sketch below.

It is also of interest to specialize our presentation of $*$-autonomous categories to the case of preorders. Again, only the existence of the maps themselves must be ensured, which gives:

Proposition 5.2. *A preorder is a $*$-autonomous category if and only if it has two symmetric tensors $\otimes$ and $\oplus$ and an object map $(-)^{\perp}$ such that*

(i) $x \otimes (y \oplus z) \leq (x \otimes y) \oplus z$,
(ii) $x \otimes x^{\perp} \leq \perp$,
(iii) $\top \leq x \oplus x^{\perp}$.

Suppose that M is a shift monoid equipped with a map $(-)^{\perp}$ such that $x + x^{\perp} = a$ ("tertium non datur") then we have

$$x \cdot x^{\perp} = x + x^{\perp} - a = a - a = 0$$

which is "contradiction". So M is a discrete $*$-autonomous category. Note that moreover M is a group, with $-x = x^{\perp} - a$; in fact shift groups (shift monoids with M a group) are the same as discrete $*$-autonomous categories in this way: $\top = a$, $x^{\perp} = a - x$, and conversely, a discrete $*$-autonomous category is a group (with respect to $\oplus$, with inverse given by $-x = x^{\perp} \otimes \perp$), and so a shift group (with $\top$ as designated invertible element). (A curiosity about this example: the initial shift group (also the initial shift monoid) is $\mathbf{Z}$, the integers, under addition with $\top = 1$. This structure also arises when checking the validity of proof nets [8].)

We can construct similar examples with ordered shift monoids, (for example, $\mathbf{Z}$ as above with the standard order), to get examples of $*$-autonomous posets. Note that a $*$-autonomous ordered shift monoid must be a group, since $x \cdot x^{\perp} \leq 0$ and $a \leq x + x^{\perp}$ imply that $x + x^{\perp} = a$, and so we are in the context above. Note also that by a suitable choice of a we can arrange for the poset to satisfy the mix rule, $x \otimes y \leq x \oplus y$, or its opposite, the co-mix rule $x \otimes y \geq x \oplus y$, or to be compact $x \otimes y = x \oplus y$.

5.2. Shifted tensors

We shall briefly consider the categorical generalization of shift monoids. Given a tensor category $(\mathbf{X}, \otimes, \top, a_{\otimes}, u_{\otimes}^{L}, u_{\otimes}^{R})$ a *tensor inverse* for the object $\perp$ is given by an object with two isomorphisms $(\perp^{-1}, s^{L}, s^{R})$ where

$$s^{L} : \perp \otimes \perp^{-1} \longrightarrow \top \qquad\qquad s^{R} : \perp^{-1} \otimes \perp \longrightarrow \top$$

such that

$$\begin{tikzcd}
(\perp^{-1} \otimes \perp) \otimes \perp^{-1} \arrow[rr, "a_\otimes"] \arrow[d, "s^R \otimes 1"'] & & \perp^{-1} \otimes (\perp \otimes \perp^{-1}) \arrow[d, "1 \otimes s^L"] \\
\top \otimes \perp^{-1} \arrow[dr, "u^L_\otimes"'] & & \perp^{-1} \otimes \top \arrow[dl, "u^R_\otimes"] \\
& \perp^{-1} &
\end{tikzcd}$$

It is worth remarking that the tensor inverse of an object $\perp$ is determined to equivalence and is of course the analogue of an inverse in a monoid. If a tensor category is the proof theory of a monoid, a tensor category, in which every element has a tensor inverse, is the proof theory of a group.

Given such a tensor inverse of $\perp$ we may create a new tensor

$$X \oplus Y = X \otimes (\perp^{-1} \otimes Y)$$

with unit $\perp$ and natural transformations:

$$A \oplus \perp \xrightarrow{u^R_\oplus} A$$
$$= A \otimes (\perp^{-1} \otimes \perp) \xrightarrow{1 \otimes s^R} A \otimes \top \xrightarrow{u^R_\otimes} A$$
$$\perp \oplus A \xrightarrow{u^L_\oplus} A$$
$$= \perp \otimes (\perp^{-1} \otimes A) \xrightarrow{a^{-1}_\otimes} (\perp \otimes \perp^{-1}) \otimes A \xrightarrow{s^L \otimes 1} \top \otimes A \xrightarrow{u^L_\otimes} A$$
$$(A \oplus B) \oplus C \xrightarrow{a_\oplus} A \oplus (B \oplus C)$$
$$= (A \otimes (\perp^{-1} \otimes B)) \otimes (\perp^{-1} \otimes C) \xrightarrow{a_\otimes ; 1 \otimes a_\otimes} A \otimes (\perp^{-1} \otimes (B \otimes (\perp^{-1} \otimes C)))$$

This we call this new tensor the $\perp$-*shifted tensor*.

The pentagon for $\oplus$ commutes from the coherence theorem for the tensor $\otimes$. It is less immediate that the diagram for the unit commutes and involves a simple diagram chase using the coherence diagram for tensor inverses.

