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Representing Abstract Groups by Powers of a Graph

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We deal with the problem of representing several abstract groups simultaneously by one graph as automorphism groups of its powers. We call subgroups $\Gamma_1,..., \Gamma_n$ of a finite group Γ representable iff there is a graph G and an injective mapping ϕ from $\bigcup_{i=1}^n \Gamma_i$ into the symmetric group on $V(G)$ such that for $i=1,..., n$ $\phi|_{\Gamma_i}$ is a monomorphism onto $\operatorname{Aut} G^i$. We give a necessary and a sufficient condition for groups being representable, the latter implying, e.g., that finite groups $\Gamma_1 \leqslant \cdots \leqslant \Gamma_n$ are representable. © 1986 Academic Press, Inc.

1. INTRODUCTION

By a result of Frucht [1] we can represent any finite group by a graph as its automorphism group, i.e., for each finite (abstract) group Γ there is a graph G with $\operatorname{Aut} G \simeq \Gamma$. We want to generalize this and represent several groups simultaneously by a graph as automorphism groups of its powers. More precisely, we call subgroups $\Gamma_1,..., \Gamma_n$ of a finite group Γ representable (by G) iff there is a graph G and an injective mapping ϕ: $\bigcup_{i=1}^n \Gamma_i \to \operatorname{Aut} G^0$ such that for $i=1,..., n$ $\phi|_{\Gamma_i}$ is a monomorphism of Γ_i onto $\operatorname{Aut} G^i$. A different notion of representing finite groups $\Gamma_1,..., \Gamma_n$ by powers of a graph would be to ask for a graph G with $\operatorname{Aut} G^i \simeq \Gamma_i$ for $i=1,..., n$. One easily sees that $\operatorname{Aut} G^i \subseteq \operatorname{Aut} G^{ij}$, $i, j \in N$. Thus some of the Γ_i have to have isomorphic subgroups, so first, we would face the question whether there is a group Γ and an embedding of $\Gamma_1,..., \Gamma_n$ into Γ such that certain subgroups get identified: $\operatorname{Aut} G^0$ is such a group with respect to $\operatorname{Aut} G^1,..., \operatorname{Aut} G^n$. But it is not always possible to find such a group, see Kurosch [2, p. 352]. Therefore we have chosen the above definition which includes the existence of Γ and quite naturally asks for a representation of the system of subgroups $\Gamma_1,..., \Gamma_n$ of Γ as the system of subgroups $\operatorname{Aut} G^1,..., \operatorname{Aut} G^n$ of $\operatorname{Aut} G$.

In the following all groups are finite, all graphs are finite, simple and undirected. Recall that the ith power G^i of a graph G is defined by

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$V(G^i) := V(G)$, $E(G^i) := \{vw \mid d_G(v, w) \leqslant i\}$, where $d_G(v, w)$ is the distance of v and w in G. If Aut G acts semiregularly on $V(G)$ we call G an sr-graph.

Let us recall a theorem due to Sabidussi [3], which strengthens the above result of Frucht: (One has to look into the constructions to see that the graphs of Sabidussi are sr-graphs, compare Vogler [4, Lemma 5].)

THEOREM 1. *For all $n \geqslant 3$ and all groups Γ there are infinitely many connected n-regular sr-graphs G with Aut $G \simeq \Gamma$.*

In Section 2 we give a necessary condition for groups being representable. The other sections are devoted to the proof of a sufficient condition. Consequences of this condition are that groups $\Gamma_1 \leqslant \cdots \leqslant \Gamma_n$ are representable and that our necessary condition is best possible in its kind. Here we want to outline the ideas of the main proof for the case, $n=2$, $\Gamma_1 \leqslant \Gamma_2$. Suppose we have an sr-graph H with Aut $H = $ Aut $H^2 \simeq \Gamma_2$ and a vertex v of degree 4. For $\alpha \in \Gamma_1$ (we identify Aut H and Γ_2) we replace $\alpha(v)$ by a C_4 as indicated in Fig. 1a for $\alpha = id$, for $\alpha \in \Gamma_2 \backslash \Gamma_1$ we replace $\alpha(v)$ by a K_4 as indicated in Fig. 1b. The automorphisms of H induce permutations of $V(G)$

