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Sharp regularity for general Poisson equations with borderline sources

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Abstract

This article concerns optimal estimates for nonhomogeneous degenerate elliptic equation with source functions in borderline spaces of integrability. We deliver sharp Hölder continuity estimates for solutions to p-degenerate elliptic equations in rough media with sources in the weak Lebesgue space $L^{\frac{n}{p}+\epsilon}_{\mathrm{weak}}$. For the borderline case, $f \in L^{\frac{n}{p}}_{\mathrm{weak}}$, solutions may not be bounded; nevertheless we show that solutions have bounded mean oscillation, in particular John–Nirenberg's exponential integrability estimates can be employed. All the results presented in this paper are optimal. Our approach is inspired by a powerful Caffarelli-type compactness method and it can be employed in a number of other situations.

Résumé

Dans cet article on établit des estimations optimales pour les solutions d'équations elliptiques non homogènes dégénérées lorsque les sources sont prises dans les espaces limites d'intégrabilité. On donne des estimations optimales de continuité höldérienne des solutions pour des équations elliptiques p-dégénérées dans des milieux grossiers lorsque les sources sont choisies dans des espaces de Lebesgue faibles, L^2_{faible}. Pour le cas limite, $f \in L^2_{\mathrm{faible}}$, les solutions ne sont pas nécessairement bornées, on peut néanmoins montrer que les solutions ont des oscillations bornées en moyenne ; en particulier on peut utiliser les estimations exponentielles d'intégrabilité de John–Nirenberg. Tous les résultats obtenus ici sont optimaux. Notre approche utilise un outil assez puissant de type compacité de Caffarelli, outil qui pourrait être utilisé dans d'autres nombreux cas.

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Keywords: Regularity theory; Borderline integrability; Degenerate elliptic equations

1. Introduction

Central theme in the theory of elliptic partial differential equations, the classical Poisson equation

$$-\Delta u = f(X), \tag{1.1}$$

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models important problems from theoretical physics, mechanical engineering to biology, economics, among many other applications. One of the key objectives in the analysis of Poisson equations is to assure regularity of u based on smoothness or integrability properties of its Laplacian, f. In this context, Schauder estimates are a fundamental result. It assures that the Hessian of u, D^2u, is as regular as f, provided f has an appropriate modulus of continuity. More precisely, if $f \in C^\alpha(B_1)$, $0<\alpha<1$ then $u \in C^{2,\alpha}(B_{1/2})$, and

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \leqslant C_n\big\{\|f\|_{C^\alpha(B_1)} + \|u\|_{L^\infty(B_1)}\big\}, \tag{1.2}$$

for a dimensional constant C_n. Schauder estimate is sharp in several ways. Clearly if $u \in C^{2,\alpha}$, then its Laplacian is α-Hölder continuous. Also if f is merely continuous, one cannot assure $u \in C^2$, nor even $C^{1,1}_{\mathrm{loc}}$ bounds are available. Schauder estimates also fail in the upper extreme, $\alpha = 1$, i.e., if $f \in \mathrm{Lip}$, it is *not true* in general that $u \in C^{2,1}_{\mathrm{loc}}$.

Establishing regularity of solutions to (1.1) reduces to understanding the behavior of the Newtonian potential of f,

$$N_f(X) := \int \frac{1}{|X-Y|^{n-2}} f(Y)\,dY. \tag{1.3}$$

The kernel that appears in (1.3), $\Gamma(X) = |X-Y|^{2-n}$, is the fundamental solution of the Laplacian. The second derivative of Γ, $D_{ij}\Gamma \sim |X-Y|^{-n}$ is not integrable, but it is almost integrable, in the sense that $|X-Y|^\epsilon D_{ij}\Gamma$ is integrable for any $0<\epsilon$. This is the key observation that explains why Schauder estimates hold when $f \in C^\alpha$, $0<\alpha<1$, and it fails when f is merely bounded or continuous.

In several applications, the source function f is not continuous, but only q-integrable, i.e., $f \in L^q(B_1)$, for some $1<q<\infty$. In this case, the corresponding regularity theory, due to Calderón and Zygmund, asserts that $u \in W^{2,q}(B_{1/2})$ and

$$\|u\|_{W^{2,q}(B_{1/2})} \leqslant C_n\big\{\|f\|_{L^q(B_1)} + \|u\|_{L^q(B_1)}\big\}. \tag{1.4}$$

In particular, if $f \in L^\infty$, then $u \in W^{2,q}$ for all $q<\infty$ and by Sobolev embedding, $u \in C^{1,\alpha}$ for any $\alpha<1$. This type of thesis is usually called *almost optimal regularity result*. Heuristically, for borderline hypotheses, *almost optimal regularity result* is the best one should hope for.

Regularity theory for problems in rough heterogeneous media, i.e., when governed by elliptic equations with measurable coefficients, is rather more sophisticated, and even for the homogeneous equation

$$\nabla \cdot \big(a_{ij}(X)Du\big) = 0,$$

solutions are, in general, known to be only Hölder continuous. This is the content of De Giorgi, Moser and Nash regularity theory. Calderón–Zygmund regularity estimates are not available in this setting. In even more complex models, the Laplacian in (1.1) is replaced by further involved nonlinear elliptic operators,

$$-\nabla \cdot a(X, Du) = f(X), \tag{1.5}$$

where $a : B_1 \times \mathbb{R}^n \to \mathbb{R}^n$ is p-degenerate elliptic vector field. Throughout this paper we shall always assume the following standard structural assumption on the vector field a:

$$\begin{cases} \big|a(X,\xi)\big| + \big|\partial_\xi a(X,\xi)\big||\xi| \leqslant \Lambda|\xi|^{p-1}, \\ \lambda|\xi_1|^{p-2}|\xi_2|^2 \leqslant \big\langle \partial_\xi a(X,\xi_1)\xi_2, \xi_2\big\rangle, \end{cases} \tag{1.6}$$

for positive constants $0<\lambda\leqslant\Lambda<+\infty$. As usual in the literature, we could also include a parameter $s\geqslant 0$ as to distinguish the model p-Laplacian operator ($s=0$) from the nondegenerate one ($s>0$), see for instance [14,15]. Throughout this paper, constants that depend only upon n, p, λ and Λ will be called *universal*.

