International Journal of Solids and Structures 48 (2011) 680–686

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

Compressibility and shear compliance of spheroidal pores: Exact derivation via the Eshelby tensor, and asymptotic expressions in limiting cases

E.C. David, R.W. Zimmerman *

Department of Earth Science and Engineering, Imperial College, London SW7 2AZ, United Kingdom

ARTICLE INFO

Article history:
Received 10 June 2010
Received in revised form 5 September 2010
Available online 5 November 2010

Keywords:
Porous media
Elasticity
Isotropic
Pore compliance
Pore aspect ratio
Eshelby's tensor
Crack

ABSTRACT

We explicitly calculate the elastic compliance of a spheroidal pore in an isotropic solid, starting from Eshelby's tensor. The exact expressions found for the pore compressibility, *P*, and the shear compliance, *Q*, are valid for any value of the aspect ratio α, from zero (cracks) to infinity (needles). This derivation clarifies previous work on this problem, in which different methods were used in different ranges of α, or typographical errors were present. The exact expressions obtained for *P* and *Q* are quite complex and unwieldy. Simple expressions for both *P* and *Q* have previously been available for the limiting cases of infinitely thin-cracks ($\alpha = 0$), infinitely long-needles ($\alpha = \infty$), and spherical pores ($\alpha = 1$). We have now calculated additional terms in the asymptotic expansions, yielding relatively simple approximations for *P* and *Q* that are valid for crack-like pores having aspect ratios as high as 0.3, needle-like pores having aspect ratios as low as 3, and nearly spherical pores. Their relatively simple forms will be useful for incorporation into various schemes to estimate the effective elastic moduli.

1. Introduction

A spheroidal inclusion in an infinite, isotropic elastic solid provides a useful model to calculate elastic properties of rocks, ceramics, bones, or other porous materials. Indeed, the ellipsoid is the only three-dimensional shape amenable to analytical treatment, and exact expressions for the elastic field of an ellipsoidal inclusion were obtained by Eshelby (1957). The exact solutions for ellipsoids are, however, very unwieldy and expressed in terms of elliptic integrals, therefore they require to be computed numerically. Nevertheless, some recent papers have indeed used the ellipsoidal pore as the basis of their calculations (Markov et al., 2005; Giraud et al., 2008). However, most modelling efforts have utilised the *spheroid*, which is simply a degenerate ellipsoid with two axis of equal length, as it offers simpler expressions. In its various forms, a spheroid can represent a wide variety of pore shapes, such as thin cracks, cylinders, or spheres (Fig. 1).

Exact expressions for the elastic compliance of spheroidal pores, i.e., the pore compressibility *P* and the shear compliance *Q*, have been presented by Wu (1966) in terms of two invariants (T_{iijj}, T_{ijij}) of Wu's tensor $\mathbb{T}$, where $\mathbb{T}$ is a fourth-order "strain-concentration" tensor relating the strain in the inclusion to the homogeneous strain applied at infinity. Wu's derivation (based on Eshelby's results) is exact, but the expressions given in his paper contain several typographical errors. The correct results were presented later by Kuster and Toksoz (1974), but their derivation used a wave-scattering approach, and it is not entirely obvious that their results should be the same as those derived from Wu's tensor. Later, such expressions for (T_{iijj}, T_{ijij}) were also presented in Cheng and Toksoz (1979) or Berryman (1980), but they are based on the previous results of Kuster and Toksoz. Close-form expressions for *P* and *Q*, based on Eshelby' results, are given by Dunn and Ledbetter (1995), in the limit of a dry spheroid. Unfortunately, their expressions again contain some typographical errors. Kachanov et al. (2003) present expressions for the pore compliances using the $\mathbb{H}$-tensor formalism, which is completely equivalent to the $\mathbb{T}$-tensor one. For a spheroid in an isotropic medium, $\mathbb{H}$ is a *transversely isotropic* tensor, whose components are explicitly given as functions of Eshelby's components, and from which the pore compressibility and shear compliance can be extracted by some algebraic manipulations. However, their expressions contain also some typographical errors, which seem to start with an extraneous factor of 2 that appears outside of the parenthesised term in the first equation on p. 263. Note that the same errors were included previously in Shafiro and Kachanov (1997). Hence, it seems difficult to find in the literature exact expressions for the pore compliances of spheroids that are derived from Eshelby's results in a clear manner. Our first preliminary goal is therefore to explicitly derive the correct expressions for the pore compliances, starting from Wu's tensor (Section 2).

The expressions we find for the elastic compliances are exact and applicable for the entire range of spheroid aspect ratios, from zero (flat cracks) to infinity (long needles). But these expressions are too cumbersome to be easily interpreted, and are too complex

* Corresponding author.
E-mail addresses: emmanuel.david08@imperial.ac.uk (E.C. David), r.w.zimmerman@imperial.ac.uk (R.W. Zimmerman).

0020-7683/$ - see front matter © 2010 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ijsolstr.2010.11.001

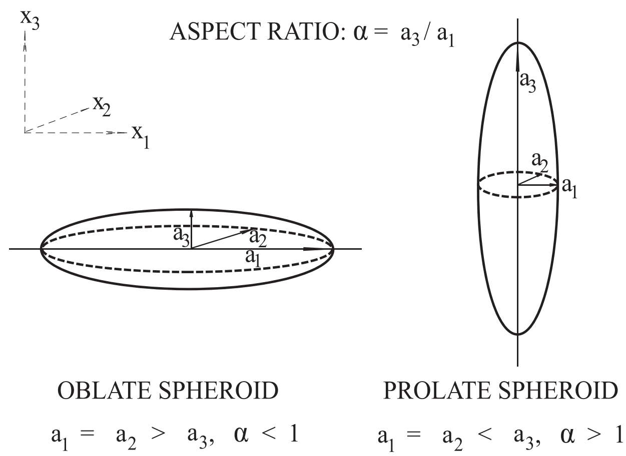

Fig. 1. Geometry of an oblate (left) and a prolate (right) spheroid. The dotted line represents the plane normal to the axis of revolution; (a_1, a_2, a_3) are the lengths of the axes of the spheroids in the directions (x_1, x_2, x_3), respectively. When "flat" (i.e., for small values of α), oblate spheroids are "crack-like"; when "long" (i.e., for large values of α), prolate spheroids are "needle-like".