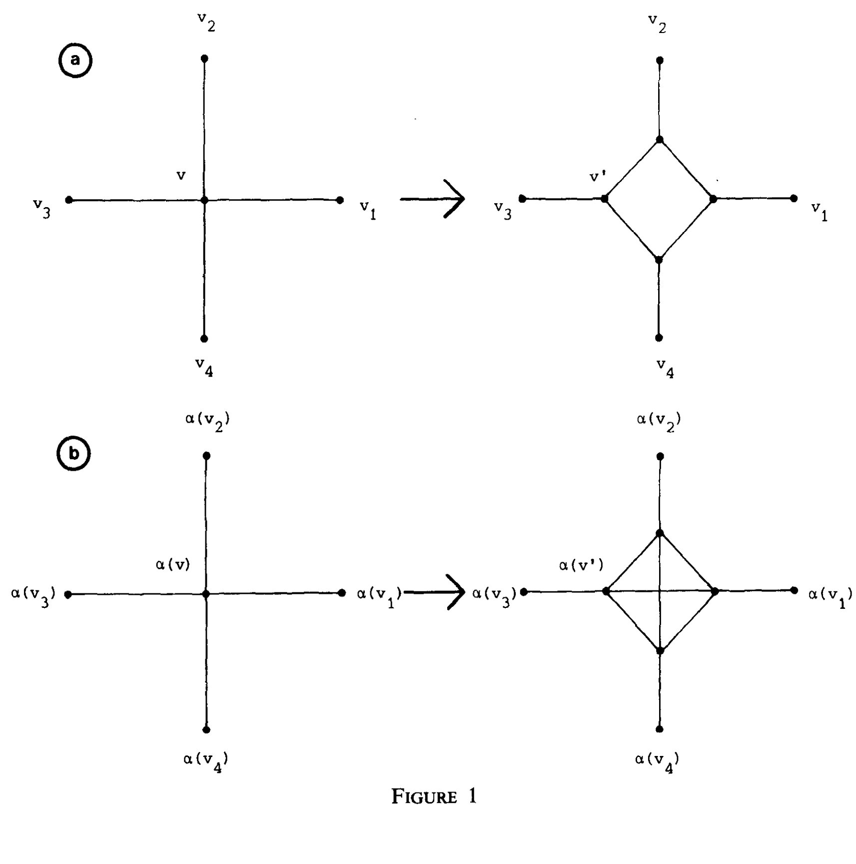

FIGURE 1

for the new graph G in an obvious way. Let us assume that all automorphisms of G and G^2 are induced by automorphisms of H. Then we see that only elements of Γ_1 can induce automorphisms of G, whereas we hope that all elements of Γ_2 induce automorphisms of G^2 since $K_4 = C_4^2$ (see below).

For a generalization of this idea we need a criterion for a graph G to have $\mathrm{Aut}\, G = \mathrm{Aut}\, G^n$—this will be given in Section 3—and we need pairs of graphs like (C_4, K_4) which have equal nth powers iff $n \in A$ for some given $A \subseteq \mathbb{N}_0$: these will be constructed in Section 4. In Section 5 we describe how to replace vertices of a graph H by some graphs without changing the automorphism group of H: up to isomorphism. The attentive reader will have noticed an obstacle for the method outlined above: The permutation of $V(G)$ induced by α of Fig. 1b is not an automorphism of G^2 since $v_1 v' \notin E(G^2)$ but $\alpha(v_1)\,\alpha(v') \in E(G^2)$. The difference between the inserted graphs has an effect "on the outside." We will deal with this by introducing "cakes" in Section 5.

2. A Necessary Condition

We have already noticed

Lemma 1. *$\mathrm{Aut}\, G^i \subseteq \mathrm{Aut}\, G^{ij}$ for all graphs G and natural numbers i, j.*

This lemma can be strengthened as follows:

Theorem 2. *$\mathrm{Aut}\, G^k \cap \mathrm{Aut}\, G^j \cap \mathrm{Aut}\, G^l \subseteq \mathrm{Aut}\, G^{j+l-k}$ for all graphs G and $k, j, l \in \mathbb{N}_0$ with $0 \leqslant k < j, l$.*

Proof. Let $\alpha \in \mathrm{Aut}\, G^k \cap \mathrm{Aut}\, G^j \cap \mathrm{Aut}\, G^l$ and $vw \in E(G^{j+l-k})$, i.e., $d_G(v, w) \leqslant j + l - k$. We have to show $d_G(\alpha(v), \alpha(w)) \leqslant j + l - k$. Choose a vertex x on a shortest v,w-path with $d_G(v, x) \leqslant j$, $d_G(x, w) \leqslant l - k$. Since $\alpha \in \mathrm{Aut}\, G^j$ we have $d_G(\alpha(v), \alpha(x)) \leqslant j$. Choose $\alpha(y)$ on a shortest $\alpha(v)$, $\alpha(x)$-path with $d_G(\alpha(v), \alpha(y)) \leqslant j - k$ and $d_G(\alpha(y), \alpha(x)) \leqslant k$. Since $\alpha \in \mathrm{Aut}\, G^k$ we have $d_G(y, x) \leqslant k$, hence $d_G(y, w) \leqslant l$. Since $\alpha \in \mathrm{Aut}\, G^l$ we get $d_G(\alpha(y), \alpha(w)) \leqslant l$ and with $d_G(\alpha(v), \alpha(y)) \leqslant j - k$ we have $d_G(\alpha(v), \alpha(w)) \leqslant j + l - k$. ∎

Corollary 1. *If $\Gamma_1, \ldots, \Gamma_n \leqslant \Gamma$ are representable, we have for all k, j, l with $0 \leqslant k < j \leqslant n$, $k < l \leqslant n$ and $j + l - k \leqslant n$ $\Gamma_k \cap \Gamma_j \cap \Gamma_l \subseteq \Gamma_{j+l-k}$, where $\Gamma_0 := \Gamma$.*

3. Graphs G with Aut $G = $ Aut G^n

For a vertex v of a graph G link(v, G) is the subgraph induced by the neighbours of v. We call v an extremal vertex iff there is a vertex w with $d_G(v, w) = \operatorname{diam} G$.

Lemma 2. *For a graph G, $v \in V(G)$ and $m \geqslant 2$ we have*:

(i) $\operatorname{diam} \operatorname{link}(v, G^m) \leqslant 3$.

(ii) *If* $\operatorname{diam} \operatorname{link}(v, G^m) = 3$ *and x is an extremal vertex of* $\operatorname{link}(v, G^m)$, *then $d_G(v, x) = m$.*

Proof. In link(v, G^m) each vertex is adjacent to a neighbour of v in G; except perhaps the only neighbour of v in G. All neighbours of v in G are adjacent in link(v, G^m). Hence (i).

If $d_G(v, x) < m$, then in link(v, G^m) x is adjacent to each neighbour of v in G; except possibly to itself. With the above consideration we see that x cannot be an extremal vertex if $\operatorname{diam} \operatorname{link}(v, G^m) = 3$. ∎

Lemma 3. *Let G be a graph, $m \geqslant 2$ and E' a set of edges of G which are not on a cycle of length $\leqslant 3$m. V' is the set of vertices being incident to any edge of E', G' is the subgraph induced by V'. Suppose*:

(i) *$d^{\min} G' \geqslant 2$, where $d^{\min} G'$ is the minimum degree of G',*

(ii) *$E(G - G') = \varnothing$,*

(iii) *$\forall w \in V(G) \setminus V'$:* $\operatorname{diam} \operatorname{link}(w, G) \leqslant 2$.