We recall that Eq. (1.5) appears for instance as the Euler–Lagrange equation of the minimization problem

$$\int F(X,\nabla u) + f(X)u\,dX \to \min,$$

where the variational kernel $F(X,\xi)$ is convex in ξ, $F(X,\xi) \sim |\xi|^p$ and $F(X,\lambda\xi) = |\lambda|^p F(X,\xi)$. A typical operator to keep in mind is p-Laplacian in rough media,

$$-\nabla \cdot \big(a_{ij}(X)|\nabla u|^{p-2}\nabla u\big),$$

where a_{ij} is a bounded, positive definite matrix.

The regularity theory for Eq. (1.5) is nowadays fairly well established; however it is considerably more subtle than the corresponding linear, uniform elliptic theory. For instance, it is well known that p-harmonic functions are locally $C^{1,\alpha}$ for some α that depends on dimension and p. The precise value of optimal α is, in general, unknown.

The main goal of the present article is to determine optimal and *almost optimal* regularity estimates for solutions to Eq. (1.5), based on integrability properties (or more generally on the behavior of the distributional function) of the source f. The regularity estimates presented in this paper do not depend much on the concept of weak solution used. Indeed, they can be understood as *a priori* estimates that do not depend on any further regularity property of f or u. In the proofs, though, we shall always work with distributional solutions. However the same arguments go through, with no change, if one chooses to use the notion of entropy solutions, see [4] or any appropriate approximation scheme. Also we mention that, per our primary motivations, throughout the whole paper we shall only consider the range

$$1 < p < n, \tag{1.7}$$

where n is the dimension and p is the degeneracy exponent of the vector field $a(X, Du)$.

For L^∞-bounds of solutions to Eq. (1.5), the borderline integrability condition on the source function f is $L^{\frac{n}{p}}$. More precisely, if $f \in L^{\frac{n}{p}+\epsilon}$, for any tiny $\epsilon > 0$, solutions are bounded; however one cannot bound the L^∞-norm of u by the $L^{\frac{n}{p}}$ norm of f. The first result we show in this paper, Theorem 3.1, is an optimal BMO estimate of solutions with source functions in the weak Lebesgue space $L^{\frac{n}{p}}_{\text{weak}}$. Under slightly different structure assumptions, a similar result has been obtained by G. Mingione, Theorem 1.12 in [14], as a consequence of potential analysis considerations (see also [18]). Our proof is neither based on potential analysis nor on singular integral considerations. Instead, it is inspired by a powerful compactness type of argument, see [5,6], and also [1,2]. The case $p = n$, i.e., for the n-Laplacian equation, with f being a finite measure relates to the article [10]. These results could be delivered by our methods as well. We emphasize that in the case $p = n$, L^1_{weak} functions may not define a finite measure. Nevertheless, Theorem 3.1 provides *a priori* estimates for *a priori* regular solutions. When f is also a measure then this implies an existence and regularity theorem together with known approximation machineries.

As soon as the source function f becomes $(\frac{n}{p} + \epsilon)$-integrable, we show that solutions are in fact continuous. Not only do we show continuity of solution, but actually we provide the precise sharp Hölder exponent of continuity of u based only on the integrability of f and the regularity theory available for a-harmonic functions. Once more, the proof of such a result is based on compactness method and explores only the behavior of the distributional function of the source f, that is, f needs only to belong to the weak Lebesgue space $L^{\theta\cdot\frac{n}{p}}_{\text{weak}}$, $1 < \theta < p$. In this case, we show, see Theorem 4.1 and Theorem 4.2, that

$$u \in C^{\min\{\frac{p}{p-1}\cdot\frac{\theta-1}{\theta}, \alpha_0^-\}}_{\text{loc}},$$

where α_0 is the universal optimal Hölder exponent for solutions to $-\nabla \cdot a(X, Du) = 0$. Furthermore, we obtain the appropriate *a priori* universal estimate. Such a result brings important novelties. The first one is the optimal regularity space u lies. In many applications, for instance in free boundary and geometric problems, it is important to determine accurately how fast the solution grow away from its zero level set. In such a setting, knowing the precise regularity estimate is crucial for the program. Examples of such problems are equations with singular terms, $-\Delta_p u \sim u^{-\gamma}$, $\gamma > 1$. For these free boundary geometric problems, solutions are expected to behave like $|X|^\beta$, near a free boundary point. Thus it is important to establish regularity estimates where potentials are assumed to belong to $L^{\frac{n}{\beta\gamma}}_{\text{weak}}$, but not in the classical Lebesgue space $L^{\frac{n}{\beta\gamma}}$. Another important advantage of our approach concerns its flexibility, which allows further generalizations, for instance to equations with measure data, to systems, or even to p-degenerate equations in nondivergence form, $\mathcal{F}(X, u, \nabla u, D^2u) = f$, where $\mathcal{F}(X, \xi s, \xi p, \xi M) \sim \xi^{p-1}\mathcal{F}(X, s, p, M)$, for $\xi > 0$. For this class of problems, compactness is consequence of Harnack type inequality as in the original approach in [5]. When projected to the constant coefficient case, the optimal C^α estimate established in this paper is in accordance to the gradient estimates obtained in [15,9] through a powerful and sophisticated nonlinear potential theory. Indeed, for the model equation $-\Delta_p u = f \in L^{\theta\frac{n}{p}}_{\text{weak}}$, it follows from [15], Theorem 1 and [9], Theorem 1.1 that $\nabla u \in L^{\frac{\theta n(p-1)}{p-\theta}}_{\text{weak}}$, thus by the Morrey embedding theorem, $u \in C^{\frac{p}{p-1}\cdot\frac{\theta-1}{\theta}}$.

