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INJECTIVE OPERATIONS OF THE TORAL GROUPS

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§1. INTRODUCTION

THIS paper is a continuation of our earlier "Actions of Compact Lie Groups on Aspherical Manifolds" [3]. Our emphasis here is shifted away from aspherical manifolds. We find that a satisfactory hypothesis about the action (T^k, X) is the requirement that at each $x \in X$ the map $f^x : T^k \to X$ given by $f^x(t) = tx$ induce a monomorphism $f_*{}^x : \pi_1(T^k, 1) \to \pi_1(X, x)$. We call such an action *injective*. If X happens to be a closed aspherical manifold then any effective action of a toral group will satisfy the hypothesis.

Let us explain one of our principal results in terms of circle group actions. Suppose (S^1, X) is a circle group acting on a space for which $H_1(X; Z)$ is finitely generated. If $f^x : S^1 \to X$ induces a monomorphism $f_*{}^x : H_1(S^1; Z) \to H_1(X; Z)$ then for a suitable value of $n > 0$ X can be fibred over S^1 with structure group Z_n. We go on to show that the condition is equivalent to the existence of a weighted map on (S^1, X); that is, a map $g : X \to S^1$ which for some integer $n > 0$ satisfies the identity $g(tx) \equiv t^n g(x)$. This is meant to be the generalization of weighted homogeneous polynomials. It turns out that all weighted maps of least positive weight are fibrations of X over S^1 with structure group Z_n.

All injective actions can be lifted to the covering space associated with $\mathrm{im}(f_*{}^x)$. Here we show the action "splits". Corresponding to this splitting there is a properly discontinuous action (Y, N) of $N = \pi_1(M)/\mathrm{im}(f_*{}^x)$ on a simply connected space Y with $Y/N = X/T^k$. The injective actions which yield the same (Y, N) action can be described by certain elements of $H^2(N; Z^k)$. We show that (T^k, X) fibers (equivariantly) over T^k if and only if this cohomology class has finite order.

We apply some of these results to flat manifolds and discuss some of the elementary bordism properties of manifolds admitting toral actions without fixed points in the last section.

The present paper provides a new viewpoint for the classification of S^1-actions on 3 manifolds [10]. In addition our fibering theorems apply directly to the study of the Brieskorn examples [7]. We expect to discuss these topics later.

Our space X is always assumed to be paracompact, path connected and locally path connected and either (1) locally compact and semi 1-connected, or (2) has the homotopy type of the CW-complex.

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§2. FACTORS

A factor of a central extension $0 \to Z^k \xrightarrow{\alpha} \pi \xrightarrow{\beta} N \to 1$ is a pair (L, ϕ) consisting of an epimorphism $\phi : \pi \to L \to 1$ for which the composition

$$\alpha_{\#} = \phi \circ \alpha : Z^k \to L$$

is a monomorphism. If we count two factors (L, ϕ) and (L', ϕ') as being equivalent whenever there is an isomorphism $L \simeq L'$ yielding a commutative diagram

$$\begin{array}{ccc} & \overset{\phi}{\nearrow} & L \\ \pi & & \downarrow \simeq \\ & \underset{\phi'}{\searrow} & L' \end{array}$$

then the resulting set of equivalence classes is in one-to-one correspondence with the normal subgroups $K \subset \pi$ for which $\mathrm{im}(\alpha) \cap K = \{e\}$.

Since $\phi : \pi \to L$ is an epimorphism we see that $\mathrm{im}(\alpha_{\#})$ is still a central subgroup of L, thus we may define $H \subset \pi$ to be the kernel of the induced homomorphism $\pi \to L/\mathrm{im}(\alpha_{\#})$.

2.1 LEMMA. *The normal subgroup H is canonically isomorphic to the direct product $Z^k \times K$, where $K \subset \pi$ is the kernel of $\phi : \pi \to L$.*

An element $h \in \pi$ lies in H if and only if there is a $t \in Z^k$ for which $\alpha_{\#}(t) = \phi(h) \in L$. The element t is unique since $\alpha_{\#}$ is a monomorphism, thus we may introduce an epimorphism $P : H \to Z^k$ by the formula $\alpha_{\#} P(h) = \phi(h)$. Of course $P(\alpha(t)) = t$ and K is also the kernel of P. Next we define $Q : H \to K$ by $Q(h) = h(\alpha P(h^{-1}))$. Using the fact that $\mathrm{im}(\alpha)$ is a central subgroup, it follows that Q is also an epimorphism with kernel $\mathrm{im}(\alpha)$ and furthermore $Q(h) = h$ for all $h \in K$. Now $h = \alpha P(h) \cdot Q(h)$ so that $Z^k \times K \simeq H$ via the isomorphism $(t, h) \to \alpha(t) \cdot h$.

A factor (L, ϕ) is said to be abelian if and only if L is a free abelian group of rank k. These abelian factors play a role in our later fibration theorem. For now we shall prove

2.2 LEMMA. *If $\pi/[\pi, \pi]$ is finitely generated, then $0 \to Z^k \to \pi \to N \to 1$ admits an abelian factor if and only if* $\mathrm{im}(\alpha) \cap [\pi, \pi] = \{e\}$.

Let us argue this as follows. Suppose that $B \subset A$ denotes a subgroup of a finitely generated free abelian group. Consider $\nu : A \to A/B$, which we express as a direct sum $F \oplus D$ where F is a finite group and D is free abelian with $\mathrm{rank}(A) - \mathrm{rank}(B) = \mathrm{rank}(D)$. Let $\nu^{-1}(F) = C$, then $B \subset C$ and $A/C \simeq D$ so that C is a direct summand of A with $\mathrm{rank}(C) = \mathrm{rank}(B)$. This establishes sufficiency if $\pi/[\pi, \pi]$ is a finitely generated free abelian group.