to be used, for example, as input in effective medium theories, such as, for example, the Differential Effective Medium scheme. This is why most works have used degenerated cases such as penny-shaped cracks, spherical pores, or needle-like pores. In fact, such results are more easily derived by starting directly with spherical or cylindrical geometries, rather than taking the limits of the expressions for spheroids. Nevertheless, cracks are never infinitely thin, cylindrical pores are never infinitely long, and spherical pores are never perfectly spherical. In particular, rocks contain such a variety of pore shapes that it seems unrealistic to deal only with such idealized inclusion geometries. Hence, it would be useful to have simple analytical expressions for the compliances that are valid for more realistic pore geometries *between* the limiting cases, such as cracks and needles of finite aspect ratio, or for nearly spherical pores. Aside from an asymptotic expression derived by Zimmerman (1991) for the pore compressibility of a needle-like pore of finite aspect ratio, few such results seem to be available. Our second goal is to present additional asymptotic expressions for both pore compressibility and shear compliance of "crack-like" and "needle-like" pores of finite aspect ratio, and nearly spherical pores (Section 3).

2. Explicit derivation of the elastic compliance of a spheroidal pore via Eshelby's method

The purpose of this section is to derive exact expressions for the compressibility and shear compliance of a spheroidal pore, starting from Eshelby's tensor. These coefficients depend on the two invariants (T_{iijj}, T_{ijij}) of the tensor $\mathbb{T}$ introduced by Wu (1966). $\mathbb{T}$ is defined as follows: in the presence of an homogeneous strain $\boldsymbol{\epsilon}$ applied at infinity, the subsequent strain of an isolated pore, $\boldsymbol{\epsilon} + \Delta\boldsymbol{\epsilon}$, is given by

$$\boldsymbol{\epsilon} + \Delta\boldsymbol{\epsilon} = \mathbb{T} : \boldsymbol{\epsilon}, \tag{1}$$

where : denotes the double inner product of two tensors, e.g., in components, since $\mathbb{T}$ and $\boldsymbol{\epsilon}$ are respectively two tensors of rank four and two, $(\boldsymbol{\epsilon} + \Delta\boldsymbol{\epsilon})_{ij} = \mathbb{T}_{ijkl}\boldsymbol{\epsilon}_{kl}$, where the summation convention for repeated indices is used. The left-hand side of Eq. (1), $\boldsymbol{\epsilon} + \Delta\ \boldsymbol{\epsilon}$, is the total strain inside the inclusion. Because $\Delta\boldsymbol{\epsilon}$ is then the *additional* strain due to the presence of the inclusion, $\mathbb{T}$ has often been referred as a "strain-concentration" tensor (see for example Benveniste (1987)).

Knowledge of $\mathbb{T}$ is sufficient for calculating the pore compliances. This can be easily seen by remembering that in a Representative Elementary Volume (REV), comprising a solid of elastic compliance tensor $\mathbb{S}^0$, and an inclusion, in proportions ϕ and $(1-\phi)$, respectively, the effective compliance tensor $\mathbb{S}$ is given by Kachanov et al. (2003)

$$\mathbb{S} = \mathbb{S}^0 + \phi\mathbb{H}, \tag{2}$$

where $\mathbb{H}$ is the additional compliance due to the presence of the inclusion (n.b.: in some previous studies, $\mathbb{H}$ is defined so as to already contain the porosity term ϕ). Note that the use of the volume fraction in Eq. (2) should not give the impression that the volume fraction should always be used as a microstructural parameter: for strongly oblate spheroids or "cracks", alternative normalizations of the pore compliances to a^3 ($a = a_1 = a_2$, where a is then the largest semiaxis of the spheroid, see Fig. 1), in other words, to the *crack density parameter*, are more appropriate in effective media schemes (Sevostianov and Kachanov, 2008). Also, in the case of inclusions having arbitrary elastic properties (such as fluid-filled pores), when the contributions of individual pores have to be summed up in effective medium theories, $\mathbb{H}$-tensors should be preferred, since they represent by definition the additional compliance of an individual pore (Kachanov and Sevostianov, 2005). For dry pores, however, $\mathbb{H}$ and $\mathbb{T}$ are simply related as follows:

$$\mathbb{H} = \mathbb{T} : \mathbb{S}^0, \tag{3}$$

e.g., in components, $\mathbb{T}$ and $\mathbb{S}^0$ being both four rank tensors, $\mathbb{H}_{ijkl} = \mathbb{T}_{ijmn}\mathbb{S}^0_{mnkl}$. The result given in Eq. (3) for dry pores is easily shown starting from well-known relations between $\mathbb{H}$ and $\mathbb{T}$, such as given by Kachanov et al. (2003) in the more general case of an inclusion composed of an arbitrary elastic material, and then taking the limit as the stiffness of this material vanishes. Combining the two previous Eqs. (2) and (3), we obtain

$$\mathbb{S} : (\mathbb{S}^0)^{-1} = \mathbb{I} + \phi\mathbb{T}, \tag{4}$$

where $\mathbb{I}$ denotes the fourth-rank identity tensor with components $\mathbb{I}_{ijkl} = (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj})/2$, and where δ is the Kronecker symbol, i.e., $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$. The left-hand side of Eq. (4) is simply the normalized effective compliance, and so it is clear from the equation above that $\mathbb{T}$ is the *normalized pore compliance tensor*.

