Local Triviality of Proper $G_a$ Actions

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Regular actions of the additive group of complex numbers on complex surfaces and on complex affine space are considered. A proper action on an affine surface admits a geometric quotient which is an affine curve. A proper action on a normal quasiaffine surface is equivariantly trivial. New criteria for local and "global" triviality of proper actions on a complex affine space of arbitrary dimension are presented. © 1999 Academic Press

1. PRELIMINARIES

Let $G_a$ denote the additive group of complex numbers, let $X$ denote a quasiaffine variety over $\mathbb{C}$, and let $\sigma: G_a \times X \to X$ be a regular (sometimes referred to as rational, polynomial, or algebraic) action of $G_a$ on $X$. The action is said to admit an equivariant trivialization if there is a variety $Y$ and a $G_a$ equivariant isomorphism $X \to Y \times \mathbb{C}$, with the group acting trivially on $Y$ and by addition on the second. In that case, the action is conjugate to a global translation and $Y$ is a geometric quotient.

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The action is said to be locally trivial (in the Zariski topology) if $X$ is covered by $G_a$ stable affine open subsets on each of which the action admits an equivariant trivialization. If $X$ is a factorial affine variety, in the sense that $\mathbb{C}[X]$ is a unique factorization domain, then a locally trivial $G_a$ action admits a quasiaffine geometric quotient $Y$. This was proved in [2] for $X = \mathbb{C}^n$, but the assertion and its proof are valid for any factorial affine variety. Since this observation will be used in Section 4, the argument is summarized in Proposition 1.1 below.

Let $x : G_a \times X \to X$ be a regular action of $G_a$ on the irreducible quasi-affine variety $X$ and let $\tilde{x}$ be the morphism $G_a \times X \to X \times X$ given by $(t, x) \mapsto (x, x(t, x))$. With $\mathbb{C}[X]$ denoting the ring of globally defined regular functions on $X$, we have the induced $\mathbb{C}$-algebra homomorphism $\mathcal{O}_x : \mathbb{C}[X] \to \mathbb{C}[x(t, x)]$. Moreover, $\tilde{x}$ induces a $\mathbb{C}$-algebra homomorphism $\tilde{x} : \mathbb{C}[X \times X] \to \mathbb{C}[X \times G_a] \cong \mathbb{C}[X, t]$. Differentiating $\tilde{x}$ yields a locally nilpotent derivation $\delta$ of $\mathbb{C}[X]$: $\delta(P) = \left. \frac{\tilde{x}(P) - P}{t} \right|_{t=0}$, $\tilde{x} = \exp(\delta t)$.

Every $\tilde{x}$, hence every regular $G_a$ action, arises as the exponential of a locally nilpotent derivation.

It should be noted that the ring of invariants of the $G_a$ action is identical to the kernel of $\delta$. The field of $G_a$ invariants of the action extended to $\mathbb{C}[X]$ is the quotient field of the ring of invariants.

The authors thank the referee for improving the argument in the proof of the following proposition.

**Proposition 1.1.** Let $X$ be a factorial affine variety with locally trivial $G_a$ action. Then the geometric quotient exists as a quasiaffine variety.

**Proof.** Let $X = \bigcup_{i=1}^m X_i$ be a cover by affine open $G_a$ stable subsets giving the local trivialization. Thus, for each $i$, $X_i \cong Y_i \times \mathbb{C}$, with $Y_i$ an affine variety, and $G_a$ acts on $\cong Y_i \times \mathbb{C}$ as the identity on the first factor and by translation on the second. In terms of coordinate rings $\mathbb{C}[X_i] \cong \mathbb{C}[Y_i][s_i]$, with $\tilde{x}(s_i) = s_i + t$.

Regard $s_i$ as a rational function on $X$. Since $X$ is factorial, there is a regular function, $h_i$, on $X$ for which $\text{div}(h_i)$ is precisely the divisor of poles of $s_i$. Since $\tilde{x}(s_i) = s_i + t$, $s_i$, and $\tilde{x}(s_i)$ have the same divisor of poles, from which it follows that $\text{div}(h_i)$ is invariant. For $x \in X$, define the function $\zeta_x : \mathbb{C} \to \mathbb{C}$ by $\zeta_x : t \mapsto \frac{h_i(x)}{h_i(x)}$.