More generally, let $B \subset A$ be a free abelian subgroup of a finitely generated abelian group. By factoring out the torsion subgroup of A and applying the above considerations to the image of B we find a direct summand $C' \subset A$ with $B \subset C'$ and $\mathrm{rank}(B) = \mathrm{rank}(C')$. If $p : A \to C$ denotes a projection of A onto the free part of C', then $p \mid B \to C$ is a monomorphism since B has no elements of finite order and $\mathrm{rank}(B) = \mathrm{rank}(C)$. This establishes sufficiency in general, while the necessity is trivial.

2.3 *Remark*. We may weaken the assumption on our extension by only requiring tha

$$1 \to G \xrightarrow{\alpha} \pi \xrightarrow{\beta} N \to 1$$

is an exact sequence. That is, we do not assume that G is a central subgroup but just a normal subgroup. If (L, ϕ) is a factor of this extension then

$$1 \longrightarrow K \longrightarrow H \underset{\chi}{\overset{P}{\rightleftarrows}} G \longrightarrow 1$$

is a split exact sequence, Of course here $\text{im}(\alpha_{\#})$ is normal. The onto map Q is not, in general, a homomorphism.

The Lemma 2.2 remains valid in this context. We would now like to discuss the *finite generation* of K in this setting. This result is not needed for our fibering theorems but it sheds light on certain 3-dimensional examples and the obstructions to fibering a manifold over a torus.

2.4 Proposition. *Let (L, ϕ) be an abelian factor of the extension*

$$1 \to G \xrightarrow{\alpha} \pi \xrightarrow{\beta} N \to 1,$$

with π finitely generated. Then the kernel $\phi = K$ is finitely generated.

The argument is similar to one employed in Satz 7 of [8]. It is not difficult to adopt the first part of their theorem to the situation here. We omit the details.

§3. A SPLITTING THEOREM FOR INJECTIVE ACTIONS

An action of a toral group (T^k, X) is *injective* if and only if at each $x \in X$ the map $f^x : T^k \to X$ given by $f^x(t) = tx$ induces a monomorphism $f_*{}^x : \pi_1(T^k, 1) \to \pi_1(X, x)$. It is readily seen that in an injective action the isotropy subgroups are all finite.

A *factor (L, ϕ) of the injection action (T^k, X)* is simply a factor of the associated central extension

$$0 \to Z^k \xrightarrow{f_*{}^x} \pi_1(X, x) \to N \to 1.$$

An important factor to keep in mind is $(\pi_1(X, x)$, identity map).

If (T^k, X) is an injective action so that (T^k, X) is equivariantly homeomorphic to $(T^k, T^k \times Y)$ with the latter action just left translation on the first factor and trivial on the Y factor we say *the action (T^k, X) splits.*

In [3; §4] we defined the *lifting* of actions of pathwise connected groups G to covering spaces B_H for which H contains $\text{im}(f_*{}^x)$, where

$$f_*{}^x : \pi_1(G, e) \to \pi_1(X, x),$$

and B_H is the covering space of X corresponding to the subgroup H. If H is normal the lifted action (G, B_H) covering (G, X) commutes with the covering transformations $\pi_1(X, x)/H$.

If (T^k, X) is an injective action, and H is a normal subgroup, $H \supseteq \text{im} f_*{}^x$, so that the covering action (T^k, B_H) splits, then we say that the action (T^k, X) *splits on B_H*, or simply that *(T^k, B_H) is a splitting action for (T^k, X).*

3.1 SPLITTING THEOREM. *Let (T^k, X) be an injective action and (L, ψ) a factor. If H is the kernel of $\pi_1(X, x) \to L/\text{im} f_\#{}^x$, then the action (T^k, X) splits on B_H.*

In particular, every injective action splits on $B_{\text{im}(f_*{}^x)}$, and $B_{\text{im}(f_*{}^x)}/T^k$ is simply connected. This is our extension of [3; 7.3] in the present context.

The proof is by induction on k and so we begin with $T^1 = S^1$. If $\omega \in \pi_1(S^1) \cong Z$ is the usual generator, then $f_*{}^x(\omega)$ is represented by $\exp(2\pi it)x$. Let us lift the action (S^1, X) to B_H based at b_0 (b_0 is the point in B_H corresponding to the trivial path at x).

3.2 LEMMA. *The covering action (S^1, B_H) is free.*

First we claim that the orbit through b_0 has trivial stability group. Clearly, $f_*{}^{b_0}(\omega) = \exp(2\pi it)b_0$ projects onto $f_*{}^x(\omega)$. Furthermore, $f_*{}^{b_0}(\omega)$ yields the generator of $\text{im}(f_*{}^x)$. Suppose that n is the largest integer ≥ 1 so that $\exp(2\pi it/n)b_0$ is a closed loop, $0 \leq t \leq 1$. Then $f_*{}^{b_0}(\omega) = n \exp(2\pi it/n)b_0$. In other words, in $H = Z \times K$, the generator of the Z-factor is divisible by n which is impossible if $n \neq 1$. The impossibility follows from the naturality of the splitting of H. For, the splitting isomorphism is given by $(mw, k) \to \alpha(mw) \cdot k$ $w \xrightarrow{\alpha} (\exp(2\pi it))x = f_*{}^x(\omega) = \alpha(w)$. Thus $n = 1$.

Now suppose y is any point in X and b is any point in B_H over y. Let $p(t)$ be a path from b_0 to b. This yields a path from x to y. We are interested in describing the loop $(\exp(2\pi it))b$. This covers the loop $(\exp(2\pi it))y$. Let m be the largest integer ≥ 1 such that $(\exp(2\pi it/m))b$ is a closed loop at b. (This projects to a loop at y.) The path $p^{-1} \circ \exp(2\pi it/m)b \circ p$ is a closed loop in H. Furthermore, m times this loop is homotopic to $p^{-1} \circ \exp(2\pi it)b \circ p$ which is homotopic to $\exp(2\pi it)b_0$ by [3; 4.1]. Thus, once again, there is an element of H which is an mth root of the generator $\text{im} f_*{}^x$ which is impossible unless $m = 1$. We have shown that the action (S^1, B_H) is principal. Next, we apply induction.