The paper is organized as follows. In Section 2 we prove a basic compactness Lemma which assures that if $\|f\|_{L^{\frac{n}{p}}_{\mathrm{weak}}}$ is small, then there exists an α_0-Hölder continuous function close to u in $L^p(B_{1/2})$. Section 3 is devoted to the proof BMO estimates. In Section 4 we address the optimal C^α regularity theory.

2. Compactness of solutions

In this section, we establish a compactness result for solutions to nonhomogeneous Poisson equations (1.5) that will play a fundamental role in the proof of Theorem 3.1 and Theorem 4.1. In fact Lemma 2.1 follows as a consequence of Lemma 3.2 in [8]. We include here a proof for completeness purposes.

Lemma 2.1. *Let $u \in W^{1,p}(B_1)$ be a weak solution to (1.5), with $\fint_{B_1} |u|^p\,dX \leqslant 1$. Given $\delta > 0$, there exists an $0 < \varepsilon \ll 1$, depending only on p, n, λ, Λ, ν and δ, such that if*

$$\|f\|_{L^{\frac{n}{p}}_{\mathrm{weak}}(B_1)} \leqslant \varepsilon, \tag{2.1}$$

then there exists a function h in $B_{1/2}$ satisfying

$$-\nabla \cdot \bar{a}(X, \nabla h) = 0, \quad \text{in } B_{1/2}, \tag{2.2}$$

for some vector field $\bar{a}$ satisfying (1.6) with the same ellipticity constants λ and Λ such that

$$\fint_{B_{1/2}} \left|u(X) - h(X)\right|^p dX < \delta^p.$$

Proof. Let us assume, for the purpose of contradiction, that the thesis of the lemma fails. If so, there would exist a $\delta_0 > 0$ and sequences

$$u_k \in W^{1,p}(B_1), \quad \text{and} \quad f_k \in L^{\frac{n}{p}}_{\mathrm{weak}}(B_1),$$

satisfying

$$\fint_{B_1} \left|u_k(X)\right|^p dX \leqslant 1, \tag{2.3}$$

for all $k \geqslant 1$,

$$-\nabla \cdot a_k(X, \nabla u_k) = f_k \quad \text{in } B_1, \tag{2.4}$$

where a_k satisfies (1.6) with constants λ and Λ, and

$$\|f_k\|_{L^{\frac{n}{p}}_{\mathrm{weak}}(B_1)} = \mathrm{o}(1), \tag{2.5}$$

as $k \to 0$; however

$$\fint_{B_{1/2}} \left|u_k(X) - h(X)\right|^p dX \geqslant \delta_0^p, \tag{2.6}$$

for any solution h to the homogeneous problem (2.2) in $B_{1/2}$ and all $k \geqslant 1$.

Now by standard Caccioppoli's type energy estimates, see for instance Theorem 6.5 and Theorem 6.1 in [11] (notice that $\frac{p^*}{p^*-1} < \frac{n}{p}$ within the range $1 < p < n$), we verify that there exists a constant $C = C(n, \lambda, \Lambda)$ such that

$$\int_{B_{1/2}} |\nabla u_k|^p\,dX \leqslant C,$$

for all $k \geqslant 1$. Thus, up to a subsequence, there exists a function $u \in W^{1,p}(B_{1/2})$ for which

$$u_k \rightharpoonup u \quad \text{in } W^{1,p}(B_{1/2}) \quad \text{and} \quad u_k \to u \quad \text{in } L^p(B_{1/2}). \tag{2.7}$$

In addition, in view of (2.4) and (2.5), by classical truncation arguments, see for instance [3], we know

$$\nabla u_k(X) \to \nabla u(X) \quad \text{for a.e. } X \in B_{1/2}. \tag{2.8}$$

Furthermore, by the Ascoli theorem, up to a subsequence, the sequence of vector fields $a_k(X,\cdot)$ converges locally uniformly to a vector field $\bar{a}$ satisfying (1.6). Given a test function $\phi \in W_0^{1,p}(B_{1/2})$, from (2.5), (2.7) and (2.8) we have

$$\begin{aligned}\int_{B_{1/2}} \bar{a}(X, Du)\cdot \nabla\phi\, dX &= \int_{B_{1/2}} a_k(X, Du_k)\cdot\nabla\phi\, dX + \mathrm{o}(1)\\ &= \int_{B_{1/2}} f_k\varphi\, dX + \mathrm{o}(1)\\ &= \mathrm{o}(1),\end{aligned}$$

as $k\to\infty$. Since ϕ was arbitrary, we conclude that u is a solution to the homogeneous equation in $B_{1/2}$. Finally we reach a contradiction in (2.6) for $k \gg 1$. The proof of Lemma 2.1 is concluded. □

Remark 2.2. Arguing as in [8], Lemma 3.2, it is possible to avoid the passage to the limit in the proof of Lemma 2.1, obtaining therefore a function h, solution to the homogeneous equation $\nabla\cdot a(X,\nabla h) = 0$, for the original vector field a. For our purposes though, it suffices to obtain an equation within the same universal class of a, (1.6).