We see that $\zeta_x$ has neither zeros nor poles and is therefore constant for every $x \in X$ and each $i$. Thus $h_i$ is an invariant; i.e., $\delta(h_i) = 0$. Let $g_i = s_i h_i$, and observe that $\delta(g_i) = h_i$ and that $(h_1, \ldots, h_m) \mathbb{C}[X] = \mathbb{C}[X]$. 
We may assume that \( C[X] = \mathbb{C}[X] \) with \( a_{ij} \in \mathbb{C}[X] \). Set \( R = \mathbb{C}[a_{ij} | 1 \leq j \leq N, 1 \leq i \leq m]([h_1, \ldots, h_m]) \subseteq \mathbb{C}[X] \), and denote by \( Y \) the affine variety with coordinate ring \( R \). The induced morphism \( \pi: X \rightarrow Y \) has as its image an open subset \( Y_0 \) of \( Y \) isomorphic to \( \bigcup_{i=1}^{m} Y_i \), and it is clear that \( \pi: X \rightarrow Y_0 \) is a quotient morphism. In particular, \( \pi \) is flat, with all its fibers \( G_a \) orbits, which are one dimensional. 

For affine \( X \), the fixed point set for the action is the set of common zeros of \( \{ \delta X_i : 1 \leq i \leq n \} \) where \( x_i, 1 \leq i \leq n \), are coordinates on \( X \). If \( s \in C[X] \) satisfies \( \delta s = 1 \), then the \( G_a \) action is equivariantly trivial. In this case \( s \) is said to be a slice. If \( \ker \delta \cap \text{im} \delta \) generates the unit ideal in \( C[X] \) the action is locally trivial [2]. A \( G_a \) action \( \sigma \) is separated if \( \sigma \) has closed image. If \( \sigma: G_a \times X \rightarrow X \times X \) is a proper morphism of varieties, we say that \( \sigma \) is a proper action. The following criterion for local (equivariant) triviality has been established:

**Theorem 1.2** [2, Theorem 2.8]. An action of \( G_a \) on \( X = \mathbb{C}^n \) is locally (equivariantly) trivial if and only if the action is proper and \( C[X] \) is a (faithful) flat extension of its subring of \( G_a \) invariants.

A criterion for local triviality of a separated action, along with an algorithm to determine local triviality, is given in [3].

Note that if \( s \) is a slice for an equivariantly trivial action then, with \( z \) an indeterminate,

\[
\exp(z\delta) |_{z=s} : C[X] \rightarrow C[X]^{G_a}
\]

is a surjective ring homomorphism.

An assertion which would imply that any proper \( G_a \) action on a normal variety is locally trivial and admits a quasiprojective quotient appears in [10], and an example of a proper \( G_a \) action on \( \mathbb{C}^5 \) which is not locally trivial appears in [5]. In that example, the quotient of \( \mathbb{C}^5 \) by \( G_a \) exists as an algebraic space but does not have the structure of a quasiprojective variety. In fact the quotient does not exist as an algebraic scheme, since properness would imply that a quotient which is an algebraic scheme is separated [18, pp. 13–14] and this is easily seen not to be the case in the example. The error in [10] is pointed out in [9].

On the other hand, the assertion that proper actions are locally trivial is nearly correct for the following reasons. For one, a stronger conclusion holds for normal quasiaffine surfaces as shown in Section 2. The main result in Section 3 proves that proper actions on \( \mathbb{C}^n \) are locally trivial when the ring of invariants is affine and regular. If, in addition, the variety associated to the ring of invariants is contractible in the Euclidean topology, then the action is equivariantly trivial. In the holomorphic setting, a proper
$G_a$ action on a complex manifold admits a quotient which is again a complex manifold [13]. The issue therefore is that there may not be sufficiently many invariant regular functions to separate orbits. However, there are sufficiently many algebraic functions. Indeed, a result of Seshadri [22, Theorem 6.1] shows that given a proper action on a normal variety $X$, there exists a normal variety $Z$, finite over $X$, with a locally trivial $G_a$ action and separated quotient $Y$, so that the morphism $Z \to X$ is $G_a$ equivariant. Moreover, $C(Z)$ is a Galois extension of $C(X)$. If $\Gamma$ is the Galois group, then the actions of $G_a$ and $\Gamma$ commute on $Z$, inducing an action of the finite group $\Gamma$ on $Y$. As was pointed out in [9], the quotient of $X$ by $G_a$ can be identified with $Y/\Gamma$ and therefore exists as an algebraic space.

All fixed point free $G_a$ actions on $C^i$ for $i = 1, 2$ are equivariantly trivial, as are all proper $G_a$ actions on $C^3$. For $n = 1$ this is obvious, the case $n = 2$ follows from the triangulability of all $G_a$ actions on $C^2$ [20], and the assertion about $n = 3$ is [6, Theorem 2.4]. As indicated above, there are proper actions on $C^5$ which are not even locally trivial, but the situation for $C^4$ is a mystery at this point. We give an example of a smooth factorial fourfold with a proper $G_a$ action that is not locally trivial. The authors thank Professors Peter Russell and Shulim Kaliman for their assistance in showing that this fourfold is not isomorphic to $C^4$.