3.3 LEMMA. *The covering action (T^k, B_H) is free.*

We have proved 3.3 in the case $k = 1$. Let us assume that we have established the lemma for all $j \leq k - 1$ and suppose that we are given an injective action (T^k, X) together with a normal subgroup K of $\pi_1(X, x)$ with $\text{im}(f_*{}^x) \cap K = \{e\}$. Let T^{k-1} be any summand of T^k and let $g^x : T^{k-1} \to X$ be the restriction of $f_*{}^x$ to T^{k-1}. We have $\text{im}(f_*{}^x) = \text{im}(g_*{}^x) \times Z$.

We have the factor (L, ϕ) with kernel K. The factor is also a factor with respect to the subgroups $\text{im}(g_*{}^x)$ and Z. Write $H = \text{kernel}(\pi_1(X, x) \to L/\text{im}(f_*{}^x))$. Thus $H = \text{im}(g_*{}^x) \times Z \times K$. On B_H we have the action $(T^1 \times T^{k-1}, B_H)$. The action (T^{k-1}, B_H) is free since we can choose kernel $Z \times K$ for the factor $\pi \to L/Z$. The action (T^1, B_H) is free since we can choose the factor $\pi \to L/\text{im}(g_*{}^x)$. Thus the action (T^k, B_H) is free.

To complete the proof of the theorem we show that the principal fibering $B_H \to B_H/T^k = Y$ admits a cross section. This fibration is uniquely determined by a map $g : Y \to K(Z^k, 2)$, or equivalently by the "Chern class" $c \in H^2(Y; Z^k)$. To prove this class vanishes we need only show $H^2(Y; Z) \to H^2(B_H; Z)$ is a monomorphism.

Now $\pi_1(B, b_0) \simeq H$ and according to (2.1), $Z^k \times K \simeq H$, thus $f_*{}^{b_0} : \pi_1(T^k, 1) \to \pi_1(B, b_0)$ is an isomorphism onto a direct factor. Dually, $f^* : H^1(B_H; Z) \to H^1(T^k; Z)$ is an epimorphism. We may use the Leray-Hirsch–Dold Theorem [4] to see that the spectral sequence of the principal fibration is trivial. Hence $H^*(Y; Z) \to H^*(B_H; Z)$ is a monomorphism and the cross-section exists.

§4. A FIBERING THEOREM

If $(T^k, B_H) = (T^k, T^k \times Y)$ is a splitting action for (T^k, X) we receive a properly discontinuous action $(Y, \pi_1/H)$. The action $(T^k \times Y, \pi_1/H)$ is given by $(t, y)\alpha = (tm(y, \alpha), y\alpha)$ where $m : Y \times \pi_1/H \to T^k$ is a map satisfying $m(y, \alpha\beta) = m(y, \alpha)m(y\alpha, \beta)$ [3]. Furthermore if the abelian group Maps(Y, T^k) is given a left π_1/H-module structure by $(\alpha f)(y) = f(y\alpha)$, then $m(y, \alpha)$ determines a cohomology class in $H^1(\pi_1/H;$ Maps$(Y, T^k))$. We point out that although §8–11 of [3] were formulated in the context of aspherical manifolds, many of the results carry over into the present context with fairly obvious changes except that $H^1(\pi_1/H;$ Maps$(Y, T^k))$ will not be isomorphic to $H^2(\pi_1/H; Z^k)$ unless $H^1(Y; Z) = 0$.

Conversely, given (Y, N), a properly discontinuous action, and $m \in H^1(N;$ Maps $(Y, T^k))$ we can form a $(T^k - N)$-action $(T^k, T^k \times Y, N)$ which projects to the (Y, N)-action by $t'(t, y) = (t't, y)$ and $(t, y)\alpha = (tm(y, \alpha), y\alpha)$. Details are explored in [3] for the case of aspherical manifolds. It suffices to say for the moment that the expected correspondence between cohomology classes and equivariant homeomorphism holds.

Let us now suppose that the action (Y, N) is a properly discontinuous action of N on Y and that the toral action (T^k, X) arises from a cohomology class in $H^1(N;$ Maps$(Y, T^k))$ of *finite order*, say n. That is, $(T^k, X) = (T^k, (T^k \times Y)/N)$. Then if $m(y, \alpha)$ is a 1-cocycle representing this class, there is a map $g : Y \to T^k$ for which

$$g(y)g(y\alpha)^{-1} \equiv m(y, \alpha)^n, \text{ for all } \alpha \in N.$$

Observe, in particular, *if N is a finite group* of order n, then *every element* in the group $H^1(N;$ Maps$(Y, T^k))$ has *finite order* dividing n.

It may be useful to keep in mind the situation described in §3 where $(T^k, T^k \times Y)$ arises as a splitting action for an *injective* action (T^k, X). However, it is not necessary for our purpose to make this last assumption; in fact, the N-group action induced by $m(y, \alpha)$, $(T^k \times Y, N)$ which gives rise to $(T^k, (T^k \times Y)/N) = (T^k, X)$ does not necessarily have to be free.

4.1 LEMMA. *The space X fibers over T^k with structure group $(Z_n)^k$.*

We use the maps $g : Y \to T^k$ to produce this fibering. We regard $(Z_n)^k \subseteq T^k$ as the k-fold direct product of the cyclic group of nth roots with itself.

In $T^k \times Y$ we introduce the subset

$$C = \{(\tau, y) \mid \tau^n g(y) = 1\}.$$

Clearly if $\lambda \in (Z_n)^k$ and $(\tau, y) \in C$, then $(\lambda\tau, y) \in C$. Thus we have a principal left action $((Z_n)^k, C)$ exhibiting Y as a quotient of C. At the same time C is invariant under $(T^k \times Y, \pi_1/H)$ for since $g(y)g(y\alpha)^{-1} = m(y, \alpha)^n$ we have $\tau^n m(y, \alpha)^n g(y\alpha) = \tau^n g(y) = 1$

provided that $(\tau, y) \in C$. Thus we have $((Z_n)^k, C, \pi_1/H)$. Put $W = C/(\pi_1/H)$ to obtain an induced action $((Z_n)^k, W)$. Actually W is going to be our fibre.