3. Optimal BMO estimates

In this section we shall establish optimal *a priori* estimates for solutions to

$$-\nabla\cdot a(X, Du) = f \in L^{\frac{n}{p}}_{\mathrm{weak}}(B_1),$$

which corresponds to the lower borderline integrability condition on f. In particular, L^∞ bounds cannot be achieved under such a weak hypothesis. We recall that a measurable function f is said to belong to the weak-$L^p(B_1)$ space, denoted by $L^p_{\mathrm{weak}}(B_1)$, if there exists a constant $K>0$ for which

$$\mathcal{L}^n\big(\big\{X\in B_1\colon \big|f(X)\big| > \tau\big\}\big) \leqslant \frac{K^p}{\tau^p}. \tag{3.1}$$

The infimum of all $K>0$ for which (3.1) holds is defined to be the weak-L^p norm of u and it is denoted by $\|u\|_{L^p_{\mathrm{weak}}(B_1)}$. Weak-$L^p$ spaces play a fundamental role in Harmonic Analysis, in particular in the theory of singular integrals. It is well known that $L^p \subsetneq L^p_{\mathrm{weak}}$. Also, if $\mathcal{M}$ denotes the Hardy–Littlewood maximal operator, then $\mathcal{M}(f)\in L^1_{\mathrm{weak}}$ provided $f\in L^1$, and such a result is optimal in the sense that $\mathcal{M}(f)$ may not belong to L^1. This is the main reason for which Calderón–Zygmund theory fails for sources in L^1.

To motivate the result of this section, we invite the readers to notice that a careful inference in the kernel from (1.3) revels a lower borderline condition for the source function f. In fact, $\Gamma\in L^r$ for any $r<\frac{n}{n-2}$, but $\Gamma\notin L^{\frac{n}{n-2}}$. That is, by Hölder inequality,

$$N_f\in L^\infty \quad \text{whenever } f\in L^{\frac{n}{2}+\epsilon},$$

since $\frac{n}{2}$ is the dual exponent of $\frac{n}{n-2}$. When $f\in L^{\frac{n}{2}}$, $n\geqslant 3$, Calderón–Zygmund estimate (1.4) reveals that

$$u\in W^{2,\frac{n}{2}} \hookrightarrow W^{1,n}\hookrightarrow L^q, \tag{3.2}$$

for any $1<q<\infty$. That is, it provides an *almost optimal regularity result*. By a duality argument, one finds out that it is impossible to bound the L^∞_{loc}-norm of u by the $L^{\frac{n}{2}}$-norm of f. However, an application of Poincaré inequality combined with (3.2) gives

$$\fint_{B_r} |u - u_r|^n \, dX \leqslant C_n \int_{B_r} |\nabla u|^n \, dX \leqslant C_n \|f\|_{L^{\frac{n}{2}}}, \tag{3.3}$$

where, u_r denotes the mean of u over B_r, i.e., $u_r := \fint_{B_r(X_0)} u \, dY$.

Recall a function $u \in L^1(B_1)$ for which there exists a constant $K > 0$ such that

$$\fint_{B_r(X_0)} |u - u_r| \leqslant K, \tag{3.4}$$

for every $X_0 \in B_1$ and $0 < r < \operatorname{dist}(X_0, \partial B_1)$, is said to belong to the BMO space. The infimum of all $K > 0$ for which (3.4) holds is defined to be the BMO-norm of u and it is denoted by $\|u\|_{\mathrm{BMO}}$.

The BMO space was originally introduced by John and Nirenberg in [13]. In that very same paper, John and Nirenberg proved the following fundamental estimate: if $\|u\|_{\mathrm{BMO}} \leqslant 1$, then there exist positive dimensional constants α and β such that

$$\fint_{B_1} e^{\alpha |u - u_1|} \, dX \leqslant \beta. \tag{3.5}$$

The original motivation for studying these functions apparently came from the theory of elasticity, [12]. Interestingly enough, John–Nirenberg's estimate for BMO functions (3.5) is used by Moser as a key ingredient in his striking proof of Harnack inequality for divergence form uniform elliptic equations. Both John–Nirenberg and Moser works were published simultaneously in the same issue: Comm. Pure Appl. Math., vol. XIV, back in 1961.

Through the years, BMO space and its analogues have been shown to enjoy many other properties, with deep applications in analysis. For our purposes, it is elucidative to think the BMO space as the correct substitute for L^∞ as the endpoint of the L^p spaces as $p \uparrow +\infty$.

In what follows, we will establish the corresponding sharp BMO estimate for solutions to p-degenerate elliptic equations

$$-\nabla \cdot a(X, Du) = f \in L^{\frac{n}{p}}_{\mathrm{weak}}(B_1), \tag{3.6}$$

where a satisfies the standard structural condition (1.6).

Theorem 3.1. *Let $u \in W^{1,p}(B_1)$ be a solution to*

$$-\nabla \cdot a(X, Du) = f(X).$$

Assume a satisfies (1.6) and $f \in L^{\frac{n}{p}}_{\mathrm{weak}}(B_1)$. Then $u \in BMO(B_{1/2})$. Furthermore,

$$\|u\|_{\mathrm{BMO}(B_{1/2})} \leqslant C\Big(\|f\|^{\frac{1}{p-1}}_{L^{\frac{n}{p}}_{\mathrm{weak}}(B_1)} + \|u\|_{L^p(B_1)}\Big),$$

for a constant C that depends only on n p, λ and Λ.

In view of the parallel described above to the linear theory, the estimate from Theorem 3.1 should be optimal. Indeed, this is the case. For instance, say, for $p < n$, if we set $f(X) = |X|^{-p}$, it is easy to see that $f \in L^{\frac{n}{p}}_{\mathrm{weak}}$. Solving $\Delta_p u = f$ with constant boundary data on ∂B_1 one finds $u(X) = c_{n,p} \cdot \ln|X|$, which is in BMO but not in L^∞.