2. SURFACES

Fauntleroy and Magid show [11] that a fixed point free $G_a$ action on a normal quasiaffine surface admits a geometric quotient. The quotient need not be separated but, if the surface is factorial affine, they show that the action is equivariantly trivial and the quotient is an affine curve. This conclusion holds for a proper action on any normal quasiaffine surface. Moreover, for an arbitrary affine surface, the geometric quotient by a proper action exists and is an affine curve.

Central to the arguments of this section is the theorem of Zariski (e.g., [19, p. 52]) asserting, for any affine domain $R$ over a field $k$ with quotient field $K$ and a field $L$ of transcendence degree 1 over $k$ and $k \subset L \subset K$, that $R \cap L$ is an affine ring. This result will be applied in the context $k = C$, $R = C[X]$, $L = C(X)^{G_a}$ for $X$ a complex surface.

**Proposition 2.1.** If $G_a$ acts properly on a normal quasiaffine surface $X$, then $X$ is in fact affine and the action is equivariantly trivial.

**Proof.** Let $Z$ denote a normal variety finite over $X$, for which a separated quotient $W = Z/G_a$ exists as in [22, Theorem 6.1]. Since $W$ is smooth and one dimensional, $W$ is either projective or affine. Since the action is nontrivial, there is an $f \in C[Z]$ such that $\delta^2(f) = 0 \neq \delta(f)$. Set $g = \delta(f)$
and observe that \( g \) is defined globally on the quotient \( W \). This implies that \( W \) is affine. Indeed, either \( g \) is nonconstant or we may assume \( g = 1 \). But \( \delta(f) = 1 \) easily implies that \( C[Z] = R[f] \) where \( R \) is the one dimensional ring of \( G_a \) invariants. In either case \( W \) has nonconstant globally defined regular functions and therefore is affine.

We have, moreover, that \( Z \) is a principal \( G_a \) bundle over \( W \) and therefore that \( G_a \) is equivariantly isomorphic to \( W \times \mathbb{C}^1 \) with \( G_a \) acting trivially on the first factor and by translation on the second. In particular, \( Z \) is affine. Since \( Z \to X \) is a finite surjection [22, Theorem 6.1], a theorem of Chevalley [12, p. 222] shows that \( X \) is also affine. Let \( \Gamma \) denote the Galois group of \( C[Z] \) over \( C(X) \), and \( n \) its order. There exists \( f \in C[Z] \) satisfying \( \delta(f) = 1 \). Since the actions of \( \Gamma \) and \( G_a \) commute on \( C[Z] \), \( \delta(\frac{\text{tr}(f)}{n}) = 1 \), where \( \text{tr} \) denotes the trace function from \( C[Z] \) to \( C[X] \). The action on \( X \) is therefore equivariantly trivial.

**Corollary 2.2.** Let \( X \) be an affine surface with a proper \( G_a \) action and let \( X_0 \) be its smooth locus. Then \( X_0 \) is affine, and the \( G_a \) action restricts to an equivariantly trivial action on \( X_0 \).

**Proof.** It suffices to show that \( X_0 \) is stable under the group action, but this is obvious as the action is by automorphisms of the surface.

**Corollary 2.3.** Let \( X \) be an affine surface with a proper \( G_a \) action and let \( \tilde{X} \) be its normalization. Denote the affine curve \( \text{Spec} \ C[X]^{G_a} \) by \( Y \) and its normalization by \( \tilde{Y} \). Then the \( G_a \) action lifts to \( \tilde{X} \), where it is equivariantly trivial with quotient \( \tilde{Y} \).

**Proof.** That the action lifts to \( \tilde{X} \) follows from [23]. Moreover, it remains proper and therefore the action is equivariantly trivial. Call the quotient \( W \). To show \( \tilde{Y} \cong W \) is equivalent to showing that \( C[W] = C[\tilde{X}]^{G_a} \) is integral over \( C[Y] \) and that \( W \) and \( Y \) are birational.

Let \( I \) denote the conductor ideal \( \{ a \in C[X] \mid aC[\tilde{X}] \subset C[X] \} \). A calculation shows that \( I \) is stable under the derivation generating the \( G_a \) action and therefore contains a nonzero invariant \( h \). We thus obtain \( C[\tilde{X}] \) as a submodule of the \( C[X] \) module generated by \( \frac{1}{h} \). From this it follows that any invariant in \( C[X] \) can be expressed as \( \frac{f}{h} \), where \( f \in C[Y] \). Birationality of \( W \) and \( Y \) is immediate and \( C[W] \) is integral over \( C[Y] \) because the latter ring is affine, hence noetherian.