Denote by $((\tau, y)) \in W$ the equivalence class of (τ, y) under the action of π_1/H on C. Let $\pi : T^k \times Y \to X$ be the π_1/H orbit map. Define now a T^k-equivariant map

$$G : T^k \times W \to X$$

by $G(t, ((\tau, y))) = \pi(t\tau, y) = t\tau\pi(1, y)$. This is well defined since $\pi(t\tau m(y, \alpha), y\alpha) = t\tau\pi(1, y)$ also. Now suppose there is a $(t_0, ((\tau_0, y_0)))$ with $G(t_0, ((\tau_0, y_0))) = \pi(t_0\tau_0, y_0) = \pi(t\tau, y) = G(t, ((\tau, y)))$. There is then an $\alpha \in \pi_1/H$ with

$$y\alpha = y_0$$
$$t\tau m(y, \alpha) = t_0\tau_0 .$$

Now $t^n = t^n\tau^n m(y, \alpha)^n g(y\alpha)$ and $t_0{}^n = t_0{}^n\tau_0{}^n g(y_0) = t_0{}^n\tau_0{}^n g(y\alpha)$. Therefore $t_0{}^n = t^n$ and there is a $\lambda \in (Z_n)^k$ for which $\lambda t_0 = t$, while $\lambda\tau m(y, \alpha) = \tau_0$. That is, $(t\lambda^{-1}, ((\lambda\tau, y))) = (t_0, ((\tau_0, y_0)))$, so if $(Z_n)^k$ acts on $T^k \times W$ by $\lambda(t, ((\tau, y))) = (t\lambda^{-1}, ((\lambda\tau, y)))$ then G induces a T^k-equivariant homeomorphism of $(T^k \times W)/(Z_n)^k$ with X. But $(t, ((\tau, y))) \to t^n \in T^k$ is the fibration of $(T^k \times W)/(Z_n)^k$ over T^k with fibre W and structure group $(Z_n)^k$. Unfortunately the construction of the fibration is not entirely canonical for at the outset we had to choose a map $g : Y \to T^k$ such that $g(y)g(y\alpha)^{-1} = m(y, \alpha)^n$. Altering the map g will even change the homotopy type of the fibre.

4.2 THEOREM. *Let (T^k, X) be an action on a space for which $H_1(X; Z)$ is finitely generated. If $f_* : H_1(T^k; Z) \to H_1(X; Z)$ is a monomorphism then for a suitable value of n the space X can be fibred over T^k with structure group $(Z_n)^k$.*

According to (2.2) the action (T^k, X), which is certainly injective, admits an abelian factor $\pi_1(X, x) \to Z^k$ so that $\pi_1/H \simeq Z^k/\mathrm{im}(f_\#{}^x)$ is a finite group. We may then apply 4.1.

§5. WEIGHTED MAPS

In this section we describe a convenient way of recognizing when an injective action (T^k, X) fibers over T^k.

5.1. First we recall some standard material on slices, see [1; Chap. VIII and Chap. XII]. Let H be a closed subgroup of the compact Lie group G and S be an H-slice in the G-space Y. Then the G-image $G(S)$ of the slice in the total space of a fiber bundle over G/H with fiber S and structure group H. (We have altered standard bundle terminology slightly in describing this structure group since H does not necessarily act effectively on S.)

In terms of the usual diagram we have:

$$\begin{array}{ccccc}
S & \longleftarrow & G \times S & \longrightarrow & G \\
\big\downarrow{\scriptstyle H} & & \big\downarrow{\scriptstyle H} & & \big\downarrow{\scriptstyle H} \\
G(S)/G = S/H & \longleftarrow & G(S) = G \times_H S & \longrightarrow & G/H
\end{array}$$

The bundle $(G(S), G/H, S, H)$ is the associated bundle to the principal H bundle $(G, G/H, H)$. If (G, X) is a G-space and $f: X \to Y$ is a G-map then $S' = f^{-1}(S)$ is also an

H-slice in X. The natural map $G \times f^{-1}(S) \to G \times S$ induces an equivariant map of the diagram associated with the H-slice S' in X to that associated with the H-slice S in Y.

Consider the special case where Y is the coset space G/H and $(G, G/H)$ is the usual left action. Let $\{1\}$ be the coset H. Then $\{1\}$ is an H-slice in G/H. Suppose $f: (G, X) \to G/H$ is an equivariant map, then $f^{-1}(S)$ is an H-slice in X, and *the equivariant map* $f: G(f^{-1}(S)) = X \to G/H$ *is a fiber bundle map with fiber* $f^{-1}(S)$ *and structure group* H.

We shall interpret section 4 in this context now. Let $G = T^k$, H a finite subgroup and consequently G/H is again a torus. We saw in §4 *the map*

$$(t, ((\tau, Y))) \to t^n \in T^k/((Z_n)^k)$$

is an equivariant map from $(T^k, (T^k \times W)/(Z_n)^k) = (T^k, X)$ to $(T^k, T^k/(Z_n)^k)$. (Since $T^k/(Z_n)^k$ is also a torus we can write this action simply as $(t, \tau) \to t^n\tau$.)

5.2 *Definition.* Let (T^k, X) be a toral action and $\{n\} = (n_1, n_2, \ldots, n_k)$ a k-tuple of integers. We say that f is a *weighted map* of weight $\{n\}$ if $f: (T^k, X) \to (T^k, T^k/Z_{n_1} \times \cdots \times Z_{n_k})$ is a T^k-equivariant map. In terms of coordinates this says that

$$f((t_1, \ldots, t_k)(x)) = (t_1^{n_1}, t_2^{n_2}, \ldots, t_k^{n_k})f(x) = (t_1^{n_1}f_1(x), \ldots, t_k^{n_k}f_k(x)).$$

In other words the coordinate functions are n_i weighted maps in terms of the circle factors.