The proof of Theorem 3.1 will be based on the compactness result granted in Lemma 2.1 and an iterative scheme. The following lemma is pivotal to our strategy.

Lemma 3.2. *Let $u \in W^{1,p}(B_1)$ be a weak solution to (1.5), with $\fint_{B_1} |u|^p \, dX \leqslant 1$. There exist constants $0 < \varepsilon_0 \ll 1$, $0 < \lambda_0 \ll 1/2$, that depend only on n, p, λ and λ, such that if*

$$\|f\|_{L^{\frac{n}{p}}_{\mathrm{weak}}(B_1)} \leqslant \varepsilon_0, \tag{3.7}$$

then

$$\fint_{B_{\lambda_0}} \left| u(X) - \left(\fint_{B_{\lambda_0}} u\,dY \right) \right|^p dX \leqslant 1. \tag{3.8}$$

Proof. Initially let us recall a general inequality:

$$\fint_{B_r} \left| u - \fint_{B_r} u\,dY \right|^p dX \leqslant 2^p \fint_{B_r} |u-\gamma|^p\,dX, \tag{3.9}$$

for any $u \in L^p$ and any real number γ. Indeed, by triangular inequality,

$$\begin{aligned} \left(\fint_{B_r} \left| u - \fint_{B_r} u\,dY \right|^p dX \right)^{1/p} &\leqslant \left(\fint_{B_r} |u-\gamma|^p\,dX \right)^{1/p} + \left| \fint_{B_r} u\,dY - \gamma \right| \\ &\leqslant \left(\fint_{B_r} |u-\gamma|^p\,dX \right)^{1/p} + \fint_{B_r} |u-\gamma|\,dX \\ &\leqslant 2\left(\fint_{B_r} |u-\gamma|^p\,dX \right)^{1/p}. \end{aligned}$$

In view of Lemma 2.1, let h be a solution to the homogeneous equation in $B_{1/2}$ such that

$$\fint_{B_{1/2}} \left| u(X) - h(X) \right|^p dX \leqslant \frac{7\lambda_0^n}{9 \cdot 2^{2p-1}}, \tag{3.10}$$

for $\lambda_0 \ll 1/2$ to be regulated soon. Such a choice will determine ε_0. Notice that (3.10) implies $\fint_{B_{1/2}} |h(X)|^p\,dX \leqslant C$, thus, by regularity theory for homogeneous equation, there exists a constant $C > 0$ universal such that

$$\left| h(X) - h(0) \right| \leqslant C|X|^{\alpha_0},$$

where C depends only on n, p, λ and Λ. Next, for $\lambda_0 \ll 1/2$ to be chosen, we estimate

$$\begin{aligned} \fint_{B_{\lambda_0}} \left| u(X) - h(0) \right|^p dX &\leqslant 2^{p-1} \left(\fint_{B_{\lambda_0}} \left| u(X) - h(X) \right|^p dX + \fint_{B_{\lambda_0}} \left| h(X) - h(0) \right|^p dX \right) \\ &\leqslant \frac{7}{9 \cdot 2^p} + C 2^{p-1} \cdot \lambda_0^{\alpha_0 p}. \end{aligned} \tag{3.11}$$

Now we can choose λ_0, depending on dimension n and p, λ and Λ so small that

$$C 2^{p-1} \cdot \lambda_0^{\alpha_0 p} \leqslant \frac{2}{9 \cdot 2^p}, \tag{3.12}$$

and the proof of Lemma 3.2 follows from (3.11) and (3.9). □

Proof of Theorem 3.1. Let u be a weak solution to

$$-\nabla \cdot a(X, Du) = f(X), \quad \text{in } B_1.$$

The proof starts off with a renormalization. Let ε_0 be the universal constant from Lemma 3.2. If we change u by κu, with $\kappa \ll 1$, so small that

$$\kappa^{p-1} \leqslant \frac{\varepsilon_0}{\|f\|_{L^{\frac{n}{p}}_{\text{weak}}(B_1)}} \quad \text{and} \quad \fint_{B_1} |\kappa \cdot u|^p\,dX \leqslant 1,$$

we can assume u and f are under the hypotheses of Lemma 3.2. In the sequel, we will show

$$\fint_{B_{\lambda_0^k}} |u - c_k|^p \, dX \leqslant 1, \quad \forall k \in \mathbb{N}. \tag{3.13}$$

Here λ_0 is the universal number from Lemma 3.2 and c_k denotes the average of u over the ball of radio λ_0^k, i.e.,

$$c_k := \fint_{B_{\lambda_0^k}} u(X) \, dX.$$

We show (3.13) by induction. The case $k = 1$ follows directly from Lemma 3.2. Assume we have verified (3.13) for k. We define the real function $v : B_1 \to \mathbb{R}$ by

$$v(X) := u\big(\lambda_0^k X\big) - c_k \tag{3.14}$$

We also define

$$a_{\lambda_0^k}(X, \xi) := a\big(\lambda_0^k X, \xi\big) \quad \text{and} \quad L_{\lambda_0^k}\phi := -\nabla \cdot a_{\lambda_0^k}(X, D\phi). \tag{3.15}$$

Notice that $a_{\lambda_0^k}$ is also p-degenerate elliptic, with the same ellipticity constants as a. From the induction assumption, we have

$$\fint_{B_1} \big|v(X)\big|^p \, dX = \fint_{B_{\lambda_0^k}} \big|u(Y) - c_k\big|^p \, dY \leqslant 1. \tag{3.16}$$

Easily one verifies that

$$\big|L_{\lambda_0^k} v(X)\big| \leqslant \lambda_0^{kp} \big|f\big(\lambda_0^k X\big)\big|, \quad \text{a.e. in } B_1. \tag{3.17}$$