**Theorem 2.4.** Let \( G_a \) act properly on an affine surface \( X \). Then the geometric quotient \( X/G_a \) exists and is an affine curve.

**Proof.** We show that the geometric quotient exists as an algebraic scheme. Then, since the action is proper, hence separated, an application of [18, pp. 13–14] shows that the quotient is separated. However, the quotient is then a curve and affine because of the existence of globally defined
nonconstant regular functions. The construction of the quotient follows the construction in the proof of [11, Theorem 2].

Let $X$, $Y$, $X_0$ have the meanings as above, and let $Y_0$ denote the smooth affine curve Spec $\mathbb{C}[X_0]^{G_a}$ which is the quotient $X_0/G_a$. The morphisms $\widetilde{\pi}$ and $\pi_0$ are the quotient morphisms from $X$ to $Y$ and $X_0$ to $Y_0$, and $\pi: X \to Y$ is the morphism induced by the inclusion of rings. The diagram of $G_a$ morphisms

$$
\begin{array}{ccc}
\tilde{X} & \longrightarrow & X \\
\downarrow \quad \tilde{\pi} & & \downarrow \pi \\
Y & \longrightarrow & Y
\end{array}
$$

is commutative by the universal property for quotients, and therefore the morphism $X \to Y$ is surjective.

We claim that $\pi$ is open. First observe that the inclusion of $X_0$ in $X$ is $G_a$ equivariant and therefore we obtain a morphism $Y_0 \to Y$ which is clearly a birational isomorphism. Let $Y_1$ denote the smooth locus of $Y$ and let $Y_2$ denote the inverse image of $Y_1$ in $Y_0$. Zariski’s main theorem shows that $Y_2 \to Y_1$ is an open immersion. Since $Y_0$ and $Y$ are affine curves, so that open sets are those with finite complement, $Y_0 \to Y$ is an open morphism. Now let $U$ be an open subset of $X$ and $U_0 = U \cap X_0$. Note that $U_0 \neq \emptyset$ and $\pi_0(U_0)$ is open in $Y_0$, the quotient morphism being open. In particular the image of $U_0$ in $Y$ is open, but again using the fact that $Y$ is an affine curve, the image of $U$ is open as well.

Let $Y_3$ be the image of $Y_2$ in $Y$, and $Y - Y_3 = \{y_1, \ldots, y_m\}$. Note that $\pi^{-1}y_i = \bigcup_{i=1}^{m} C_{ij}$, a disjoint union of $G_a$ orbits in $X$, and $W = \pi^{-1}Y_2 \cong Y_2 \times \mathbb{C}$. Then $X - W = \bigcup_{i,j} C_{ij}$. For each integer-valued function $\alpha$ on $\{1, \ldots, m\}$ satisfying $1 \leq \alpha(i) \leq n_i$, set $X_\alpha = W \cup \bigcup_{i=1}^{m} C_{\alpha(i)}$. Then $X_\alpha$ is open in $X$, and $\pi |_{X_\alpha}$ is a quotient map onto $Y$. Since $X$ is union over all possible $\alpha$ of the $X_\alpha$ and a quotient exists locally for all $X_\alpha$, the geometric quotient exists for $X$.

3. LOCAL TRIVIALITY

The main tool is a deep result of Miyanishi [17] which relies on the notion of geometric irreducibility in codimension 1 (GICO) of a morphism of algebraic schemes. Since our concern is with affine varieties over the complex field, this condition can be expressed via

**Definition 1.** Let $\phi: X \to Y$ be a morphism of affine varieties. Then $\phi$ is GICO over $Y$ provided that for any height 1 prime ideal $p$ of $\mathbb{C}[Y]$...
and prime ideal $P$ of $C[X]$ minimal over $pC[X]$, defining a codimension 1 subvariety $T$ of $X$, the field $C(\phi T)$ is algebraically closed in $C(T)$.

This condition will be examined in the context of the morphism $X = C^n \to Y = \text{Spec} \ C[X]^{G_e}$ induced by the inclusion of rings $C[X]^{G_e} \to C[X]$ under the assumption that $C[X]^{G_e}$ is affine. It should be noted that $C[X]^{G_e}$ is a factorially closed subring of $C[X]$, hence also a unique factorization domain. Thus we are concerned with the extension of the quotient field of $C[X]^{G_e}/(p)$ to the quotient field of $C[X]/pC[X]$ for all principal prime ideals $(p)$ of $C[X]^{G_e}$.