Let us denote the set of weighted maps of weight $\{n\}$ by $E_{\{n\}}$. Suppose $f \in E_{\{n\}}$ and $g \in E_{\{m\}}$, then $fg^{-1} \in E_{\{n\}-\{m\}}$. Thus $E_{\{0\}} = E_{\{0,0,\ldots,0\}}$, which can be identified with the abelian group Maps$(X/T^k, T^k)$, acts transitively on $E_{\{n\}}$, if $E_{\{n\}} \neq \phi$. In fact each $E_{\{n\}}$ is in $1:1$ correspondence with $E_{\{0\}}$.

Let us suppose that $H_1(X, Z)$ is finitely generated.

5.3 THEOREM. *The class* $E_{\{n\}} \neq \phi$ *for some* $\{n\}$ *with all* $n_i > 0$ *if and only if* $f_*^x: H_1(T^k; Z) \to H_1(X; Z)$ *is a monomorphism.*

Thus, we obtain an equivariant fibering with finite abelian structure group (a subgroup of T^k) if and only if $f^x{}_*$ is a monomorphism.

If $g \in E_{\{n\}}$, $n_i > 0$, then there is an x so that $g(x) = 1$. The composite map $g \circ f^x: T^k \to T^k$ is a map of form $t \to t^{\{n\}}$ of degree $\{n\}$.

If f_*^x is a monomorphism apply 4.3 and obtain a fibering with $g \in E_{\{n\}}$.

The weighted map $g \in E_{\{n\}}$ obtained from 4.3 has the special form $\{n\} = (n, n, \ldots, n)$. For a given $g \in E_{\{n\}}$ the numbers $\{n\}$ may not be "minimal". This will mean that the slice $g^{-1}(S)$, that is the fiber, will not be connected. However, we can lift the map g to a covering torus $'T^k$ so that the lifted weighted map G satisfies

$$G_*: \pi_1(X) \to \pi_1('T^k)$$

is an epimorphism. Thus the fiber can be chosen connected.

(We can do this in such a way that we get a new map

$$G: X \to T^k/(Z_{m_1} \times Z_{m_2} \times \cdots \times Z_{m_k})$$

where each Z_{m_i} is a subgroup of Z_{n_i}. Let $g \in E_{\{n\}}$ and write it in terms of coordinates $g = (g_1, \ldots, g_k)$. If $\mathrm{im}(g_{1*}: \pi_1(X) \to \pi_1(S^1/Z_{n_1})) = l_1 Z$, then we shall prove that there is a map of weight n_1/l_1. Choose a map, by covering space theory $G_1 : X \to S^1/Z_{m_1}$ with $(G_1(x))^{l_1} \equiv g_1(x)$. Let $l_1 m_1 = n_1$ and introduce a map of $S^1 \times X \to S^1/Z_{m_1}$ by

$$(t, x) \to t^{-m_1} G_1(tx)G_1(x)^{-1}.$$

Noting that $t^{-m_1 l_1} G_1(tx)^{l_1} G(x)^{-l_1} = t^{-n_1} g_1(tx) g(x)^{-1} \equiv 1$, we conclude, since $S^1 \times X$ is connected, that $t^{-m_1} G_1(tx) G_1(x)^{-1}$ is constant and equal to a l_1-th root of unity. By taking $t = 1$ we see that this root is 1, so $t^{-m_1} G_1(tx) G_1(x)^{-1} \equiv 1$, and G_1 is a map of weight m_1. Continuing with each factor we obtain $Z_{m_1} \times \cdots x Z_{m_k} \subseteq Z_{n_1} \times \cdots \times Z_{n_k}$ and a map $G = (G_1, \ldots, G_k) \in E_{\{m_1, \ldots, m_k\}}$. The homomorphism $G_* : \pi_1(X, x) \to \pi_1(T^k/(Z_{m_1} \times \cdots \times Z_{m_k}), 1)$ is now onto.)

Finally corresponding to an equivariant map $(T^k, X) \xrightarrow{g} (T^k, T^k/H)$ we can find a weighted map. We alter T^k of (T^k, X) by an automorphism so that g is weighted. (Of coursethe new action on X is only equivalent to the old one up to an automorphism of T^k.) However, we may go back to the original action by now invoking 5.3.

5.4 A well known example of weighted maps for (S^1, X) arises in connection with the Brieskorn examples and the affine complex varieties defined by weighted homogeneous polynomials [7]. In particular 5.1 implies immediately that *the complement* $\mathbb{C}^n - V$ *in* $\mathbb{C}^n$ *of this type of* affine variety V *fibers over the circle with finite cyclic structure group.*

To see this more clearly, consider a Brieskorn variety V defined by the zeroes of the polynomial $p(z) = z_0^{a_0} + z_1^{a_1} + \cdots + z_{n-1}^{a_{n-1}}$. (The more general situation of weighted homogeous varieties is exactly similar.) Let $c_i = c/a_i$ where c = least common multiple of $\{a_0, \ldots, a_{n-1}\}$. Define $S^1 \times \mathbb{C}^n \to \mathbb{C}^n$ by

$$(t, z_0, \ldots, z_{n-1}) \to (t^{c_0} z_0, \ldots, t^{c_{n-1}} z_{n-1}).$$

Notice that the variety V, the unit sphere S^{2n-1}, and $S^{2n-1} - V$ are invariant. Let $f: S^{2n-1} - V \to S^1$ be defined by $p(z)/\|p(z)\|$. Observe that $f(tz) = t^c f(z) = t^c p(z)/\|p(z)\|$. That is, the map f: $(S^1, S^{2n-1} - V) \to (S^1, S^1)$ is equivariant. The action (S^1, S^1) is defined by $(t, \tau) \to t^c \tau$. *Thus* $f^{-1}(S^1) = S^{2n-1} - V$, *the complement of the variety, fibers over the* circle S^1/Z_c, *with fiber* $f^{-1}\{1\}$, *and structure group* Z_c. This is a special case of Milnor's fibering theorem.