If we label $f_{\lambda_0^k} := \lambda_0^{pk} |f(\lambda_0^k X)|$, a direct computation reveals

$$\begin{aligned}\mathcal{L}^n\big(\big\{X \in B_1 \colon |f_{\lambda_0^k}| > \tau\big\}\big) &= \mathcal{L}^n\left(\left\{X \in B_1 \colon |f| \geqslant \frac{\tau}{\lambda_0^{kp}}\right\}\right) \cdot \lambda_0^{-nk} \\ &\leqslant \frac{\|f\|^{\frac{n}{p}}_{L^{\frac{n}{p}}_{\mathrm{weak}}}}{\tau^{\frac{n}{p}}}.\end{aligned} \tag{3.18}$$

That is

$$\|f_{\lambda_0^k}\|_{L^{\frac{n}{p}}_{\mathrm{weak}}} \leqslant \|f\|_{L^{\frac{n}{p}}_{\mathrm{weak}}} \leqslant \varepsilon_0. \tag{3.19}$$

We have verified that v is under the hypotheses of Lemma 3.2, which assures

$$\fint_{B_{\lambda_0}} \left|v(X) - \left(\fint_{B_{\lambda_0}} v \, dY\right)\right|^p dX = \fint_{B_{\lambda_0^{k+1}}} \big|u(X) - c_{k+1}\big|^p \, dX \leqslant 1. \tag{3.20}$$

This concludes the proof of (3.13). Finally, given $0 < r \ll 1$, let $m \in \mathbb{N}$ be such that

$$\lambda_0^{m+1} \leqslant r < \lambda_0^m.$$

If we label $u_r := \fint B_r u \, dY$, we estimate,

$$\begin{aligned}\fint_{B_r} |u - u_r|^p \, dX &\leqslant 2^p \fint_{B_r} |u - \lambda_0 u_r|^p \, dX \\ &\leqslant \frac{2^p}{\lambda_0} \fint_{B_{\lambda_0^m}} |u - c_m|^p \, dX \\ &\leqslant C.\end{aligned}$$

The proof of Theorem 3.1 can be now concluded my means of a standard covering argument, which we shall omit here. □

We finish up this section by highlighting once more that the strategy used in our reasoning to establish Theorem 3.1 is indeed quite flexible. It is based on a fine scaling balance between the norm of the source f and the homogeneity of the equation itself. This indicates that similar analysis should be possible to be carried on for equations with measure data, provided the solution already lies in a proper Sobolev space, under the classical diffusion assumption $|f|(B_r) \leqslant Cr^{n-p}$, for any ball B_r of radius r. For that, though, one needs to revisit the proof of Lemma 2.1 and work under appropriate notion of solutions through truncation. We do not intend to pursue that in this present paper.

4. C^α regularity

In this section we turn our attention to optimal regularity estimates to Eq. (1.5) when the source function f lies in a slightly better space, say, $f \in L^{\frac{n}{p}+\epsilon}_{\text{weak}}$. In this case, heuristic scaling methods indicate that weak solutions should be locally bounded. Indeed, under slightly stronger assumptions on f, boundedness or even continuity of solutions can be delivered by known methods, for instance through Serrin's Harnack inequality [16]. Nevertheless this approach hardly reveals the sharp Hölder exponent of continuity of the solution.

In this section we still work under assumption (1.6). As we have already invoked, it is classical, see for instance [16], that $W^{1,p}$ solutions to the homogeneous equation

$$-\nabla \cdot a(X, Du) = 0, \quad \text{in } B_1, \tag{4.1}$$

are α_0-Hölder continuous in $B_{1/2}$, and

$$\|u\|_{C^{\alpha_0}(B_{1/2})} \leqslant C(n, \lambda, \Lambda, p)\|u\|_{L^p(B_1)}. \tag{4.2}$$

The optimal exponent α_0 in (4.2) depends only upon dimension, p and ellipticity constants λ, and Λ. In general $\alpha_0 < 1$ and its precise value is unknown.

Theorem 4.1. *Let $u \in W^{1,p}(B_1)$ be a solution to*

$$-\nabla \cdot a(X, Du) = f(X). \tag{4.3}$$

Assume (1.6) and $f \in L^{\theta\cdot\frac{n}{p}}_{\text{weak}}(B_1)$, $1 < \theta < p$. Then $u \in C^\alpha(B_{1/2})$, for

$$\alpha = \min\left\{\frac{p}{p-1}\cdot\frac{\theta-1}{\theta}, \alpha_0^-\right\}, \tag{4.4}$$

where α_0 is the universal optimal Hölder exponent for solutions to $-\nabla \cdot a(X, Du) = 0$. Furthermore,

$$\|u\|_{C^\alpha(B_{1/2})} \leqslant C(n, \lambda, \Lambda, p, \theta)\Big(\|f\|^{\frac{1}{p-1}}_{L^{\theta\cdot\frac{n}{p}}_{\text{weak}}(B_1)} + \|u\|_{L^p(B_1)}\Big).$$

The sharp relation in (4.4) should be read as follows:

$$\left|\begin{array}{ll} \text{If } \dfrac{p}{p-1}\cdot\dfrac{\theta-1}{\theta} < \alpha_0 & \text{then } u \in C^{\frac{p}{p-1}\cdot\frac{\theta-1}{\theta}}_{\text{loc}}. \\ \text{If } \dfrac{p}{p-1}\cdot\dfrac{\theta-1}{\theta} \geqslant \alpha_0 & \text{then } u \in C^{\alpha}_{\text{loc}}, \text{ for any } \alpha < \alpha_0. \end{array}\right. \tag{4.5}$$

The proof of Theorem 4.1 will be given in Section 4.1 below. Optimality of the thesis of Theorem 4.1 can be checked directly by computing in the unit ball, B_1

$$\Delta_p|X|^{\frac{p}{p-1}\cdot\frac{\theta-1}{\theta}} = c|X|^{-\frac{p}{\theta}} \in L^{\theta\cdot\frac{n}{p}}_{\text{weak}}.$$

It is interesting to notice that $|X|^{-\frac{p}{\theta}}$ is not in the classical Lebesgue space $L^{\theta\cdot\frac{n}{p}}$. A valuable feature of Theorem 4.1 is the fact that it provides universal bounds, i.e., Hölder estimates that depend only on ellipticity and p-degeneracy feature of the operator. This is particularly important in homogenization problems. However, under continuity (or some sort of VMO condition) on the medium, we can show that solutions to the homogeneous equation

$$-\nabla \cdot a(X, Du) = 0,$$

are C^α for every $\alpha < 1$. Indeed this fact is an immediate consequence of our next theorem.