Another reason for using Wu's tensor is that it is very simply related to Eshelby's tensor, $\mathbb{E}$:

$$\mathbb{T} = (\mathbb{I} - \mathbb{E})^{-1}, \tag{5}$$

which has also been obtained starting from general results found for an inclusion having arbitrary elastic properties, and passing to the limit of a dry pore. The components of $\mathbb{E}$ are explicitely known for spheroidal pores (Kachanov et al., 2003). For a spheroid of aspect ratio α, embedded in an isotropic solid having Poisson's ratio ν, and according to the conventions of orientations adopted in Fig. 1, where the two directions 1 and 2 are equivalent, the components of $\mathbb{E}$ are:

$$\mathbb{E}_{1111} = -\frac{3\alpha^2}{8(1-\nu)(1-\alpha^2)} + \frac{1}{4(1-\nu)}\left[1 - 2\nu + \frac{9}{4(1-\alpha^2)}\right]g, \tag{6}$$

$$\mathbb{E}_{3333} = \frac{1}{1-\nu}\left[2 - \nu - \frac{1}{1-\alpha^2}\right] + \frac{1}{2(1-\nu)}\left[-2(2-\nu) + \frac{3}{1-\alpha^2}\right]g, \tag{7}$$

$$\mathbb{E}_{1122} = \frac{1}{8(1-\nu)}\left[1 - \frac{1}{1-\alpha^2}\right] + \frac{1}{16(1-\nu)}\left[-4(1-2\nu) + \frac{3}{1-\alpha^2}\right]g, \tag{8}$$

$$\mathbb{E}_{1133} = \frac{\alpha^2}{2(1-\nu)(1-\alpha^2)} - \frac{1}{4(1-\nu)}\left[1 - 2\nu + \frac{3\alpha^2}{1-\alpha^2}\right]g, \tag{9}$$

$$\mathbb{E}_{3311} = \frac{1}{2(1-\nu)}\left[-(1-2\nu)+\frac{1}{1-\alpha^2}\right] + \frac{1}{4(1-\nu)}\left[2(1-2\nu)-\frac{3}{1-\alpha^2}\right]g, \tag{10}$$

$$\mathbb{E}_{1212} = -\frac{\alpha^2}{8(1-\nu)(1-\alpha^2)} + \frac{1}{16(1-\nu)}\left[4(1-2\nu)+\frac{3}{1-\alpha^2}\right]g, \tag{11}$$

$$\mathbb{E}_{1313} = \frac{1}{4(1-\nu)}\left[1-2\nu+\frac{1+\alpha^2}{1-\alpha^2}\right] - \frac{1}{8(1-\nu)}\left[1-2\nu+3\frac{1+\alpha^2}{1-\alpha^2}\right]g, \tag{12}$$

with the following symmetries

$$\mathbb{E}_{1111} = \mathbb{E}_{2222}, \tag{13}$$
$$\mathbb{E}_{1122} = \mathbb{E}_{2211}, \tag{14}$$
$$\mathbb{E}_{1133} = \mathbb{E}_{2233}, \tag{15}$$
$$\mathbb{E}_{3311} = \mathbb{E}_{3322}, \tag{16}$$
$$\mathbb{E}_{1212} = \mathbb{E}_{2121}, \tag{17}$$
$$\mathbb{E}_{1313} = \mathbb{E}_{2323} \tag{18}$$

and where g is a function of α, defined as follows for the two cases of oblate and prolate spheroids:

$$g = \frac{\alpha}{(1-\alpha^2)^{3/2}}\left(\arccos\alpha - \alpha\sqrt{1-\alpha^2}\right) \text{ for an oblate spheroid } (\alpha<1), \tag{19}$$
$$g = \frac{\alpha}{(1-\alpha^2)^{3/2}}\left(\alpha\sqrt{1-\alpha^2} - \text{arccosh}\alpha\right) \text{ for a prolate spheroid } (\alpha>1). \tag{20}$$

The other non-null components (such as E_{2121}, E_{1221}, ...) are obtained using the fundamental symmetry relations for Eshelby's tensor, $E_{ijkl} = E_{jikl} = E_{ijlk}$.

$\mathbb{T}$-tensor components are obtained from Eshelby's tensor components by inverting Eq. (5). Some of the $\mathbb{T}$-components are related to compression, others to shear. For example, T_{1133} relates the total compressive strain in the inclusion ($\epsilon_{11} + \Delta\epsilon_{11}$) to an applied compressive strain ϵ_{33}. The "trace" components related to compression are

$$[T] = \begin{bmatrix} T_{1111} & T_{1122} & T_{1133} \\ T_{2211} & T_{2222} & T_{2233} \\ T_{3311} & T_{3322} & T_{3333} \end{bmatrix}, \tag{21}$$

which, after inverting Eq. (5) by Cramer's rule, are given by

$$[T] = \frac{1}{\Delta}\begin{bmatrix} (1-E_{1111})(1-E_{3333})-E_{1133}E_{3311} & E_{1133}E_{3311}+E_{1122}(1-E_{3333}) & E_{1133}[(1-E_{1111})+E_{1122}] \\ E_{1133}E_{3311}+E_{1122}(1-E_{3333}) & (1-E_{1111})(1-E_{3333})-E_{1133}E_{3311} & E_{1133}[(1-E_{1111})+E_{1122}] \\ E_{3311}[(1-E_{1111})+E_{1122}] & E_{3311}[(1-E_{1111})+E_{1122}] & (1-E_{1111})^2-E_{1122}^2 \end{bmatrix}, \tag{22}$$

where

$$\Delta = \det\begin{bmatrix} 1-E_{1111} & -E_{1122} & -E_{1133} \\ -E_{1122} & 1-E_{1111} & -E_{1133} \\ -E_{3311} & -E_{3311} & 1-E_{3333} \end{bmatrix} \tag{23}$$