A preliminary version of this section did not use slices. The tying in of weighted maps with slices was suggested to us by A. G. Wasserman.

§6. $H^2(N; Z^k)$

Let (T^k, X) be an injective action. Introduce the group N so that the sequence

$$0 \to \pi_1(T^k, 1) \xrightarrow{f_* x} \pi_1(X, x) \to N \to 1$$

is exact.

6.1 Theorem. *There is an equivariant fibering* $(T^k, X) \to (T^k, T^k/(Z_n)^k)$ *for a suitable value of n if and only if the element in* $H^2(N; Z^k)$ *represented by the central extension*

$$0 \to \pi_1(T^k, 1) \xrightarrow{f_* x} \pi_1(X, x) \to N \to 1$$

has finite order.

Use the covering action $(T^k, B_{\mathrm{im}(f_*{}^x)})$ corresponding to the image of $f_*{}^x$ and recall there is a splitting action $(T^k, T^k \times Y, N)$ where $Y = B_{\mathrm{im}(f_*{}^x)}/T^k$. Since Y is simply connected and (Y, N) is a properly discontinuous group of transformations it follows, [3; §8], that

$$H^1(N; \mathrm{Maps}(Y, T^k)) \xrightarrow[\cong]{\delta} H^2(N; Z^k).$$

Furthermore, since each $(T^k \times Y, N)$ action is given by a cocycle representative m in $H^1(N; \mathrm{Maps}(Y, T^k))$ the cohomology class $\delta[m] \in H^2(N; Z^k)$ will describe a particular central extension of Z^k by N. If the action $(T^k, B_{\mathrm{im}(f_*{}^x)}) = T^k, T^k \times Y)$ is the splitting action of an injective action (T^k, X) on $B_{\mathrm{im}(f_*{}^x)}$ then the covering action $T^k \times Y \to (T^k \times Y)/N = X$ is determined by m and $\delta[m]$ is the central extension

$$0 \to \mathrm{im}(f_*{}^x) \to \pi_1(X, x) \to N \to 1.$$

Thus assuming that this central extension is an element of finite order in $H^2(N; Z^k)$ we have that $[m]$ is of finite order in $H^1(N; \mathrm{Maps}(Y, T^k))$. We apply 4.1.

To see the converse consider the equivariant projection

$$p : (T^k, X) \to (T^k, T^k/(Z_n)^k).$$

Clearly the map $p_* f_*{}^x : \pi_1(T^k, 1) \to \pi_1(T^k/(Z_n)^k)$ is "multiplication by $\{n\}$". Corresponding to the fibering

$$\begin{array}{ccc} X' = T^k \times W & \longrightarrow & T^k \\ \downarrow & & \downarrow \\ X = T^k \times_{(Z_n)^k} W & \longrightarrow & T^k/(Z_n)^k \end{array}$$

we have the two central extensions

$$\begin{array}{ccccccccc} & & & & 1 & & 1 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Z^k & \longrightarrow & \pi' = Z^k \times N' & \longrightarrow & N' & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \text{identity}} & & \downarrow & & \downarrow{\scriptstyle q} & & \\ 0 & \longrightarrow & Z^k & \longrightarrow & \pi = Z^k \times_{(Z_n)^k} N & \longrightarrow & N & \longrightarrow & 1 \\ & & & & & \searrow & \swarrow & & \\ & & & & & & (Z_n)^k & & \\ & & & & & & \downarrow & & \\ & & & & & & 1 & & \end{array}$$

We may regard the first as the trivial element of $H^2(N'; Z^k)$ and the second as an element $[m] \in H^2(N; Z^k)$. Since $q : N' \subset N$, there is a natural map $q^* : H^2(N; Z^k) \to H^2(N'; Z^k)$ which sends $[m]$ onto $q^*[m] = 0$ by restriction. From the exact sequence $0 \to Z^k \to Q^k \to Q^k/Z^k \to 0$ the homomorphism $q^* : H^*(N; Q^k) \to H^*(N'; Q^k)$ is injective since $(Z_n)^k$ is a finite group.

Thus $[m]$ goes to zero in $H^2(N; Z^k) \to H^2(N; Q^k)$ and consequently had finite order in $H^2(N; Z^k)$. This completes the proof of the theorem.

We are now going to obtain a condition that guarantees every element of $H^2(N; Z^k)$ will have finite order. If (T^k, X) is an injective action on an aspherical n-manifold then there is a splitting action $(T^k, B_{\mathrm{im}(f_* x)}, N) = (T^k, T^k \times Y, N)$ where Y is a *contractible* $(n-k)$-cohomology manifold over Z.

Let us now assume that (Y, N) is a properly discontinuous action on an acyclic separable metric finite dimensional space with a finite number of orbit types. The latter holds if Y/N is compact, for example.

We obtained in [3; 9.2] an exact sequence

$$0 \to H^2(Y/N; Z^k) \to H^2(N; Z^k) \xrightarrow{\nu} H^0(Y/N; h^2) \to .$$

The image of ν is a subgroup of sections of a certain sheaf h^2 over Y/N. The stalks $h_{y_*}{}^2$, $y^* \in Y/N$ are $H^2(N_y; Z^k)$. Thus every element of $H^0(Y/N; h^2)$ has finite order. If now we add the assumption that $H^2(Y/N; Z^k)$ has only elements of finite order we obtain

6.2 PROPOSITION. *Every element of* $H^2(N; Z^k)$ *has finite order.* In particular, we have the following

6.3 COROLLARY. *If* (T^k, X) *denotes an effective action on a closed aspherical manifold and* $H^2(X/T^k; Z)$ *is a torsion group then* X *fibers over the torus* T^k.