Then we have Aut $G^m = $ Aut G.

Proof. Because of Lemma 1 we only have to show Aut $G^m \subseteq$ Aut G.

(a) Given an x,y-path W, a y,z-path W' and a z,x-path W'' in G with $l(W) + l(W') + l(W'') \leqslant 3m$ and $vw \in E'$. Then vw is either on none or on at least two of the paths. Otherwise $W \cap W' \cap W''$-vw would contain a v,w-path and vw would be on a cycle of length $\leqslant 3m$.

(b) If $d_G(x, y) \leqslant m + 1$, then $vw \in E'$ is either on none or on all shortest x,y-paths in G. Apply (a) for two shortest x,y-paths and the y,y-path of length 0. $(2m + 2 \leqslant 3m)$

(c) For $vw \in E'$ we have: For $l = 1, \dots, m + 1$ there is $x_1 \in V'$ with $d_G(v, x_l) = l$ and w on each shortest v,x_l-path. For $l = 1, \dots, m$ there is $y_l \in V'$ with $d_G(v, y_l) = l$ and w on no shortest v,y_l-path. We find x_l by induction. $x_1 := w$. Let $x_l \in V'$ for $l \leqslant m$ be given, let $x_l x \in E'$. Apply (a) to a shortest v,x_l-path, $x_l x$, a shortest x,v-path and the edge $x_l x$ $(l + 1 + (l + 1) \leqslant 2m + 2 \leqslant 3m)$.

If x_l is on one shortest v, x-path, then it is on all. Then we have the same for w and $d_G(v, x) = l+1$; put $x_{l+1} := x$. Otherwise $x_l x$ is on every shortest v, x_l-path. By (i) we can find some other neighbour $x' \in V'$ of x_l. Apply (a) to a shortest v, x_l-path, $x_l x'$, a shortest v, x'-path and the edge $x_l x$. We get that $x_l x$ is on every shortest v, x'-path, hence w, too. Thus $d_G(v, x') = l+1$ and we put $x_{l+1} := x'$.

Now we apply this to wv. Let $x'_l \in V'$, $l = 1,..., m+1$, be vertices with $d_G(w, x'_l) = l$ and v on every shortest w, x'_l-path. Put $y_l := x'_{l+1}$, $l = 1,..., m$.

Now we define: $e \in E(G^m)$ is an r-edge of G^m, iff $e = vw$ with $\operatorname{diam} \operatorname{link}(v, G^m) = 3$ and w adjacent to all nonextremal vertices of $\operatorname{link}(v, G^m)$. Obviously an automorphism of G^m maps an r-edge onto an r-edge. So we are done, if we can show that the r-edges of G^m are exactly the edges of G.

Let $vv' \in E(G)$. Without loss of generality $v \in V'$ because of (ii), so there is $vw \in E'$: possibly $w = v'$. By (c) let x and y be vertices with $d_G(v, x) = d_G(v, y) = m$, such that every shortest v,x-path but no shortest v,y-path contains w. Let z be a neighbour of x in $\operatorname{link}(v, G^m)$. Apply (a) to shortest paths in G from v to x, x to z and z to v and the edge vw. Because of $d_G(v, x) = m$ vw is not on any shortest x,z-path. Thus w is on every shortest v,z-path. Analogously, w is not on any shortest v,z'-path in G for a neighbour z' of y in $\operatorname{link}(v, G^m)$. Hence x and y do not have a common neighbour in $\operatorname{link}(v, G^m)$, so by Lemma 2 $\operatorname{diam} \operatorname{link}(v, G^m) = 3$. If $d_G(v, u) = m$, then w is on every or on no shortest v, u-path. A consideration for u and y or for x and u as above for x and y shows that u is an extremal vertex of $\operatorname{link}(v, G^m)$. Hence for every non-extremal vertex u $d_G(v', u) \leqslant m$ and vv' is an r-edge.

Now let vv' be an r-edge of G^m. If $v \in V(G) \setminus V'$, then by (iii) $\operatorname{diam} \operatorname{link}(v, G^m) \leqslant 2$. So let $v \in V'$ and $vw \in E'$. We can suppose that w is not on any shortest v, v'-path in G: the case that w is on every such path is analogous. Suppose $d := d_G(v, v') \geqslant 2$. Apply (c) to vw and (a) to shortest paths from v to v', v' to x_{m+1-d} and x_{m+1-d} to v and the edge vw $(d + (d+m+1-d) + (m+1-d) \leqslant 2m+2 \leqslant 3m)$. Thus vw is on the shortest v',x_{m+1-d}-path, so $d_G(v', x_{m+1-d}) = m+1$ and by Lemma 2 x_{m+1-d} is not an extremal vertex of $\operatorname{link}(v, G^m)$, contradicting vv' being an r-edge. ▮

COROLLARY 2. *Let G be a graph with girth $> 3m$ for $m \in \mathbb{N}$ and minimum degree $\geqslant 2$. Then* Aut G^m = Aut G.

Remark. A necessary and sufficient condition for Aut G^m = Aut G seems to be difficult to find considering: For $3m < n$: $\operatorname{Aut}(C_n \times K_2)^m = \operatorname{Aut}(C_n \times K_2) \Leftrightarrow m$ and n are relatively prime.