In the sequel, we shall slightly improve the thesis of Theorem 4.1, provided the medium has some sort of continuity property. For simplicity purposes, for the next theorem, we shall work under classical continuity assumption on the operator a with respect to the X variable. That is, there exists a modulus of continuity τ such that

$$\left|a(X, \xi) - a(Y, \xi)\right| \leqslant \tau\big(|X - Y|\big)|\xi|^{p-1}. \tag{C}$$

We remark that under the structural assumption (1.6) solutions to the homogeneous, constant coefficient equation have *a priori* $C^{1,\epsilon}$ estimates for $X_0 \in B_{1/2}$ fixed. That is

$$-\nabla a(X_0, Dh) = 0,\ B_1 \quad \text{implies} \quad \|h\|_{C^{1,\epsilon}(B_{2/3})} \leqslant C(n, p, \lambda, \Lambda)\|h\|_{L^p(B_1)}, \tag{4.6}$$

for some $0 < \epsilon < 1$ that depends only on p, n, λ and Λ, see, for instance, [7].

Theorem 4.2. *Let $u \in W^{1,p}(B_1)$ be a solution to*

$$-\nabla \cdot a(X, Du) = f(X). \tag{4.7}$$

Assume (1.6), (C) *and that $f \in L^{\theta \cdot \frac{n}{p}}(B_1)$, $1 < \theta < p$. Then $u \in C^{\frac{p}{p-1} \cdot \frac{\theta-1}{\theta}}(B_{1/2})$ and furthermore,*

$$\|u\|_{C^{\frac{p}{p-1} \cdot \frac{\theta-1}{\theta}}(B_{1/2})} \leqslant C(n, \lambda, \Lambda, p, \tau, \theta)\Big(\|f\|_{L_{\text{weak}}^{\theta \cdot \frac{n}{p}}(B_1)} + \|u\|_{L^p(B_1)}\Big).$$

Before delivering the proofs of Theorem 4.1 and Theorem 4.2, let us make few comments about our C^α regularity estimates. Initially, as in Theorem 3.1, it seems reasonable to establish the same optimal result for measure data f, provided $|f|(B_r) \leqslant Cr^{\frac{\theta n - p}{\theta}}$, for any ball of radius r. As for Theorem 4.2, continuity condition can be greatly relaxed. In fact all we need is a sort of Cordes–Nirenberg type of condition: there exists a universal constant $\delta_\star > 0$ such that

$$\left|a(X, \xi) - a(0, \xi)\right| \leqslant \delta_\star |\xi|^{p-1}.$$

The upper threshold case for continuity theory, $f \in L^n$, is a delicate issue, see [17]. At this point, though, an interesting consequence of Theorem 4.1 is that solutions to

$$-\nabla a(X, Du) = f \in L^n_{\text{weak}}(B_1),$$

for measurable coefficients equations, have *almost* the same modulus of continuity as a-harmonic functions, i.e., solutions to $-\nabla a(X, Dh) = 0$. That is, if a-harmonic functions in B_1 are locally C^{α_0}, then solutions to $-\nabla a(X, Du) = f \in L^n_{\text{weak}}(B_1)$ are locally C^β, for any $0 < \beta < \alpha_0$. The same analysis employed in Theorem 4.2 gives that for equations with continuous coefficients, solutions to $-\nabla a(X, Du) = f \in L^n_{\text{weak}}(B_1)$ are locally C^β, for any $0 < \beta < 1$.

4.1. Proof of Theorem 4.1

We revisit the proof of Lemma 3.2. Suppose $⨍ B_1 |u|^p\, dX \leqslant 1$ and for $q = \theta \cdot p/n$,

$$\varepsilon_1 \geqslant \|f\|_{L^q_{\text{weak}}(B_1)} \geqslant c_n \|f\|_{L_{\text{weak}}^{\frac{n}{p}}},$$

with $\varepsilon_1 > 0$ to be chosen. From Lemma 2.1 there exists a function h, solution to

$$-\nabla \cdot a(X, Dh) = 0, \quad \text{in } B_{1/2},$$

such that

$$\fint_{B_{1/2}} \left|u(X) - h(X)\right|^p \leqslant \delta_1.$$

The latter choice for δ_1 determines ε_1 through the compactness Lemma 2.1. Since $\|h\|_{L^p}$ is under control, the regularity theory for homogeneous equation assures $h \in C^{\alpha_0}(B_{1/3})$ and for a universal constant $C > 0$,

$$\left|h(X) - h(0)\right| \leqslant C|X|^{\alpha_0}.$$

We can readily estimate

$$\begin{aligned}\fint_{B_{\lambda_1}} \left|u(X) - h(0)\right|^p dX &\leqslant 2^{p-1}\left(\fint_{B_{\lambda_1}} \left|u(X) - h(X)\right|^p dX + \fint_{B_{\lambda_1}} \left|h(X) - h(0)\right|^p dX\right)\\ &\leqslant 2^{p-1}\delta_1\lambda_1^{-n} + 2^{p-1}\lambda_1^{p\alpha_0}.\end{aligned} \tag{4.8}$$