and the "deviatoric" components of $\mathbb{T}$ related to shear are

$$T_{1212} = 1/2(1-2E_{1212}), \quad T_{1313} = 1/2(1-2E_{1313}). \tag{24}$$

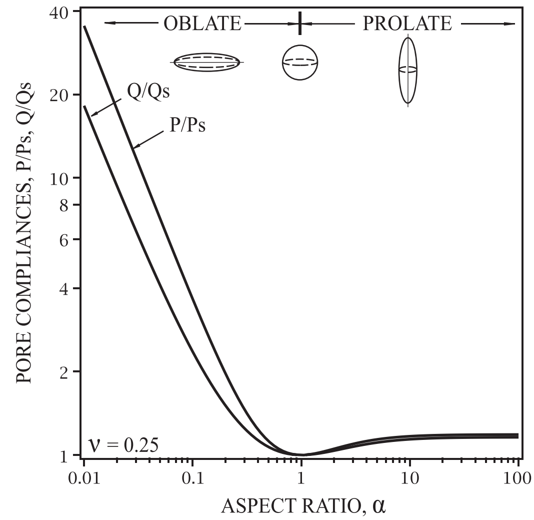

Fig. 2. Pore compressibility and shear compliance of a spheroidal pore, as a function of α, the spheroid's aspect ratio, for a solid's Poisson's ratio $\nu = 0.25$. Exact expressions for the pore compressibility, P, and the shear compliance, Q, given by Eqs. (29) and (30), have been normalized respectively to P_s and Q_s, compliances of a spherical pore given by Eqs. (31) and (32).

$\mathbb{T}$ represents the normalized pore elastic properties, but still depends on the relative orientations between the applied stress and the spheroid axes. If we assume that the solid contains numerous pores having random orientations, we can extract the pore compressibility and shear compliance by an appropriate averaging of $\mathbb{T}$-components. This averaging process yields two invariants T_{iijj} and T_{ijij}, where the summation convention for repeated indices is used. The pore compressibility P and the pore shear compliance Q are then given by

$$P = \frac{T_{iijj}}{3}, \tag{25}$$

$$Q = \frac{1}{5}\left(T_{ijij} - \frac{T_{iijj}}{3}\right), \tag{26}$$

where

$$T_{iijj} = (2T_{1111}+T_{3333}) + 2(T_{1122}+T_{1133}+T_{3311}), \tag{27}$$
$$T_{ijij} = (2T_{1111}+T_{3333}) + 2(T_{1212}+2T_{1313}). \tag{28}$$

As a result, P and Q are cumbersome combinations of Eshelby's tensor components that can be rearranged in the following forms:

$$P = \frac{(1-\nu)}{6(1-2\nu)} \times \frac{4(1+\nu)+2\alpha^2(7-2\nu)-[3(1+4\nu)+12\alpha^2(2-\nu)]g}{2\alpha^2+(1-4\alpha^2)g+(\alpha^2-1)(1+\nu)g^2} \tag{29}$$

$$Q = \frac{4(\alpha^2-1)(1-\nu)}{15\{8(\nu-1)+2\alpha^2(3-4\nu)+[(7-8\nu)-4\alpha^2(1-2\nu)]g\}} \times \left\{\frac{8(1-\nu)+2\alpha^2(3+4\nu)+[(8\nu-1)-4\alpha^2(5+2\nu)]g+6(\alpha^2-1)(1+\nu)g^2}{2\alpha^2+(1-4\alpha^2)g+(\alpha^2-1)(1+\nu)g^2} - 3\left[\frac{8(\nu-1)+2\alpha^2(5-4\nu)+[3(1-2\nu)+6\alpha^2(\nu-1)]g}{-2\alpha^2+[(2-\nu)+\alpha^2(1+\nu)]g}\right]\right\} \tag{30}$$

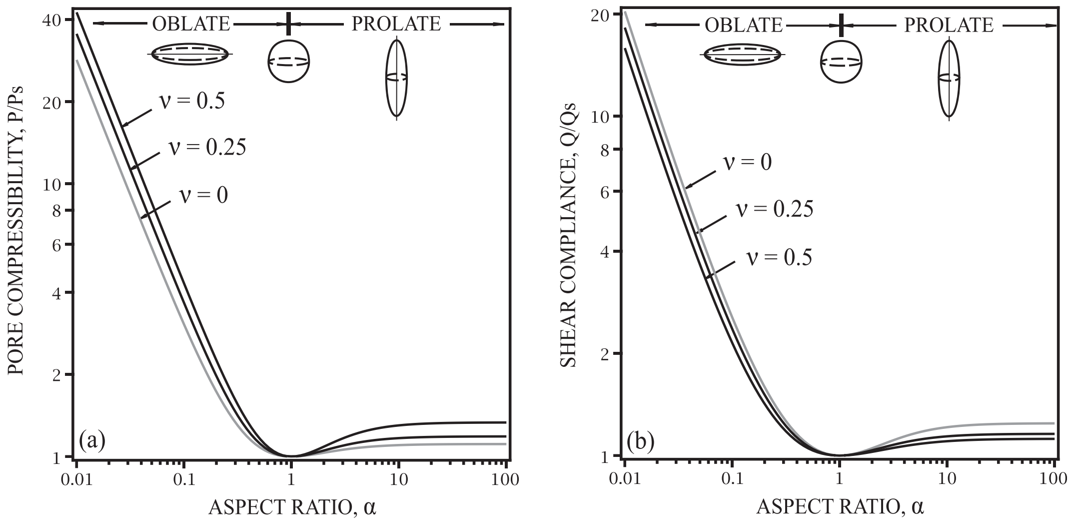

Fig. 3. Pore compressibility (a) and shear compliance (b) of a spheroidal pore, as a function of α, the aspect ratio, for three values of ν, the solid's Poisson's ratio. Exact expressions for the pore compressibility, P, and the shear compliance, Q, given by Eqs. (29) and (30), have been normalized respectively to P_s and Q_s, compliances of a spherical pore, given by Eqs. (31) and (32).