4. Graphs with Some Powers Being Equal

For a set A of natural numbers we want to find graps G and H with $G^m = H^m$ iff $m \in A$. These graphs have to be defined on the same vertex set. Now consider the mapping α: $V(G \,\dot\cup\, H) \to V(G \,\dot\cup\, H)$ that exchanges each vertex of G with the corresponding vertex of H; then $\alpha \in \mathrm{Aut}(G \,\dot\cup\, H)^m$ iff $m \in A$ and in view of Theorem 2 A has to be closed as defined below.

Definition. We call $A \subseteq \mathbb{N}_0$ closed, iff $0 \in A$ and for all $k, j, l \in A$ with $k < j,l$ $j + l - k \in A$. For $T \subseteq \mathbb{N}_0$ the closed hull $A(T)$ is the minimal closed set containing T: observe that $\mathbb{N}_0$ and the intersection of closed sets are closed. Let Ω be the set of all proper closed subsets A of $\mathbb{N}_0$ for which $\mathbb{N}_0 - A$ is finite, $\Omega_n = \{A \in \Omega \mid \max(\mathbb{N}_0 - A) = n\}$. Clearly $\Omega = \bigcup_{n \in \mathbb{N}} \Omega_n$.

We will need some more notions and the following graphs:

Definition. For $0 \leqslant l < n$, $S \subseteq \{1,..., l\}$ $F(n, l, S)$ is defined by:

$$V(F(n\, l, S)) := \{i, -i \mid i = 1,..., n\}$$

$$\begin{aligned} E(F(n, l, S)) := &\{i(-i) \mid i = 1,..., n\} \\ &\cup \{i(i+1), (-i)(-i-1) \mid i = 1,..., n-1, i \notin S\} \\ &\cup \{i(-i-1), (-i)(i+1) \mid i = 1,..., n-1, i \in S \vee i > l\}. \end{aligned}$$

Example (Fig. 2). $F(4, 2, \{1\})$.

Definition. Let $k, n \in \mathbb{N}_0$, $M = \{1,..., n\}$. A k-interval of M is a subset $\{i, i+1,..., i+k-1\}$ of M. We call a subset S of M k-translation-invariant, iff any two k-intervals of M contain the same number of elements of S—or equivalently iff for $i \in S$ $i+k$, $i-k \in S$ if they are in M. We call another subset T of M S-even (S-odd), iff $|T \cap S|$ is even (odd).

Lemma 4. *Let* $0 \leqslant l < n$, $S \subseteq \{1,..., l\}$, $F := F(n, l, \varnothing)$, $F' := F(n, l, S)$, $x, y \in V(F)$. *Then*:

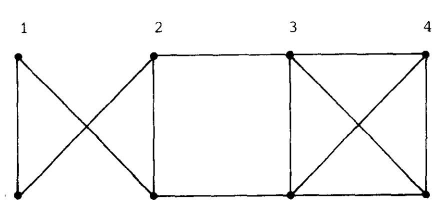

Figure 2

(i) *If* $|x| > l+1$ *or* $|y| > l+1$ *and* $x \neq -y$, *then* $d_F(x, y) = d_{F'}(x, y) = d_{F \cup F'}(x, y) = \| x| - |y\|$.

(ii) $d_F(x, y), d_{F'}(x, y), d_{F \cup F'}(x, y) \in \{||x| - |y||, ||x| - |y|| + 1\}$

(iii) $\forall m \in \mathbb{N}_0$: $F^m = F'^m \Leftrightarrow m > 1 \vee$ *each m-interval of* $\{1, ..., n\}$ *is S-even.*

Proof. (i) and (ii) are easy.

(iii) Because of (i) and (ii) it is enough to consider $x, y \in V(F)$ with $|x|$, $|y| \leqslant l+1$ and $||x| - |y|| = m$. Without loss of generality, $|x| \leqslant |y|$. Consider the path whose vertices have absolute value $|x|$, $|x| + 1, ..., |y|$. ∎

LEMMA 5. *Let* $n \in \mathbb{N}$, $A \in \Omega_n$. *Then there are* $r \in \mathbb{N}$ *and* $(l_1, S_1), ..., (l_r, S_r)$ *with* $l_i \in \mathbb{N}$, $l_i \leqslant n$, $S_i \subseteq \{1, ..., l_i\}$ *for* $i = 1, ..., r$, *such that for all* $x \in \mathbb{N}_0$: $x \notin A \Leftrightarrow \exists i \in \{1, ..., r\}$: *There is an* S_i*-odd x-interval in* $\{1, ..., l_i\}$.

Proof. Let k be the least positive element of A. We have $k \neq 1$, because with A being closed $1 \in A$ would imply $A = \mathbb{N}_0$ by induction and $(t+1) = t + 1 - 0$; but $\mathbb{N}_0 \notin \Omega$. Let j be the least positive element of A that is not a multiple of k, $j = pk + q$ with $0 < q < k$. We note if $n < pk$, we can put $r := 1$, $l_1 := n$ and $S_1 := \{i \in \{1, ..., l_1\} \mid i \equiv 1 \vee i \equiv k \bmod k\}$. If $x \in A$, then there is an x-interval in $\{1, ..., l_1\}$, only if $x < n$, x is a multiple of k. But $\{1, ..., k\}$ is S_1-even and S_1 is k-translation-invariant. On the other hand, if $x \notin A$, then $x \leqslant n$, x is not a multiple of k and $\{1, ..., x\}$ is an S_1-odd x-interval of $\{1, ..., l_1\}$.

Now we will prove the lemma by induction over k. For $k = 2$ we are done by the above consideration, because n is odd and j is the least odd number in A, so $j = n + 2$. Let the lemma be true for all $A' \in \Omega$ with smaller least positive element than k. By the above consideration we may assume $n > pk$. Now $q \geqslant 2$, because a closed set contains with a and $a+1$ also $a + 2 = (a+1) + (a+1) - a$ and analogously all larger numbers, i.e., if pk, $pk + 1 \in A$ then $n < pk$. Put $A' := \{x - pk \mid x \in A \wedge x \geqslant pk\}$.