It is now immediate that there are weak distributions (of the planar variety) for the tensor over the shifted tensor given by

$$A \otimes (B \oplus C) \xrightarrow{\delta^L_L} (A \otimes B) \oplus C = A \otimes (B \otimes (\perp^{-1} \otimes C)) \xrightarrow{a_\otimes} (A \otimes B) \otimes (\perp^{-1} \otimes C)$$

$$(B \oplus C) \otimes A \xrightarrow{\delta^R_R} B \oplus (C \otimes A) = (B \otimes (\perp^{-1} \otimes C)) \otimes A \xrightarrow{a_\otimes ; 1 \otimes a_\otimes} B \otimes (\perp^{-1} \otimes (C \otimes A))$$

Using the coherence of the tensor $\otimes$ it is immediate that all the diagrams of weak distributivity, which do not involve the cotensor unit, will commute. Those which involve the cotensor unit are also easily checked. We therefore have:

Proposition 5.3. *Any monoidal category with an object $\perp$ with a tensor inverse has a $\perp$-shifted tensor $\oplus$ which together with the tensor forms a weakly distributive category in which the weak distributions are natural isomorphisms.*

We can obtain the non-planar weak distributions if the monoidal category is symmetric or, more interestingly, when the original category was braided (some care has to be taken over the sense of the twistings). This suggests a further variant on group theory: braided tensor categories in which every object has a tensor inverse!

Starting at the weakly distributive end, we might reasonably be curious about the effect of demanding that the weak distributions be isomorphisms. In the planar case we note that $\top \oplus \top$ is the natural candidate for the tensor inverse of $\perp$ as

$$
\begin{aligned}
&\perp \otimes (\top \oplus \top) \xrightarrow{s^L} \top \\
&= \perp \otimes (\top \oplus \top) \xrightarrow{\delta_L^L} (\perp \otimes \top) \oplus \top \xrightarrow{u_\otimes^R \oplus 1} \quad \perp \oplus \top \xrightarrow{u_\oplus^L} \top
\end{aligned}
$$

and symmetrically for s^R. It is a substantial diagram chase to establish that when the weak distributions are isomorphisms this defines a tensor inverse for $\perp$. It then follows more easily that the $\oplus$ is a shifted tensor (to natural equivalence).

Proposition 5.4. *Any weakly distributive category whose weak distributions are natural isomorphisms has a tensor inverse for $\perp$ whose shift tensor is naturally equivalent to the cotensor.*

Finally, if we had the non-planar distributions and they were isomorphisms as well then there would be a twist map:

$$A \oplus B \xleftarrow{\delta_L^L;\delta_R^R} A \otimes (\top \oplus \top) \otimes B \xrightarrow{\delta_L^R;\delta_R^L} B \oplus A$$

where we have omitted the required units and associative maps. This natural isomorphism introduces a braiding on the cotensor (the proof involves ferocious diagram chasing and is beyond the scope of this article) and, by shifting, therefore a braiding on the tensor. Thus, the non-planar weakly distributive categories with all the weak distributions isomorphisms give rise to braided monoidal categories with an object with a tensor inverse. When the unit $\top$ and counit $\perp$ are isomorphic the non-planar weakly distributive category "degenerates" into a braided monoidal category with the "mix" rule giving a natural isomorphism of the tensors.

To show that a category is $*$-autonomous may be made more simple *via* our characterization in situations whenever the weakly distributive nature of the setting is known. We now discuss two examples of this: the span construction and modules of a bialgebra.

5.3. Span categories

Let $\mathbf{X}$ be any finitely complete category, then it is well-known that $\mathsf{Span}(\mathbf{X})$ has a symmetric tensor corresponding to the product. A little less well-known is the fact that the span category is $*$-autonomous, more precisely compact closed. To show this we identify the tensor and cotensor with the product and regard the distributions as the natural maps given by associativity and symmetry. This is immediately a weakly distributive category, albeit a rather trivial example. To make it $*$-autonomous it remains only to identify the involution and complementation maps. We shall let the involution be the obvious span reversal: note that this is the identity on objects. The complementation maps are given by

$$A \times A \xleftarrow{\Delta} A \xrightarrow{!} 1$$

and its span reversal. To show $*$-autonomy it suffices to check the complementation diagrams. Note that the following square is always a pullback:

$$\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & A \times A \arrow[d, "\Delta \times 1"] \\
A \times A \arrow[r, "1 \times \Delta"] & A \times A \times A
\end{tikzcd}$$

This allows the following interpretation of the complementation diagram (in which the distributions and unit maps have been suppressed):

$$\begin{tikzcd}
& & A \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & & \\
& A \times A \arrow[dl, "p_0"'] \arrow[dr, "1 \times \Delta"] & & A \times A \arrow[dl, "\Delta \times 1"'] \arrow[dr, "p_1"] & \\
A & & A \times A \times A & & A
\end{tikzcd}$$

This is transparently the identity map in the span category; hence the span category is $*$-autonomous.

Before passing on it is worth considering whether the span category has products. Because of the involution any product will be, at the same time, a coproduct and the coprojections will be the span reversals of the projections.

Furthermore, as the product of the original category when lifted to the span category has a right adjoint it must distribute over any coproducts the span category has. Thus, if the span biproduct arises from the original category's coproduct, the original category must have been distributive.

Lindner [12] observed a converse to this that the coproduct of extensive distributive categories gives rise to a biproduct in the span category.

5.4. Modules of bialgebras

It is generally known that the modules of a Hopf algebras of a $*$-autonomous category form a $*$-autonomous category. Perhaps this is one of the reason why the Hopf algebra's poorer cousin the bialgebra receives scant attention. However, one might reasonably stop to ask whether the category of bialgebra modules of a $*$-autonomous category enjoys any special properties. We can now provide an answer: it forms a weakly distributive category.

In fact, we may start with any symmetric (or braided) weakly distributive category, form the modules of a bialgebra thereof, and the resulting category will be weakly distributive. It is easily checked that the tensors of the bialgebra module category are the underlying tensors as are the weak distributivities.

Even if one is not convinced of the value of bialgebra modules, this observation does now provide an alternative way to establish the $*$-autonomy of Hopf modules. What remains is, of course, only to provide the complementation diagrams, which is straightforward.

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