The notations (P,Q) for the pore compressibility and shear compliance correspond to the ones used by Mavko et al. (1998). The expressions given above for P and Q are exact, and are valid for the entire range of aspect ratios, from zero to infinity. The only difference between the expressions for oblate spheroids $(0 < \alpha < 1)$ and prolate spheroids $(1 < \alpha < \infty)$ is contained in the function g, which is defined for oblate and prolate spheroids in Eqs. (20) and (19), respectively. We have verified that the expressions of T_{iijj} and T_{ijij} exactly match the ones found by Kuster and Toksoz (1974), although they are expressed in a different form; this was not *a priori* obvious. P and Q are functions only of the spheroid's aspect ratio, α, and the solid's Poisson ratio, ν, although the effect of these parameters is difficult to infer from the complex expressions presented above. Fig. 2 shows P and Q as functions of the aspect ratio for a Poisson ratio equal to 0.25, and Fig. 3 shows how P (Fig. 3(a)) and Q (Fig. 3(b)) vary with aspect ratio for different values of the Poisson ratio. P and Q have been normalized to P_s and Q_s, which are respectively the compressibility and the shear compliance of a spherical pore, obtained by taking the limit $\alpha = 1$ in Eqs. (29) and (30):

$$P_s = \frac{3(1-\nu)}{2(1-2\nu)}, \tag{31}$$

$$Q_s = \frac{15(1-\nu)}{7-5\nu}. \tag{32}$$

P and Q vary in very similar ways with both the aspect ratio (Fig. 2), and Poisson's ratio (Fig. 3). Both take on a minimum value for spheres $(\alpha = 1)$, where they coincide with the lower value of the Hashin–Shtrikman bounds (see Hashin and Shtrikman (1961)). For thin-cracks $(\alpha \to 0)$, P and Q become infinite. More precisely, a first-order asymptotic development reveals that

$$P \sim \frac{4(1-\nu^2)}{3\pi\alpha(1-2\nu)}, \tag{33}$$

$$Q \sim \frac{8(1-\nu)(5-\nu)}{15\pi\alpha(2-\nu)}. \tag{34}$$

For needles $(\alpha \to \infty)$, finite limits for P and Q are found:

$$P \to \frac{5-4\nu}{3(1-2\nu)}, \tag{35}$$

$$Q \to \frac{8(5-3\nu)}{15}. \tag{36}$$

All such limits recover the well-known solutions for spherical pores (MacKenzie, 1950), thin cracks (Walsh, 1965; Budiansky and O'Connell, 1976), and needles (Berryman, 1980), which are often derived starting directly with the limiting geometries rather than by evaluating limits of the expressions (29) and (30). This agreement validates our calculations. However, the exact results for arbitrary aspect ratios are too cumbersome to be used in further analytical treatment, such as input in the effective medium schemes. Hence, it would be useful to have asymptotic expressions that are valid for cracks or needles having *finite* aspect ratios, or for *nearly* spherical pores. Asymptotic expressions for these limiting cases are derived in the next section.

3. Asymptotic expressions in the limiting cases

3.1. Crack-like pores

The bulk and shear compliances of an infinitely thin crack are each of the form $A(\nu)\alpha^{-1}$ (Eqs. (34) and (33)). But the comparison with the exact solutions clearly indicates that taking the thin-crack approximation is only valid when the value of the pore aspect ratio is less than 0.1 for the pore compressibility, and less than 0.01 for the shear compliance (Fig. 4). Hence, the thin crack approximation is not valid for calculating elastic properties of rocks whose cracks have aspect ratios greater than 0.01. We have derived asymptotic expressions for the next two terms in the expansions: $B(\nu)\alpha^0$ and $C(\nu)\alpha^1$. The resulting three-term approximations are accurate for aspect ratio as high as 0.3, with less than 2% error (Fig. 5):

$$P \sim \frac{P_{-1}}{\alpha} + P_0 + P_1\alpha, \tag{37}$$

$$Q \sim \frac{Q_{-1}}{\alpha} + Q_0 + Q_1\alpha, \tag{38}$$

where (P_{-1}, Q_{-1}) follow from Eqs. (34) and (33):

$$P_{-1} = \frac{4(1-\nu^2)}{3\pi(1-2\nu)}, \tag{39}$$

$$Q_{-1} = \frac{8(1-\nu)(5-\nu)}{15\pi(2-\nu)} \tag{40}$$

and where (P_0, Q_0, P_1, Q_1) are found to be

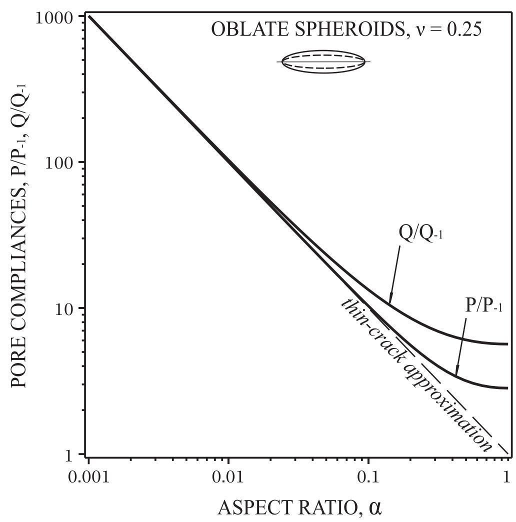

Fig. 4. Pore compressibility and shear compliance of an oblate spheroid as a function of aspect ratio, for a Poisson ratio $\nu = 0.25$. Exact expressions for the pore compressibility, P, and the shear compliance, Q, given by Eqs. (29) and (30), have been normalized respectively to P_{-1} and Q_{-1}, the thin-crack limit coefficients given by Eqs. (39) and (40).