We have $A' \in \Omega_{n-pk}$ and $A = \{ik \mid i = 0, ..., p-1\} \cup \{y + pk \mid y \in A'\}$. The least positive number of A' is q and $q < k$. By induction there are r', $(l'_1, S'_1), ..., (l'_{r'}, S'_{r'})$, such that for A' the lemma is true. Put $r := r' + 1$, $l_r := pk$, $S_r := \{x \in \{1, ..., pk\} \mid x \equiv 1 \vee x \equiv k \bmod k\}$ and for $i = 1, ..., r'$ $l_i := l'_i + pk$ and $S_i := \{x \in \{1, ..., l_i\} \mid x \equiv y \bmod k$ for some $y \in S'_i$ with $0 < y \leqslant k\} \cup \{x \in \{1, ..., l_i\} \mid x \equiv 0 \bmod k$ if $l'_i < k \wedge \{1, ..., l'_i\}$ is S'_i-odd$\}$.

First, let $x \in A$. If $x \leqslant pk$, then x is a multiple of k; all S_i are by definition k-translation-invariant and $\{1, ..., k\}$ is S_i-even because of $k \in A'$. If $x > pk$, put $x' := x - pk$, hence $x' \in A'$. If there is an x-interval in $\{1, ..., l_i\}$, then $i \neq r$, $x' \leqslant l'_i$ and every x'-interval in $\{1, ..., l'_i\}$ is S'_i-even. But $S_i \cap \{1, ..., l'_i\} = S'_i$, because S'_i is k-translation-invariant. Hence an x-interval in $\{1,..., l_i\}$ is the union of an x'-interval in $\{1,..., l'_i\}$ and p k-intervals, thus it is S_i-even. On the other hand, let $x \notin A$. If $x < pk$, then x is not a multiple of k, i.e., $\{1,..., x\}$ is an S_r-odd x-interval in $\{1,..., l_r\}$. If $x > pk$, then $x' := x - pk \notin A'$. There is an i and an S'_i-odd x'-interval in $\{1,..., l'_i\}$; we can complete this with p k-intervals to get an S_i-odd x-interval in $\{1,..., l_i\}$. ∎

5. Inserting Graphs Without Changing the Automorphism Group

Our first method of construction describes how to replace the vertices of an sr-graph H by copies of some graphs G_i and to join the edges of H to some vertices of the inserted graphs given by mappings f_i such that the automorphism group remains the same up to isomorphism.

Construction 1. Let H be an sr-graph and $v_1,..., v_n$ representatives of the orbits of Aut H acting on $V(H)$. Furthermore for $i = 1,..., n$, let G_i be a graph and f_i: $E_i \to V(G_i)$ a mapping, where E_i is the set of edges incident with v_i. We call the vertices in $f_i(E_i)$ vertices of attachment. We construct G as follows:

$$\begin{aligned} V(G) &:= \{(\alpha(v_i), w) \mid \alpha \in \text{Aut } H, i \in \{1,..., n\}, w \in V(G_i)\}, \\ E(G) &:= \{(\alpha(v_i), w)(\alpha(v_i), w') \mid ww' \in E(G_i)\} \\ &\quad \cup \{(\alpha(v_i), w)(\beta(v_j), w') \mid \alpha(v_i)\, \beta(v_j) = e \in E(H), \\ &\quad f_i(\alpha^{-1}(e)) = w, f_j(\beta^{-1}(e)) = w'\}. \end{aligned}$$

Lemma 6. *Let G be constructed from an* sr*-graph H, graphs G_i and mappings f_i using Construction* 1 *with*:

(i) $\forall i \in \{1,..., n\}\ \forall \alpha \in \text{Aut } G_i$:

$$(\forall e \in E_i \colon \alpha(f_i(e)) = f_i(e)) \Rightarrow \alpha = id.$$

(ii) $\forall \gamma \in \text{Aut } G,\ i \in \{1,..., n\},\ \alpha \in \text{Aut } H\ \exists j \in \{1,..., n\},\ \beta \in \text{Aut } H$:

$$\gamma(V_{i,\alpha}) = V_{j,\beta}$$

where $V_{i,\alpha} := \{(\alpha(v_i), w) \mid w \in V(G_i)\}$.

Then Aut $G \simeq$ Aut H *and G is an* sr*-graph. G is connected, if H and all the G_i are connected.*

Proof. By $(\alpha(v_i), w) \to \alpha(v_i)$ we get a homomorphism of G onto H that also is surjective as a mapping from $E(G)$ to $E(H)$. Because of (ii) there is a corresponding group homomoprhism ϕ: Aut $G \to$ Aut H. For $\beta \in$ Aut H we get β': $(\alpha(v_i), w) \to (\beta\alpha(v_i), w)$ as an inverse image under ϕ. The subgraph of G induced by $V_{i,\alpha}$ is isomorphic to G_i. Therefore, if $\gamma \in \ker \phi$, then $\gamma|_{V_{i,\alpha}}$ is an automorphism of $\langle V_{i,\alpha} \rangle$ which leaves the vertices of attachment fixed, because each of them is the only vertex that has a neighbour in the corresponding $V_{j,\beta}$, so because of (i) γ is the identity on each $V_{i,\alpha}$ and thus on $V(G)$. Because of the form of the automorphisms of G as given above G is an sr-graph. The statement about G being connected is obvious. ∎

Remark. The conditions of Lemma 6 are also necessary. If one of them is violated, Aut G has more elements than Aut H.