In the proposition above every element of $H^2(N; Z^k)$ gives rise to a toral action $(T^k, (T^k \times Y)/N)$ with Y/N as quotient space. If, in addition, we knew that $\pi_2(Y/N) = 0$, then by dividing out T^k by the smallest subgroup F containing all the stability groups the induced action $(T^k/F, ((T^k \times Y)/N)/F)$ is free and so $\pi_1(T^k/F) \to \pi_1(((T^k \times Y)/N)/F)$ is a monomorphism. This implies that $\pi_1(T^k) \to \pi_1((T^k \times Y)/N)$ is injective. However it may not be true that $B_{\mathrm{im}(f_* x)} = T^k \times Y$.

It is easy to see that every S^1 action on a 3-manifold X yields a 2 manifold for orbit space. The surface X/S^1 is compact if and only if X is compact. If X is with boundary then X/S^1 has boundary. Assume a finite number of orbit types, no fixed points, and X/S^1 is not S^2 or P_2. Then

6.4 COROLLARY. (S^1, X) *is injective and* X *is aspherical. Furthermore it fibers over* S^1 *unless* X/S^1 *is a closed orientable surface.*

From this it also easily follows that the only time (S^1, X) is not injective is when $\pi_1(X)$ is finite and X is closed. In this case the space X is one of the well known spaces covered by the 3-sphere, *cf.* [8 and 10].

§7. SOME APPLICATIONS

Suppose that M^n is a compact flat Riemmanian manifold. Calabi has shown that if $k = \text{rank } H_1(M^n; Z)$ then there exists a flat Riemmanian manifold M^{n-k} and a finite subgroup $F \subseteq T^k$ so that M^n is the total space of a fiber bundle over T^k/F with structure group F. In fact, it is easily seen that there is an action (T^k, M^n) so that

$$\begin{array}{ccc} T^k, T^k \times M^{n-k} & \longrightarrow & (T^k, T^k) \\ \big\downarrow {\scriptstyle /F} & & \big\downarrow \\ (T^k, M^n) & \xrightarrow{M^{n-k}} & (T^k, T^k/F) \end{array}$$

is equivariant. The lower map is the fibering. Thus every compact flat n-dimensional Riemmanian manifold with $k = \text{rank } H_1(M^n; Z)$ admits an injective T^k-action which fibers over the k-torus. The structure group F acts as a group of isometries over M^{n-k}. The converse is also true (see [13; Chapter 3]). It is known that every compact flat n-manifold has fundamental group of the form

$$0 \to Z^n \to \pi \to F \to 1$$

where F is a finite group and Z^n is a maximal abelian normal subgroup.

There is a topological form of this theorem.

7.1 Topological Calabi Theorem. *Let (T^k, X) be an injective action and suppose there exists a normal finitely generated abelian subgroup $A \subset \pi_1(X, x)$ for which π_1/A is a finite quotient group. Then X fibers over T^k with a finite (abelian) structure group.*

Proof. Since $\text{im}(f_*^x)$ is central we can assume that $\text{im}(f_*^x) \subseteq A$. Hence there exists a covering action (T^k, B_A) which of course is still injective. Since $H_1(T^k) \to H_1(B_A)$ is a monomorphism, (T^k, B_A) fibers over T^k and we can choose the fibering so that the fiber W is connected. As in 6.1, the class $[m]$ in $H^2(\pi_1(X); Z^k)$ representing the action (T^k, X) maps onto the trivial class in $H^2(\pi_1(W); Z^k)$ which represents $(T^k, T^k \times W)$. Since $H^2(\pi_1(X); \mathbb{Q}^k) \to H^2((\pi_1(B_A); \mathbb{Q}^k) \to H^2(\pi_1(W); \mathbb{Q}^k)$ is a monomorphism, $[m]$ goes trivially under the homomorphism $H^2(\pi_1(X); Z^k) \to H^2(\pi_1(X); \mathbb{Q}^k)$. Hence $[m]$ has finite order and we apply 6.1.

As we mentioned before any effective action of a connected Lie group on a closed aspherical manifold must be an injective toral action. On the other hand, according to a preprint of A. Vasquez, On flat Riemannian manifolds, the existence of a normal abelian subgroup of rank n in a closed aspherical manifold of dimension n is equivalent to $\pi_1(X)$ being a Bieberbach group. Hence, X would have the homotopy type of a flat manifold.

Observe here that if X is smooth then B_A has the homotopy type of a torus and hence is a topological torus except possibly $(n, k) = (4, 1)$. If there exists an effective smooth $(T^2, B_A{}^5)$ where $B_A{}^5$ is a non-standard torus then $B_A{}^5/T^2$ is a smooth homotopy 3-torus which contains a contractible 3 manifold which is not a 3-cell contradicting the Poincaré conjecture. If $(T^1, B_A{}^5)$ exists then there are non-standard homotopy smooth 4-dimensional tori.

We now turn our attention to some of the general bordism properties that manifolds admitting injective actions must satisfy. What we have to offer are some rather elementary proofs of known results in the differentiable case [2] and some topological extensions of these results. We also make an application to the bounding problem for flat manifolds. In order to deduce these results it seems useful to use the language of cohomology manifolds. The key observation is the following.

7.2 LEMMA. *Let* (S^1, X) *denote a circle action, with all stability groups contained in a finite subgroup* F *of* S^1. *If* X *is an orientable cohomology* n-*manifold over* K, *where* K *is a field whose characteristic does not divide the order of* F, *then the orbit space* X/S^1 *is an orientable cohomology* $(n-1)$-*manifold and the mapping cylinder* $M(p)$ *of the orbit map* $p : X \to X/S^1$ *is an* $(n+1)$-*cohomology manifold with boundary* X *over* K.