The relevant theorem from [17] is Theorem 2, stated here as:

**Theorem 3.1.** Let $f: X \to Y$ be a dominant morphism from an affine variety $X = \text{Spec} A$ of dimension $n + 1$ to a smooth affine variety $Y = \text{Spec} B$ of dimension $n$. Assume that

1. general fibers of $f$ are isomorphic to $C^1$,
2. both $A$ and $B$ are factorial,
3. $A^* = B^*$,
4. $f$ is GICO over $Y$,
5. the image of $f$ is open in $Y$.

Then there exists a closed subset $Z$ of $Y$ such that

1. if $Z \neq \emptyset$ then $Z$ is pure of codimension 2 in $Y$,
2. $U \equiv Y - Z = f(X)$, and $f: X \to U$ is a $C^1$ bundle.

By “general fibers” is meant all fibers over some unspecified open subset in $Y$.

If $X$ in the above theorem is $C^n$ and $B = C[X]^{G_e}$ for some $G_e$ action on $X$, and $f$ is the morphism induced from the inclusion of $B$ in $A$, then conditions 1–3 of the hypothesis are always met (assuming that $B$ is affine and regular). Conditions 2 and 3 are clear, and condition 1 follows from the consideration of an open subset of the form $Y_h$, where $0 \neq h = \delta g$ for some $g \in C[X]$. Since $h$ is a $G_e$ invariant the action restricts to an action on $X_h$. There the action is equivariantly trivial, with slice $\mathfrak{a}_h$. On the other hand, condition 4 need not hold for a $G_e$ action with a finitely generated ring of invariants, as the following example shows.

**Example.** Let $\delta$ be the locally nilpotent derivation of $A = C[x_1, x_2, x_3]$ given by $x_3 \mapsto \delta x_3, x_2 \mapsto \delta x_2, x_1 \mapsto \delta x_1$. The kernel of $\delta$ is easily seen to be equal to $B = C[x_1, x_2^2 - 2x_1x_3]$. But $B/(x_1) \cong C[x_2^2] \hookrightarrow A/x_1A \cong C[x_2, x_3]$, so that the morphism $\text{Spec} A \to \text{Spec} B$ is not GICO.
We state as a lemma the following criterion for properness:

**Lemma 3.2 [2, Theorem 2.3].** A regular $G_a$ action on $\mathbb{C}^n$ is proper if and only if $\overline{\sigma}$ is surjective (i.e., if $t$ is in the image of $\overline{\sigma}$).

**Proposition 3.3.** If $G_a$ acts properly on $X = \mathbb{C}^n$ and $\mathbb{C}[X]^{G_a}$ is finitely generated as a $\mathbb{C}$ algebra, then the morphism $\mathbb{C}^n \to \text{Spec} \mathbb{C}[X]^{G_a}$ is GICO.

**Proof.** Let $\overline{\delta}$ denote the locally nilpotent derivation of $\mathbb{C}[X]$ generating the action, let $C_0$ be the ring of $G_a$ invariants in $\mathbb{C}[X]$, and let $I$ be the intersection of $C_0$ with the image of $\overline{\delta}$. It is clear that $I$ is an ideal in $C_0$.

If $I = C_0$ then $\overline{\delta}(s) = 1$ for some $s \in C_0$, $C_0 = C_0[s]$, a polynomial extension, and the action is equivariantly trivial. In particular the extension is GICO. If $I$ is a proper ideal, then it suffices to show that the height of $I$ is at least 2. Indeed, if $p \in C_0$ is prime and $h = \delta g \in I - (p)$, then $p$ does not divide $g$ and $C[X, \frac{1}{h}] = C_0[1/h][g]$, a polynomial extension. Taking residue classes modulo $p$ preserves the polynomiality of the extension.

Let $p$ be prime in $C_0$ and $S = C_0 - (p)$. Since $S^{-1}C_0$ is a discrete valuation ring and $S^{-1}C[X]$ is torsion free as a $S^{-1}C_0$ module, the extension $S^{-1}C_0 \to S^{-1}C[X]$ is flat. Note that $\overline{\delta}$ extends to a locally nilpotent derivation, also denoted $\overline{\delta}$, of $S^{-1}C[X]$ with kernel $S^{-1}C_0$ and that $t$ is in the image of the ring homomorphism $S^{-1}\overline{\sigma}$.