DEFINITION. Let $H_1,..., H_n$ be graphs with the same vertex set V. Put $V' := \{v' \mid v \in V\}$. The cakes $T_0, T_1,..., T_n$ belonging to the H_i are defined by: For $i = 0,..., n$: $V(T_i) := V \cup V'$, $E(T_i) := E(H_i) \cup \{vv' \mid v \in V\} \cup \{v'w' \mid \exists l: vw \in E(H_l)\}$, where $E(H_0) := \emptyset$.

We call the vertices of V upper vertices, those of V' lower vertices. (Thus the upper vertices of T_i induce H_i for $i = 1,..., n$, the lower vertices induce a graph isomorphic to $H_1 \cup \cdots \cup H_n$.)

LEMMA 7. *Let $T_0,..., T_n$ be the cakes belonging to $H_1,..., H_n$, $l \in \mathbb{N}$, $i, j \in \{1,..., n\}$, $H := H_1 \cup \cdots \cup H_n$. Then*:

(i) *If $H_i^l = H_j^l$, then $T_i^l = T_j^l$.*

(ii) *If for all $v, w \in V$, $k \in \{1,..., n\}$ $d_H(v, w) \geqslant d_{H_k}(v, w) - 2$, then conversely: If $T_i^l = T_j^l$, then $H_i^l = H_j^l$.*

Proof. (i) For each path in a cake containing only upper vertices there is a path containing the corresponding lower vertices. Therefore for a lower vertex v' and some vertex w there is a shortest v', w-path containing only lower vertices except possibly w. Thus for all $v, w \in V$ $d_{T_i}(v', w') = d_{T_j}(v', w')$ and $d_{T_i}(v', w) = d_{T_j}(v', w)$. Now let $v, w \in V$ with $d_{T_i}(v, w) \leqslant l$. If there is a shortest v, w-path in T_i containing a lower vertex, then there is one containing only lower vertices except v and w, therefore $d_{T_j}(v, w) \leqslant l$. On the other hand, if each shortest v, w-path in T_i contains only upper vertices, then $d_{H_i}(v, w) \leqslant l$, thus $d_{H_j}(v, w) \leqslant l$, therefore $d_{T_j}(v, w) \leqslant l$.

(ii) By the hypothesis for all $v, w \in V$ there are shortest v, w-paths in T_i and T_j containing only upper vertices. ∎

CONSTRUCTION 2. Let $n, k \in \mathbb{N}$, $k \geqslant 3$, H' a k-regular sr-graph, $v_1,..., v_l$ vertices of H' belonging to different orbits of Aut H'. Let $H_{i,\alpha}$, $i = 1,..., l$, $\alpha \in$ Aut H', be connected graphs with the same vertex set V with $|V| = k$.

For fixed i let $T_{i,0}$, $T_{i,\alpha}$, $\alpha \in \mathrm{Aut}\, H'$, be the cakes belonging to $H_{i,\alpha}$. Let N be a common multiple of $1, \ldots, n$ with $N \geqslant \mathrm{diam}\, H_{i,\alpha} + 2$ for all $\alpha \in \mathrm{Aut}\, H'$, $i = 1, \ldots, l$.

We get H from H' by subdividing each edge N times, thus girth $H \geqslant 3N + 3$ and $\mathrm{Aut}\, H \simeq \mathrm{Aut}\, H'$. We consider $\mathrm{Aut}\, H$ and $\mathrm{Aut}\, H'$ as equal. We construct G' from H with Construction 1 by replacing the vertices of the orbit of v_i by a copy of $T_{i,0}$ for $i = 1, \ldots, l$, where the lower vertices are the vertices of attachment. We leave the other vertices unchanged, i.e., we replace them by copies of K_1. We construct G from G' by adding edges $(\alpha(v_i), v)(\alpha(v_i), w)$, for $\alpha \in \mathrm{Aut}\, H$, $i \in \{1, \ldots, l\}$, $vw \in E(H_{i,\alpha})$.

LEMMA 8. *For Construction* 2 *we have*:

(i) $G^N = G'^N$.

(ii) $\forall m \leqslant n$: $\mathrm{Aut}\, G^m \subseteq \mathrm{Aut}\, G' \simeq \mathrm{Aut}\, H$.

(iii) *If we identify* $\mathrm{Aut}\, G'$ *and* $\mathrm{Aut}\, H$ *using the isomorphism given in* (ii), *then for all* $0 < m \leqslant n$, $\alpha \in \mathrm{Aut}\, G'$: $\alpha \in \mathrm{Aut}\, G^m \Leftrightarrow \forall v, w \in V$, $\beta \in \mathrm{Aut}\, H$, $i \in \{1, \ldots, l\}$:

$$[vw \in E((T_{i,\beta})^m) \Leftrightarrow vw \in E((T_{i,\alpha\beta})^m)].$$

Proof. First we have:

(a) If v, w are not upper vertices of the same copy of a cake, then $d_G(v, w) = d_{G'}(v, w)$. There is a shortest v, w-path in G, which contains no upper vertices except possibly v and w; compare the proof of Lemma 7. This path is contained in G', too. Now $G' \subseteq G$ gives (a).

(i) Because of (a) we only have to conside pairs of upper vertices form the same copy of a cake. But those vertices are adjacent in G'^N, because $N \geqslant \mathrm{diam}\, H_{i,\alpha} + 2$, and $G' \subseteq G$.