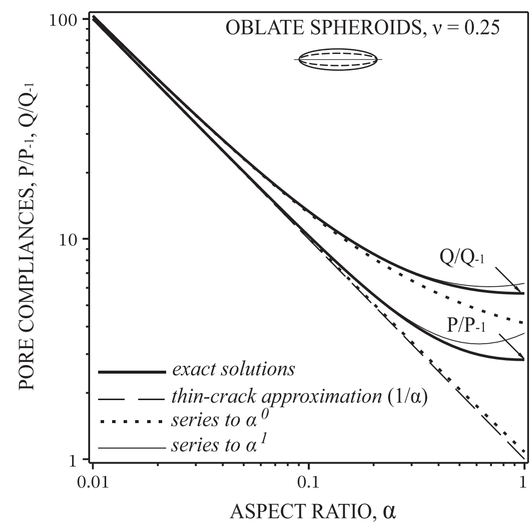

Fig. 5. Pore compressibility and shear compliance of an oblate spheroid, normalized to the thin-crack limit coefficients (P_{-1} and Q_{-1}, respectively), according to the exact expressions (29) and (30), and taking an increasing number of terms in the expansion series (37) and (38).

$$P_0 = \frac{1}{6}(1-\nu)(1-2\nu), \tag{41}$$

$$P_1 = \frac{(1+\nu)(1-\nu)}{12(1-2\nu)}\left[\pi(1-2\nu)^2 + \frac{8(7-8\nu)}{\pi}\right], \tag{42}$$

$$Q_0 = \frac{2}{15}\left[(5-2\nu^2) + \frac{48(1-\nu)(3-\nu)}{\pi^2(2-\nu)^2}\right], \tag{43}$$

$$Q_1 = \frac{\pi}{120}\left[\frac{37-8\nu(3+4\nu-2\nu^3)}{1-\nu}\right] + \frac{4(1-\nu)}{15\pi(2-\nu)^2}\left[-8(7+\nu^3)+3\nu(9\nu-1)+\frac{96(3-\nu)^2}{\pi^2(2-\nu)}\right]. \tag{44}$$

Kachanov et al. (2003) present an asymptotic expression for the $\mathbb{H}$-tensor for a crack-like oblate spheroid, but taking only terms up to order α^0 in the series expansion. After we extract P_0 from their expression for $\mathbb{H}$, we find that their P_0 coefficient is zero, which is incorrect. However, the consequences of this error are small, as P_0 is in fact very small. (It is interesting to note that Zimmerman (1991) also incorrectly calculated P_0 to be zero). Hence, adding only the constant term in the expansion series (38) does not improve the accuracy very much, as can be seen in Fig. 5. For the shear compliance, on the other hand, taking the second term Q_0 improves the limit of accuracy of the asymptotic expansion from about 0.01 to 0.1. Including the next term, $Q_1\alpha$, provides an expression that is accurate for α as large as about 0.3.

Zimmerman (1985) derived an exact expression for the "effective pore compressibility" G_0C_{pp}, where G_0 is the solid's shear modulus, by solving the elastostatic problem and integrating the normal displacement (Edwards, 1951) over the surface of the spheroidal cavity. C_{pc} quantifies how the pore volume changes with confining pressure at constant pore pressure, whereas C_{pp} quantifies how pore volume changes with pore pressure, with confining pressure being held constant. These two pore compressibilities are simply related by $C_{pc} = C_0 + C_{pp}$, where C_0 is the solid's compressibility. But P is the normalized pore compressibility, so $P = K_0C_{pc}$, where $K_0 = 1/C_0$ denotes the solid's bulk modulus.

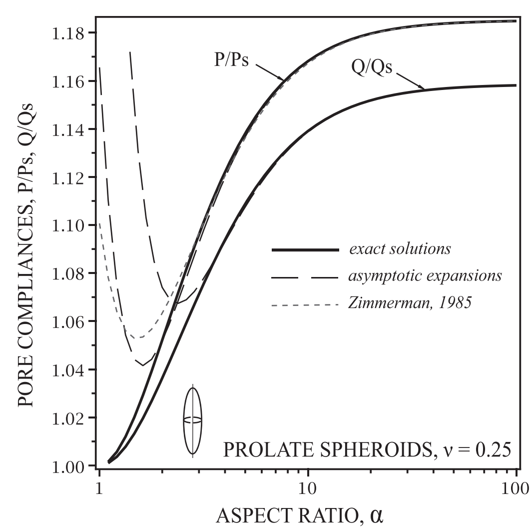

Fig. 6. Pore compressibility and shear compliance of a prolate spheroid, normalized to the sphere limit coefficients (P_s and Q_s, respectively), according to the exact Eqs. (29) and (30), and to the asymptotic expansions (49) and (48). The expression found by Zimmerman (1985) for G_0C_{pp} (Eq. (51)), converted to K_0C_{pc}, is also shown.

Remembering that $K_0/G_0 = 2(1+\nu)/3(1-2\nu)$, expressions for K_0C_{pc} can be converted to G_0C_{pp}:

$$G_0C_{pp} = \frac{3(K_0C_{pc}-1)(1-2\nu)}{2(1+\nu)}, \tag{45}$$

which yields, using the exact expression for P given by Eq. (29),

$$G_0C_{pp} = \frac{g\{6g(1-\alpha^2)(1-2\nu)+3[-3+4\nu(1-\alpha^2)]\}+4(1-\nu)(1-\alpha^2)+6\alpha^2}{4\{\alpha^2+g[-4\alpha^2+1+g(1+\nu)(\alpha^2-1)]\}}, \tag{46}$$

which exactly corresponds to the expression found by Zimmerman (1985). Using G_0C_{pp} instead of K_0C_{pc} (e.g., P) can be convenient, since it avoids Poisson's ratio's dependance in some particular cases, e.g., such as the sphere limit, where $G_0C_{pp} = 3/4$. Similarly, a series expansion for G_0C_{pp}, valid for $0 < \alpha < 0.3$ is obtained:

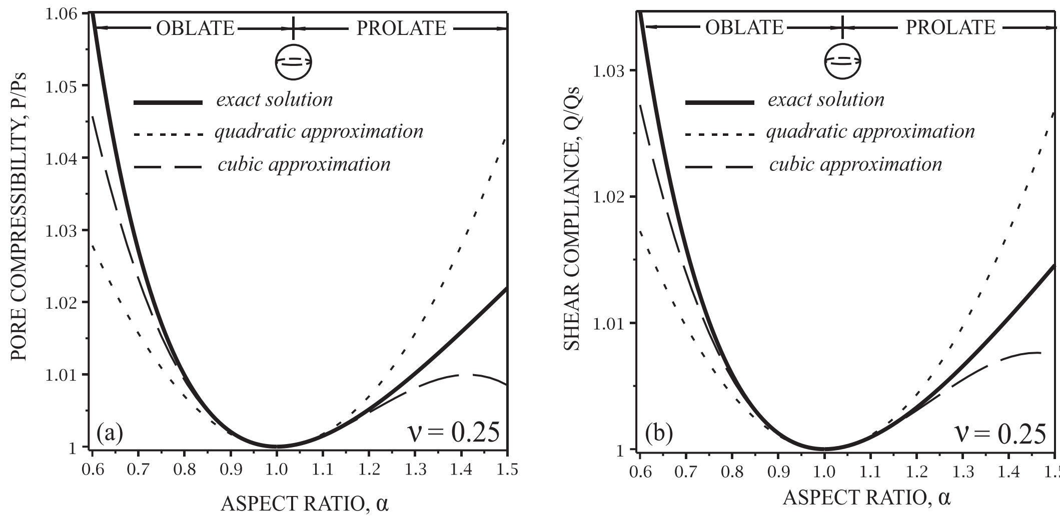

Fig. 7. Pore compressibility (a) and shear compliance (b) of a nearly spherical pore, normalized to the values for a sphere (P_s and Q_s, respectively), according to the exact expressions given by Eqs. (29) and (30), and taking an increasing number of terms in the expansion series (52) and (53).

$$G_0C_{pp} \sim \frac{2(1-v)}{\pi\alpha} + \frac{1}{4}(1-2v)(2v-5) + \frac{\alpha(1-v)}{8\pi}\left[\pi^2(1-2v)^2 + 8(7-8v)\right]. \quad (47)$$

3.2. Needle-like pores

At the other end of the aspect ratio spectrum, the limits of (P,Q) when $\alpha \to \infty$ are given by Eqs. (36) and (35). For needle-like pores of finite aspect ratio, simple asymptotic expressions involving $\ln\alpha$ are derived, that are accurate for aspect ratios as low as 3 for the shear compliance, and as low as 2 for the pore compressibility, with less than 0.5% error (Fig. 6):

$$P \sim \frac{5-4v}{3(1-2v)} + \frac{(1+v)\{4(1-v)[1-\ln(2\alpha)]-1\}}{6(1-2v)(1-v)\alpha^2}, \quad (48)$$

$$Q \sim \frac{8(5-3v)}{15} + \frac{4}{15} \times \frac{[13-43v+12v^2(5-2v)][1-\ln(2\alpha)]+3v(2-v)-1}{(1-v)\alpha^2}. \quad (49)$$

The accuracy of the asymptotic expressions is almost independent of v, the solid's Poisson ratio. Similarly, for G_0C_{pp}, we obtain

$$G_0C_{pp} \sim 1 + \frac{1}{\alpha^2}\left[\frac{3-4v}{4(1-v)} - \ln(2\alpha)\right]. \quad (50)$$

This result is close to the expression found by Zimmerman (1985),

$$G_0C_{pp} \sim \frac{2\alpha^2+1}{2[\alpha^2+\ln(2\alpha)]}, \quad (51)$$

but provides a somewhat better approximation, as shown by Fig. 6.

3.3. Nearly spherical pores

The case of a sphere is at the boundary between oblate and prolate shapes, and represents the minimum possible value of both compressibility and shear compliances (Fig. 2). Kachanov et al. (2003) have examined the case of a *slightly deformed sphere*, but their expressions for both the $\mathbb{H}$-tensor and Eshelby's components are only linear in $(1-\alpha)$. As the sphere yields the minimum values of P and Q, it is clear that the linear terms of the $\mathbb{H}$-tensor and Eshelby's components will not be sufficient to yield expressions for P and Q that are useful in the neighbourhood of a sphere. Moreover, as the compliances do not behave in a symmetric manner around $\alpha = 1$, an approximation only quadratic in $(1-\alpha)$ has a very limited range of accuracy (Fig. 7(a),(b)). Nevetheless, the series expansions of P and Q in the neighbourhood of $\alpha = 1$ converge quite rapidly so that, using only the additional term that is cubic in $(1-\alpha)$, an approximation valid for $0.7 < \alpha < 1.3$ with less than 0.5% error is obtained as follows:

$$P \sim \frac{3(1-v)}{2(1-2v)}\left\{1 + \frac{4(1+v)}{5(7-5v)}(1-\alpha)^2\left[1 + \frac{83-73v}{7(7-5v)}(1-\alpha)\right]\right\}, \quad (52)$$

$$Q \sim \frac{15(1-v)}{7-5v}\left[1 + \frac{4(1-\alpha)^2}{175(7-5v)^2}\left\{299 - 7v(98-65v) + \frac{138079 - 7v[54357 - 7v(7293-2225v)]}{49(7-5v)}(1-\alpha)\right\}\right] \quad (53)$$

and

$$G_0C_{pp} \sim \frac{3}{4}\left\{1 + \frac{12(1-v)}{5(7-5v)}(1-\alpha)^2\left[1 + \frac{(83-73v)(1-\alpha)}{7(7-5v)}\right]\right\}. \quad (54)$$

4. Conclusions

By deriving exact analytical expressions for the compressibility P and shear compliance Q of a spheroidal pore with arbitrary aspect ratio, starting from Eshelby's tensor, we have clarified a certain number of previous works where typographical mistakes were present (Wu, 1966; Dunn and Ledbetter, 1995; Kachanov et al., 2003), or where such results had been derived using other formalisms, such as wave-scattering (Kuster and Toksoz, 1974) or by directly solving the elastostatic problem (Zimmerman, 1985), among others. While it seems that the particular case of the shear compliance has not received nearly as much attention as has the compressibility, we have presented an extensive study showing that the pore aspect ratio, α, and the solid's Poisson ratio, v, have a very similar influence on both P and Q. Our analysis also clearly shows that the commonly used approximations for the limiting cases of infinitely "thin" cracks, and infinitely long needles, are no longer valid for crack-like pores having aspect ratios greater than 0.01, or for needle-like pores having aspect ratios lower than 100, respectively. On the contrary, our new asymptotic expressions for both P and Q, valid for a much wider range of aspect ratios, can account for more realistic pore geometries and, nevertheless, remain simple. Such expressions can therefore be used as input in an effective medium theory, to calculate the elastic response of solids containing *numerous* pores.