We now apply an argument based on the proof of [2, Theorem 2.8]. Denote $S^{-1}C[X]$ by $D_1$, and $S^{-1}C_0$ by $D_0$. Set $D^{-(r+1)} = D_{r}$ and $D_{r}^\sigma = \overline{\delta}D$ (observe that $\overline{\sigma}$ is a homomorphism of $C_0$ algebras). Extend $\overline{\delta}$ to a derivation $\overline{\delta}$ of $D[i]$ by $i \mapsto -1$. Then $\overline{\delta}$ is locally nilpotent with kernel $D^\sigma [6, Lemma 2.7]$.

Since a proper action is necessarily fixed point free, the image of $\overline{\delta}$ generates the unit ideal in $D$. If $\overline{\delta}^2 = 0$, then the image of $\overline{\delta}$ is $ID_0$ and hence $I$ is not contained in $p$. We may therefore assume that $\overline{\delta}^2 \neq 0$.

Consider the exact sequence $0 \to D_1 \to D_r \overset{\overline{\delta}_r}{\to} D_{r-2}$ of $D_0$ modules for any $r \geq 2$. Since $\overline{\sigma}$ is a $D_0$ isomorphism, $D^\sigma$ is a flat $D_0$ algebra. Tensoring the above sequence over $D_0$ with $D^\sigma$ we obtain the exact sequence $0 \to D^\sigma \otimes D_1 \to D^\sigma \otimes D_r \overset{\overline{\delta}_r}{\to} D^\sigma \otimes D_{r-2}$.

Since $t$ is in the image of $\overline{\sigma}$, $t$ lies in $D^\sigma \otimes D_1$, and so may be expressed as $\sum_{i=0}^{k} (\overline{\delta}(P_k)/s_k)(Q_k/s'_k)$, with $P_k, Q_k \in C[X]$, and $s_k, s'_k$ in $C_0 - (p)$. Applying $\overline{\delta}$ to this expression yields $-1 = \sum_{i=0}^{k} (\overline{\delta}(P_k)/s_k)(\overline{\delta}(Q_k)/s'_k)$, with $\overline{\delta}(Q_k) \in I$. Comparing coefficients of $t$ in the last equations shows that $ID = D$ and therefore $I$ is not contained in $(p)$.

**Theorem 3.4.** Suppose that $G_a$ acts properly on $X = \mathbb{C}^n$, $\mathbb{C}[X]^{G_a}$ is finitely generated as a $\mathbb{C}$ algebra, and $Y = \text{Spec} \mathbb{C}[X]^{G_a}$ is smooth. If the morphism $\pi: X \to Y$ induced by the ring inclusion has open image, then the
action is locally trivial and admits a geometric quotient isomorphic to an open subset of $Y$. If $\pi$ is surjective, then the action is equivariantly trivial.

**Proof.** The proposition and Theorem 3.1 demonstrate that $X$ is a $C^1$ bundle over an open subset of $Y$. But this forces all fibers of the morphism $X \to Y$ to be empty or one dimensional. Since $Y$ is smooth and $X$ is Cohen-Macaulay, this morphism is flat [16, p. 179] and the action is locally trivial. That the geometric quotient exists and is isomorphic to an open subset of $Y$ follows from [2, Theorem 2.5]. If $\pi$ is surjective then it is faithfully flat and the action is equivariantly trivial.

The crucial point is that if $G_a$ acts on $X = \mathbb{C}^n$ via the locally nilpotent derivation $\delta$ and $\ker(\delta) \cap \text{im}(\delta)$ lies in no height 1 prime ideal of $\mathbb{C}[X]$ then the morphism $X \to Y$ is GICO. Properness is merely a convenient way to verify this condition. We therefore record the

**Corollary 3.5.** Suppose that $G_a$ acts on $X = \mathbb{C}^n$, $\mathbb{C}[X]^{G_a}$ is finitely generated as a $\mathbb{C}$ algebra, and $Y = \text{Spec} \mathbb{C}[X]^{G_a}$ is smooth. If $\pi: X \to Y$ has open image and $\delta$, the derivation generating the action, satisfies $\ker(\delta) \cap \text{im}(\delta)$ lies in no height 1 prime ideal of $\mathbb{C}[X]$ then the action is locally trivial.

The following strengthening of [4, Theorem 3] is obtained.

**Corollary 3.6.** With $X$, $Y$, $\pi$ as in Theorem 3.4, assume in addition that $Y$ is contractible as a topological space in the Euclidean topology (e.g., $Y \cong \mathbb{C}^{n-1}$). Then the action is equivariantly trivial.

**Proof.** The result follows immediately from the theorem and the argument of [4, Theorem 3] which we summarize. The relevant cohomological facts can be found in [7, p. 289, 301; 15, Appendix A2].