(ii) $\mathrm{Aut}\, G' \simeq \mathrm{Aut}\, H$ follows from Lemma 6. Lemma 6(i) is obvious, to see (ii) consider the subgraph of G' induced by the vertices of degree 1 and their neighbours. Its components are exactly the inserted copies of the cakes. Now we apply Lemma 3 to G'. We choose E' as the set of edges not in any copy of a cake. Because girth $H > 3N$ this is feasible. Because the $T_{i,0}$ are connected and $d^{\min} H = 2$, we have Lemma 3(i). Because the vertices not in V' have degree 1, we get (ii) and (iii). Therefore $\mathrm{Aut}\, G'^N = \mathrm{Aut}\, G'$. With (i) and Lemma 1 we now have $\mathrm{Aut}\, G^m \subseteq \mathrm{Aut}\, G^N = \mathrm{Aut}\, G'^N = \mathrm{Aut}\, G'$.

(iii) Because of (a) an automorphism of G' maps vertices adjacent in G^m onto vertices adjacent in G^m, unless they are both upper vertices of the same copy of a cake. A shortest path in G between two upper vertices of the same copy has length at most $N + 2$, since girth $H > 3N$ it is completely contained in that copy. Therefore, whether two such vertices are adjacent in G^m, only depends on the corresponding cake. Since α maps $T_{i,\beta}$ onto $T_{i,\alpha\beta}$, we are done. ▌

Considering Lemmas 7 and 8 we only have to control the powers of the graphs $H_{i,\alpha}$ in Construction 2 in order to control the groups $\operatorname{Aut} G^m$. But we have already met some graphs whose powers we can control. So we are ready to prove our sufficient condition.

6. A Sufficient Condition

THEOREM 3. *Let $n \in \mathbb{N}$, $\Gamma_1, ..., \Gamma_n$ subgroups of a finite group Γ. For $m \leqslant n$ and $A \in \Omega_m$ put $\Gamma(A) := \langle \bigcup_{x \notin A} \Gamma_x \rangle$. If for $m = 1, ..., n$ $\Gamma_m = \bigcap_{A \in \Omega_m} \Gamma(A)$, then $\Gamma_1, ..., \Gamma_n$ are representable; Indeed, there are infinitely many graphs that represent $\Gamma_1, ..., \Gamma_n$.*

Proof. By Theorem 1 let H' be a connected $(2n+2)$-regular sr-graph with $\operatorname{Aut} H' \simeq \Gamma$. Since there are infinitely many such graphs, we may assume that H' has sufficiently many orbits with respect to $\operatorname{Aut} H'$, and there are even infinitely many such graphs from which we can construct infinitely many graphs that represent $\Gamma_1, ..., \Gamma_n$. Using Construction 2 we will construct G from H'. For convenience we consider $\operatorname{Aut} H'$ and Γ as equal, and consequently as in Construction 2 and Lemma 8 we will consider Γ as a subgroup of $\operatorname{Aut} G^0$. Therefore we will not explicitly mention the mapping ϕ as needed in the definition of representable.

For $m \leqslant n$ and $A \in \Omega_m$ choose $r(A)$, $(S_1(A), l_1(A)), ..., (S_{r(A)}(A), l_{r(A)}(A))$ according to Lemma 5. By the way, it would be enough to consider the elements of some subset Ω'_m of Ω_m with $\Gamma_m = \bigcap_{A \in \Omega'_m} \Gamma(A)$. We map the pairs (A, i) with $A \in \Omega_m$, $m \leqslant n$, $i \leqslant r(A)$ injectively into the set of orbits of H' and choose representatives $v_{A,i}$ of the corresponding orbits. Now we apply Construction 2 with

$$H_{(A,i),\alpha} := F(n+1, l_i(A), \varnothing) \qquad \text{for} \quad \alpha \in \Gamma(A)$$

and

$$H_{(A,i),\alpha} := F(n+1, l_i(A), S_i(A)) \qquad \text{for} \quad \alpha \notin \Gamma(A).$$

We will use Lemma 8(ii) and (iii) to determine $\operatorname{Aut} G^m$ for $1 \leqslant m \leqslant n$. Because of Lemma 4(ii) and Lemma 7 we only have to consider the mth powers of the graphs $H_{(A,i),\beta}$. First, let $\alpha \in \Gamma_m$. We have to show that for all (A, i) and all $\beta \in \Gamma$: $H^m_{(A,i),\beta} = H^m_{(A,i),\alpha\beta}$. If $\alpha \in \Gamma(A)$, then $\beta \in \Gamma(A)$ if and only if $\alpha\beta \in \Gamma(A)$, i.e., $H_{(A,i),\beta} = H_{(A,i),\alpha\beta}$. If $\alpha \notin \Gamma(A)$, then $m \in A$. By Lemma 5 there is no $S_i(A)$-odd m-interval in $\{1, ..., l_i(A)\}$, thus because of Lemma 4(iii) all the $H^m_{(A,i),\beta}$ are equal. Therefore $\alpha \in \operatorname{Aut} G^m$.

Now let $\alpha \notin \Gamma_m$. By hypothesis there is an $A \in \Omega_m$ with $\alpha \notin \Gamma(A)$ and by definition $m \notin A$. By Lemma 5 there exists an $i \leqslant r(A)$, such that there is an $S_i(A)$-odd m-interval in $\{1,..., l_i(A)\}$. Thus by Lemma 4(iii) $H^m_{(A,i),id} \neq H^m_{(A,i),\alpha}$, i.e., $\alpha \notin \operatorname{Aut} G^m$. Therefore $\Gamma_m = \operatorname{Aut} G^m$. ∎

COROLLARY 3. *Let for $n \in \mathbb{N}$ $\Gamma_1 \leqslant \cdots \leqslant \Gamma_n \leqslant \Gamma$ be finite groups. Then $\Gamma_1,..., \Gamma_n$ are representable by infinitely many graphs.*

Proof. We have $A = \{0, m+1, m+2,...\} \in \Omega_m$ and by the hypothesis $\Gamma(A) = \Gamma_m$- observe that $\Gamma_m \subseteq \bigcap_{A \in \Omega_m} \Gamma(A)$ is always true. ∎

Now we can show that for up to five groups the necessary condition of Corollary 1 is sufficient, too.