Specifically, in the case of the No-Interaction Approximation (NIA) scheme, the effective bulk and shear moduli (K,G) are given by

$$\begin{cases} \frac{K_0}{K} = 1 + \phi P_\alpha(\nu_0), \\ \frac{G_0}{G} = 1 + \phi Q_\alpha(\nu_0), \end{cases} \tag{55}$$

where ϕ is the porosity, and we use ν_0 to denote the solid's Poisson ratio, to avoid confusion with ν, the effective Poisson ratio of the porous material. Note that this pair of equations follows directly from (4).

According to the Differential Effective Medium (DEM) scheme, the effective moduli are described by a pair of coupled differential equations (LeRavalec and Gueguen, 1996):

$$\begin{cases} (1-\phi)\frac{1}{K}\frac{dK}{d\phi} = -P_\alpha(\nu), \\ (1-\phi)\frac{1}{G}\frac{dG}{d\phi} = -Q_\alpha(\nu), \end{cases} \tag{56}$$

for a fixed value of α, with the initial conditions $K(0) = K_0$ and $G(0) = G_0$.

However, once again, analytical solutions for such differential equations have only heretofore been obtained in the limiting cases of thin-cracks, spherical pores or needles. Using the new asymptotic expressions for P and Q, simple solutions for the DEM for the case of an isotropic solid containing pores having finite values of α have been found, and will be presented in a future paper. Having access to such solutions will greatly simplify the process of inverting sonic velocity data to obtain pore aspect ratio distributions (see, for example in rocks, (Cheng and Toksoz, 1979)). Finally, it would be very interesting to extend the present study to the case of fluid-saturated pores. Such a calculation will predict a stiffer elastic behaviour than would be obtained by using results for dry pores and invoking the Gassmann equation (LeRavalec and Gueguen, 1996), since Eshelby's method considers the pores as being *isolated* with regard to fluid flow, whereas Gassmann assumes fluid pressure equilibrium between the various pores.

References

Benveniste, Y., 1987. A new approach to the application of mori-tanaka's theory in composite materials. Mech. Mater. 6, 147–157.

Berryman, J.G., 1980. Long-wavelength propagation in composite elastic media II. Ellipsoidal inclusions. J. Acoust. Soc. Am. 68, 1820–1831.

Budiansky, B., O'Connell, R., 1976. Elastic moduli of a cracked solid. Int. J. Solids Struct. 12, 81–97.

Cheng, C.H., Toksoz, M.N., 1979. Inversion of seismic velocities for the pore aspect ratio spectrum of a rock. J. Geophys. Res. 84, 7533–7543.

Dunn, M.L., Ledbetter, H., 1995. Poisson's ratio of porous and microcracked solids: Theory and application to oxyde superconductors. J. Mater. Res. 10, 2715–2722.

Edwards, R.H., 1951. Stress concentrations around spheroidal inclusions and cavities. J. Appl. Mech. 18, 19–27.

Eshelby, J.D., 1957. The determination of the elastic field of an ellipsoidal inclusion, and related problems. Proc. Roy. Soc. London A 241, 376–396.

Giraud, A., Hoxha, D., Do, D., Magnenet, V., 2008. Effect of pore shape on effective porothermoelastic properties of isotropic rocks. Int. J. Solids Struct. 45, 1–23.

Hashin, Z., Shtrikman, S., 1961. Note on a variational approach to the theory of composite elastic materials. J. Franklin Inst. 271, 336–341.

Kachanov, M., Sevostianov, I., 2005. On quantitative characterization of microstructures and effective properties. Int. J. Solids Struct. 42, 309–336.

Kachanov, M., Shafiro, B., Tsukrov, I., 2003. Handbook of Elasticity Solutions. Kluwer, Dordrecht.

Kuster, G.T., Toksoz, M.N., 1974. Velocity and attenuation of seismic waves in two-phase media: Part I. Theoretical formulations. Geophysics 39, 587–606.

LeRavalec, M., Gueguen, Y., 1996. High- and low-frequency elastic moduli for a saturated porous/cracked rock—differential self-consistent and poroelastic theories. Geophysics 61, 1080–1094.

MacKenzie, J.K., 1950. The elastic constants of a solid containing spherical holes. Proc. Roy. Soc. London B 63, 2–11.

Markov, M., Levine, V., Mousatov, A., Kazatchenko, E., 2005. Elastic properties of double-porosity rocks using the differential effective medium model. Geophys. Prospect. 53, 733–754.

Mavko, G., Mukerji, T., Dvorkin, J., 1998. The Rock Physics Handbook. Cambridge University Press, Cambridge.

Sevostianov, I., Kachanov, M., 2008. Connections between elastic and conductive properties of heterogeneous materials. Adv. Appl. Mech. 42, 69–252.

Shafiro, B., Kachanov, M., 1997. Materials with fluid-filled pores of various shapes: effective elastic properties and fluid pressure polarization. Int. J. Solids Struct. 34, 3517–3540.

Walsh, J.B., 1965. The effect of cracks on the compressibility of rocks. J. Geophys. Res. 70, 381–389.

Wu, T.T., 1966. The effect of inclusion shape on the elastic moduli of a two-phase material. Int. J. Solids Struct. 2, 1–8.

Zimmerman, R.W., 1985. Compressibility of an isolated spheroidal cavity in an isotropic elastic medium. J. Appl. Mech. 52, 606–608.

Zimmerman, R.W., 1991. Compressibility of sandstones. Elsevier, Amsterdam.