For the proof we use the fact that if a finite group F acts on an orientable cohomology manifold and preserves the orientation then X/F is an orientable cohomology manifold over K provided the order of F is not divisible by the characteristic of K, [9; Th. 1]. Let U be a $S_x{}^1$-slice at $x \in X$, and consider the following diagram

$$\begin{array}{ccccc} U & \longleftarrow & U \times S^1 & \longrightarrow & S^1 \\ \downarrow {\scriptstyle S_x{}^1} & & \downarrow & & \downarrow \\ U/S_x{}^1 & \xleftarrow{p} & U \underset{S_x{}^1}{\times} S^1 & \longrightarrow & S^1/S_x{}^1 \end{array}$$

which we can embed equivariantly into

$$\begin{array}{ccccc} U & \longleftarrow & U \times D^2 & \longrightarrow & D^2 \\ \downarrow {\scriptstyle S_x{}^1} & & \downarrow {\scriptstyle S_x{}^1} & & \downarrow {\scriptstyle S_x{}^1} \\ U/S_x{}^1 & \longleftarrow & M(p) = U \underset{S_x{}^1}{\times} D^2 & \longrightarrow & D^2/S_x{}^1. \end{array}$$

The space $U \underset{S_x{}^1}{\times} D^2$ can be identified with the mapping cylinder of the orbit map p restricted to the open invariant set $S^1(U)$. This amounts to "filling in" each orbit by a disk and extending the S^1 action on the boundary (the orbit) to the entire disk. We apply the result quoted above to $U \to U/S_x{}^1$ and $U \times D^2 \to U \underset{S_x{}^1}{\times} D^2$ to obtain the desired result locally.

Remarks. If (S^1, X) has a locally finite orbit structure and characteristic $K = 0$ tnen $M(p)$ and X/S^1 are orientable cohomology manifolds over K. Of course this would be guaranteed if X were an orientable cohomology manifold over the integers Z. The lemma is not true in the non-orientable case; also there is a fairly obvious generalization to manifolds with boundary.

Finally if the action is assumed *free* and X is a topological manifold, "filling in the orbits" yields a *topological manifold with boundary* while X/S^1 may fail to be a manifold unless X is smooth and the action is smooth . The point is that $M(p)$ near a point in X/S^1

is homeomorphic to $U \times R^2$ where U is a slice. But since $U \times R^1$ is an open subset of X, $M(p)$ is a manifold.

7.3 Corollary. *If (T^k, X) is an action on an orientable closed cohomology manifold over the rationals $\mathbb{Q}$ with finite orbit structure and without stationary points, then X bounds an orientable rational cohomology manifold and the index of X is* 0.

For the proof one needs only to choose a circle subgroup which acts without fixed points and construct the rational cohomology manifold $M(p)$ described above.

We may make several observations concerning X and $M(p)$. If X is also a cohomology manifold over Z_2 and all the stability groups for the S^1 action are of finite odd order, then $M(p)$ is a cohomology manifold over $\mathbb{Z}_2$. Now, since X bounds, it has vanishing Stiefel Whitney numbers where the classes are derived from the Wu formula and Poincaré duality.

A result of Floyd [5] states that the orbit space of an action of a finite group on an ANR is again an ANR. If we apply this result to the orbit map

$$U \times D^2 \xrightarrow{S_x^1} U \times D^2/S_x^1$$

of 7.2 we obtain that $M(p)$ is an ANR if X was an ANR.

Finally in the smooth case, one can obtain more. Note that rational Pontryagin classes can be defined for triangulable compact cohomology manifolds over Q (with or without boundary). The construction is due to René Thom and J. Milnor, see [6] and also [11]. For smooth actions it is known that the orbit space is triangulable [14]. Moreover, it can be seen that the orbit map is nice enough to triangulate the mapping cylinder $M(p)$ so that the triangulation on its boundary is compatible with a smooth triangulation of X. Consequently, for a smooth (T^k, X) action without stationary points all rational Pontryagin numbers must vanish. Moreover, if one can find a fixed point free circle subgroup for which all stability groups are of odd order, then X bounds a smooth manifold. More generally there is the result of Conner and Floyd:

7.4 Lemma. [2; 30.1] *Suppose that (T^k, X) is a smooth action on a smooth closed manifold. If $F((Z_2)^k, X) = \phi$, then X is a smooth boundary.*

7.4 offers some information on the well known conjecture that every flat manifold bounds. Let X be a closed flat manifold such that (T^k, X) acts effectively. By Calabi's theorem this will be true if and only if rank $H_1(X) \geq k$. Calabi's fibering theorem produces a diagram

$$\begin{array}{ccc} T^k & \longleftarrow & T^k \times W \\ \downarrow & & \downarrow \\ T^k/F & \longleftarrow & T^k \times_F W \end{array}$$

where $F \subseteq T^k$ and rank $T^k \leq$ rank $H_1(X)$. The manifold W is also a flat manifold and F is a group containing all the stability groups. The holonomy group Γ also contains all of the stability groups.

Thus, by applying 7.3 and 7.4 we obtain the following:

7.5 THEOREM. *Let Γ be the holonomy group of a flat orientable closed Riemannian manifold M. Suppose $(Z_2)^s$ is the largest product of Z_2's embeddable in Γ. Then* dimension *$H_1(M; Q) > s$ implies that M fibers over a torus of rank greater than s, and M bounds a closed (orientable) smooth manifold.*

This theorem appears to be closely related to recent results of A. Vasquez [12]. It suggests that his integer $n(\Phi)$ may be related to the integer s above. Of course, all but a finite number of flat manifolds with Γ as holonomy group have dimension $H_1(M; Q) > s$.

We may state a generalized form of 7.5.

7.6 THEOREM. *Let (L, ϕ) be a factor of the injective smooth action (T^k, X) where X is a closed (orientable) smooth manifold. If $(Z_2)^{s+1}$ is not contained as a subgroup of $L/\text{im}(f_*{}^x)$ and $k > s$, then X bounds a smooth (orientable) manifold.*

The point here is that $L/\text{im}(f_*{}^x)$ contains the stability groups of (T^k, X). In particular, if $(Z_2)^k \subseteq T^k$ is a stability group in X, then it will be a stability group for $(Y, L/\text{im}(f_*{}^x))$. If X, in addition, is orientable, then it is known that the rational Pontryagin numbers vanish [3; 43.8], or we may apply 7.3 to obtain the vanishing Pontryagin numbers.

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