The theorem yields local triviality, and equivariant triviality if $\pi: X \to Y$ is surjective. Assume then that $U = \pi(X)$ is a proper Zariski open subset of $Y$. Then $U$ is contractible, since the contractible space $X$ is the total space of a principal $G_a$ bundle over $U$. Set $V$ equal to the complement of $U$ in $Y$ and observe that the dimension of $V$ is $n - 3$ (part 2 of the conclusion of Theorem 3.1). Alexander duality shows that the reduced homology of $U$ in dimension 3 is isomorphic to $H^{2n-6}_c(V)$, the cohomology with compact supports of $V$ in dimension $2n - 6$. If $S$ is the union of the singular locus of $V$ with all components of $V$ of dimension < $n - 3$, then $H^{2n-6}_c(V) \cong H^{2n-6}_c(V, S)$. But $H^{2n-6}_c(V, S) \cong H^{2n-6}_c(V - S)$, and $V - S$ is the finite disjoint union of real manifolds of dimension $2n - 6$. As such, $H^{2n-6}_c(V - S) \neq 0$. Thus the assumption that $V \neq \emptyset$ leads to a contradiction of the contractibility of $U$, so that $\pi: X \to Y$ must be surjective and the result follows. ■
4. EXAMPLES

It is unknown whether proper $G_a$ actions on $C^4$ are locally trivial. The following example suggests that if they are, then it is due to some special attribute of $C^4$.

The proper but not locally trivial action of $G_a$ on $C^5$ described in [5] is generated by the locally nilpotent derivation $\delta$ of $C[x_1, x_2, y_1, y_2, z]$ given by

\[
x_2 \mapsto x_1 \mapsto 0, \quad y_2 \mapsto y_1 \mapsto 0, \quad z \mapsto (1 + x_1 y_2^2).
\]

To see that the action is proper, observe that

\[
t = \delta z - y_2^2(\delta x_2 - x_2) - y_2(\delta x_2 - x_2)(\delta y_2 - y_2) - \frac{(\delta x_2 - x_2)(\delta y_2 - y_2)^2}{3}.
\]

It is shown in [5] that the ring of invariants is generated by the five polynomials $c_1 = x_1$, $c_2 = y_1$

\[
c_3 = x_1 y_2 - x_2 y_1
\]

\[
c_4 = 3y_1 z - x_1 y_2^2 - 3y_2
\]

\[
c_5 = \frac{c_1^2 c_4 + c_2^3 + 3c_1 c_3}{c_2}.
\]

Let $\lambda$ be any complex number $\neq 0, \pm \sqrt{-3}$. Set $A = C[x_1, x_2, y_1, y_2, z]/(c_4 - \lambda)$, and $Z = \text{Spec } A$. A routine calculation shows that $Z$ is smooth and, since $c_4$ is an invariant, $Z$ is $G_a$ stable. The action is proper since it is the restriction of a proper action to a closed subset (the induced homomorphism $C[Z \times Z] \to C[Z, t]$ is surjective). We claim that $Z$ is rational and factorial but the action on $Z$ is not locally trivial.

**Rationality.** This is clear from the relation defining $Z$.

**Factoriality.** Set $B = C[x_1, y_1, y_2, z]/(c_4 - \lambda)$. $a_i$ the residue classes of $x_i$, $b_i$ the residue classes of $y_i$, and $w$ the residue class of $z$ in $A$. Since $A \cong B[a_2]$, it suffices to show that $B$ is a unique factorization domain. The criterion of [16, Theorem 20.2] is applied, showing that $b_1$ is prime in $B$ and that $B[1/b_1]$ is a ufd. To see that $b_1$ is prime observe that $B/(b_1) \cong C[x_1, y_1, y_2, z]/(y_1, c_4 - \lambda) \cong C[x_1, y_2, z]/(x_1 y_2^3 + 3y_2 - \lambda)$, an integral domain. Moreover, $B[1/b_1] \cong C[x_1, y_1, y_2, 1/y_1]$, which is a ufd.

**Local triviality.** Another application of [8] shows that the ring of $G_a$ invariants in $A$ is generated as a $C$ algebra by $\{d_1, d_2, d_3, d_5\}$, where $d_i$ is the image of $c_i$ in $A$. Note that because of the unipotency of $G_a$, this assertion is not obvious and also uses the rationality of $Z$. Embedding $Y = \text{Spec } B^{G_a}$ as a closed subset of $C^4$ via $(d_1, d_2, d_3, d_5)$ one checks that the
fiber over the unique singular point \((0, 0, 0, 0)\) of \(Y\) is two dimensional, and therefore the action cannot be locally trivial (Proposition 1.1).