COROLLARY 4. *Let $n \leqslant 5$ and $\Gamma_1,..., \Gamma_n$ be subgroups of a finite group Γ. $\Gamma_1,..., \Gamma_n$ are representable if and only if for all $k, j, l \in \{0,..., n\}$ mit $k < j, l$ and $j + l - k \leqslant n$: $\Gamma_k \cap \Gamma_j \cap \Gamma_l \subseteq \Gamma_{j+l-k}$, where $\Gamma_0 := \Gamma$. If the groups are representable, then by infinitely many graphs.*

Proof. The necessity of the criterion is clear. We will prove the sufficiency with Theorem 3, but we need a preparation first. Without loss of generality, we have $n = 5$; if necessary we can put the groups that are not given $:= \Gamma$. By [2, Sect. A.3] there is a finite group Γ' with subgroups $\Gamma'_{3,5}$, $\Gamma'_{4,5}$, respectively, which are isomorphic to $\Gamma_{3,5} := \langle \Gamma_3 \cup \Gamma_5 \rangle$, $\Gamma_{4,5} := \langle \Gamma_4 \cup \Gamma_5 \rangle$, respectively, such that both isomorphisms coincide on Γ_5 and $\Gamma'_{3,5} \cap \Gamma'_{4,5} = \Gamma'_5$, where Γ'_5 is the image of Γ_5. By hypothesis $\Gamma_3 \cap \Gamma_4 \leqslant \Gamma_5$, hence the isomorphisms give a well-defined injective mapping ϕ: $\Gamma_3 \cup \Gamma_4 \cup \Gamma_5 \to \Gamma'$- apply the first isomorphism to $\Gamma_3 \cup \Gamma_5$, the second to $\Gamma_4 \cup \Gamma_5$. Because of $\Gamma_1 \leqslant \Gamma_2 \leqslant \Gamma_4$ ϕ is an injective mapping $\bigcup_{i=1}^5 \Gamma_i \to \Gamma'$ and $\phi|_{\Gamma_i}$ is a monomorphism onto a subgroup Γ'_i of Γ' for $i = 1,..., 5$. Now we have to prove that $\Gamma'_1,..., \Gamma'_5$ are representable. These groups also satisfy the hypothesis and furthermore $\langle \Gamma'_3 \cup \Gamma'_5 \rangle \cap \langle \Gamma'_4 \cup \Gamma'_5 \rangle = \Gamma'_5$. Hence we can apply Theorem 3 since

$$\Gamma'_1 = \Gamma'(\{0, 2, 3, \quad ...\}); \qquad \Gamma'_2 = \Gamma'(\{0, 3, 4,...\}),$$
$$\Gamma'_3 = \Gamma'(\{0, 2, 4, 5,...\}), \qquad \Gamma'_4 = \Gamma'(\{0, 3, 5, 6,...\}),$$
$$\Gamma'_5 = \Gamma'(\{0, 2, 4, 6, 7...\}) \cap \Gamma'(\{0, 3, 6, 7...\}). \quad ∎$$

For $n > 5$ there exist examples of representable groups that do not satisfy the criterion of Theorem 3 and for which our "trick" of the above proof does not work. We do not know of groups that satisfy Corollary 1 but are not representable. Although we do not have a necessary and sufficient condition for groups being representable, we can at least show that the necessary condition of Corollary 1 is best possible in its kind. For k_0, $k_1, \ldots, k_r \in \mathbb{N}_0$ we can deduce the necessity of $\Gamma_{k_1} \cap \cdots \cap \Gamma_{k_r} \subseteq \Gamma_{k_0}$ from Corollary 1, i.e., we can deduce "For all graphs G: $\operatorname{Aut} G^{k_1} \cap \cdots \cap \operatorname{Aut} G^{k_r} \subseteq \operatorname{Aut} G^{k_0}$" from Theorem 2, if and only if $k_0 \in A(\{k_1, \ldots, k_r\})$. (Observe that $A(\{k_1, \ldots, k_r\}) = \bigcup_{i=1}^{\infty} A_i$, where $A_1 := \{0, k_1, \ldots, k_r\}$ and $A_{i+1} := A_i \cup \{j + l - k \mid j, k, l \in A_i, 0 \leqslant k < j, l\}$.)

COROLLARY 5. *Let* $r \in \mathbb{N}$, $k_0, k_1, \ldots, k_r \in \mathbb{N}_0$. *If for all graphs* G $\operatorname{Aut} G^{k_1} \cap \cdots \cap \operatorname{Aut} G^{k_r} \subseteq \operatorname{Aut} G^{k_0}$, *then* $k_0 \in A\{(k_1, \ldots, k_r\})$.

Proof. Put $\Gamma := S_2$ and $n := \max_i k_i$, w.l.o.g. $n \neq 0$. For $m = 1, \ldots, n$ put $\Gamma_m := S_2$ iff $m \in A := A(\{k_1, \ldots, k_r\})$, $\Gamma_m := \{id\}$ otherwise. By Theorem 3 $\Gamma_1, \ldots, \Gamma_n$ are representable, because for $m \in A$ we have $\Gamma_m = \Gamma$ and for $m \notin A$ $\Gamma_m = \Gamma(A \cup \{m+1, m+2, \ldots\})$. Hence for a corresponding graph G we have $\operatorname{Aut} G^{k_1} = \cdots = \operatorname{Aut} G^{k_r} \simeq S_2$, and if the intersection of these groups is contained in $\operatorname{Aut} G^{k_0}$ then $k_0 \in A$. ∎

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