Corollary 5.1 of \([14]\) shows that the Euler characteristic \(\chi(Y)\) of the affine variety \(Y\) with coordinate ring \(B\) is zero. Since \(Z \cong Y \times C\), \(\chi(Z) = 0\), and therefore \(Z\) is not isomorphic to \(C^4\). Indeed, viewing \(Y\) as the hypersurface \(y_1z = (y_2(x_1y_2^2 + 3) + \lambda)/3\) in \(C^4\), note that the plane curve \(W\) defined by \((y_2(x_1y_2^2 + 3) + \lambda)/3\) is isomorphic to \(C - 0\). Corollary 5.1 asserts the equality \(\chi(Y) = \chi(W)\) (= 0). Proposition 5.1 of \([14]\) shows however that \(\pi_1(Z) = 1\) and from Theorem 6.1 we see that that the Makar–Limanov invariant of \(Z\) is also trivial. 

Theorem 2.3.1 of \([21]\) gives a criterion related to Theorem 3.1 for an integral domain \(A\) to be a polynomial ring in one variable over a subring \(K\). The setup has \(S \subset k \subset K \subset A\), where \(k\) is a ring and \(S\) is a multiplicatively closed subset. The criterion relies on the condition that \(K\) is \(S\)-inert in \(A\), which means that

1. \(A \cap S^{-1}K = K\), and
2. for each height 1 prime ideal \(P\) of \(k\) containing some element of \(S\),
   (a) \(PA\) is prime;
   (b) with \(\bar{K}\) denoting the image of \(K\) in \(A/PA\), \(F\) the quotient field of \(\bar{K}\), and \(L\) the quotient field of \(A/PA\), we have \(F\) algebraically closed in \(L\);
   (c) \((A/PA) \cap F = \bar{K}\).

Note the close connection between condition 2b and GICO. With the above notations, the criterion of \([21,\ Theorem 2.3.1]\) for \(A\) to be isomorphic to a one variable polynomial ring over \(K\) is

1. \(S^{-1}A\) is isomorphic to a one variable polynomial ring over \(S^{-1}K\),
2. \(S\) is generated by prime elements of \(k\), and
3. \(K\) is \(S\)-inert in \(A\).

The above criterion is stronger than GICO as demonstrated by the following example of a locally trivial, but not equivariantly trivial, \(G_u\) action on \(C^5\). The example is due to Winkelmann \([24,\ Sect. 2]\).

Let \(\delta\) be the derivation of \(C[x, y, z, w, u]\) given by

\[\delta: \begin{align*}
  u &\mapsto 0 \\
  w &\mapsto 0 \\
  z &\mapsto u \\
  y &\mapsto w \\
  x &\mapsto 1 + uy - wz.
\end{align*}\]
Note that \( \delta^2 \) annihilates the linear span of \( \{x, y, z, w, u\} \) and that \( 1 = \delta(x) - y\delta(z) + z\delta(y) \) is in the ideal generated by the intersection of the kernel and the image of \( \delta \). Thus the action is locally trivial and the quotient quasiaffine. The algorithm in [8] shows that the ring \( C_0 \) of \( G_a \) invariants is generated by \( \{u, w, xu - z - zyw + z^2w, xw - y - uw + zyw, uy - zw\} \). However, the image of the morphism \( C^5 \to \operatorname{Spec} C_0 \) is a proper open subset, so the action is not equivariantly trivial. In particular, \( C[x, y, z, w, u] \) is not isomorphic to a one variable polynomial ring over \( C_0 \).

If we set \( k = K = C_0, A = C[x, y, z, w, u] \), and \( S = \{u^i \mid i \geq 0\} \), then condition 2c in the definition of \( S \)-inertness fails. To see this, note that \( u \) is prime in \( K \) and \( K/(u) = C[w, zw, z + z^2w, xw + y + zyw] \). Setting \( a = w, b = zw, c = z + z^2w, d = xw - y + zyw \), we see that \( K/(u) \cong C[a, b, c][d] \), i.e., isomorphic to a one variable polynomial ring over \( C[a, b, c] \). Moreover, \( C[a, b, c] \cong C[r, s, t]/(rt - b(1 + b)) \), which is not a ufd. Thus \( z \) does not lie in \( K/(u) \), for otherwise \( K/(u) \) would be isomorphic to a three variable polynomial ring over \( C \) and have unique factorization. One could also apply the subalgebra membership algorithm in [1] to check this. Clearly \( z \) is in \( F \) and this violates condition 2c for \( S \)-inertness.

**REFERENCES**