Now, fixed $\alpha < \alpha_0$ we can choose $\lambda_1 \ll 1$ universally small so that

$$2^{p-1}\lambda_1^{p\alpha_0} \leqslant \frac{1}{10}\lambda_1^{\alpha}. \tag{4.9}$$

Once λ_1 is chosen as indicated above, we select δ_1 (and therefore ε_1) as

$$2^{p-1}\delta_1 = \frac{9}{10}\lambda_1^{n+\alpha}. \tag{4.10}$$

If we combine (4.8), (4.9) and (4.10) we conclude that

$$\fint_{B_{\lambda_1}} \left|u(X) - h(0)\right|^p dX \leqslant \lambda_1^{p\alpha}, \tag{4.11}$$

provided

$$\|f\|_{L^q_{\text{weak}}(B_1)} \leqslant \varepsilon_1, \tag{4.12}$$

for $0 < \varepsilon_1 \ll 1$ that depends only on dimension, p λ, Λ and $\alpha < \alpha_0$. In addition, from the regularity theory for homogeneous equation,

$$\left|h(0)\right| \leqslant C, \tag{4.13}$$

for a universal constant $C > 0$.

We remind that the assumptions $\fint B_1 |u|^p\, dX \leqslant 1$ and $\|f\|_{L^q_{\text{weak}}(B_1)} \leqslant \varepsilon_1$ can be reached by a simple change of scaling and normalization. Thus, with no loss of generality, we can work under these hypotheses.

In the sequel we shall prove that there exists a convergent sequence $\{\mu_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ for which

$$\fint_{B_{\lambda_1^k}} \left|u(X) - \mu_k\right|^p dX \leqslant \lambda_1^{kp\alpha}. \tag{4.14}$$

As before, we will verify (4.14) by induction. The case $k = 1$ is precisely (4.11), with $\mu_1 = h(0)$. Suppose we have checked (4.14) for $k = 1, 2, \dots, m$. Define

$$v(X) := \frac{u(\lambda_1^m X) - \mu_m}{\lambda_1^{m\alpha}}. \tag{4.15}$$

With the same notation as in (3.15), we readily verify, as in (3.17), that

$$\left|L_{\lambda_1^{\alpha m}} v(X)\right| \leqslant \lambda_1^{m[p-(p-1)\alpha]}\left|f\left(\lambda_1^m X\right)\right| =: f_m(X). \tag{4.16}$$

One easily estimates, for any $\tau > 0$,

$$\mathcal{L}^n\big(\big\{X \in B_1\colon\ |f_m| > \tau\big\}\big) = \mathcal{L}^n\left(\left\{X \in B_1\colon\ |f| \geqslant \frac{\tau}{\lambda_1^{m[p-(p-1)\alpha]}}\right\}\right) \cdot \lambda_1^{-m\cdot n}$$
$$\leqslant \frac{\|f\|^q_{L^q_{\text{weak}}}}{\tau^q} \cdot \big[\lambda_1^{m[p-(p-1)\alpha]q} \cdot \lambda_1^{-m\cdot n}\big]$$
$$\leqslant \varepsilon_1^q, \tag{4.17}$$

in view of the sharp assumption (4.4). We have shown that v is entitled to the conclusion in (4.11). Let h_m be the solution to the homogeneous problem that is $\sqrt[p]{\delta_1}$-close to v in $B_{1/2}$ in the L^p-distance. We label $h_m(0) = t_m$ and, as in (4.13), $|t_m| < C$ for a universal constant. Applying (4.11) to v we find

$$⨍_{B_{\lambda_1}} \big|v(X) - t_m\big|^p\, dX \leqslant \lambda_1^{p\alpha}. \tag{4.18}$$

Rescaling (4.18) back yields

$$⨍_{B_{\lambda_1^{m+1}}} \big|u(X) - \big(\mu_m + \lambda_1^{m\alpha} t_m\big)\big|^p\, dX \leqslant \lambda_1^{p\alpha(m+1)}. \tag{4.19}$$

Therefore, the induction step for (4.14) is verified by taking

$$\mu_{m+1} := \mu_m + \lambda_1^{m\alpha} t_m.$$

Indeed $\{\mu_k\}_{k\in\mathbb{N}}$ is a convergent sequence, because we estimate

$$|\mu_{k+j} - \mu_k| \leqslant C\frac{\lambda_1^{\alpha k}}{1-\lambda_1^{\alpha}} = \mathrm{o}(1),$$

as $k \to \infty$. Finally, if we define

$$\bar{\mu} := \lim_{k\to\infty} \mu_k,$$

and $0 < r < 1$ is arbitrary, estimate (4.14) gives

$$⨍_{B_r} \big|u(X) - \bar{\mu}\big|^p\, dX \leqslant C r^{p\alpha};$$

therefore u is α-Hölder continuous at the origin. The proof of Theorem 4.1 follows now via standard covering arguments, which we omit here. □

4.2. Proof of Theorem 4.2

For convenience, let us label $q := \theta \cdot \frac{p}{n} > \frac{p}{n}$. The proof of Theorem 4.2 is based on the following refinement of the Compactness Lemma 2.1.

Lemma 4.3. *Let $u \in W^{1,p}(B_1)$ be a weak solution to (1.5), with $⨍B_1 |u|^p\, dX \leqslant 1$. Given $\delta > 0$, there exists an $0 < \varepsilon \ll 1$, depending only on p, n, λ, Λ and δ such that if*

$$\|f\|_{L^q_{\text{weak}}(B_1)} \leqslant \varepsilon, \quad \text{and} \quad \big|a(X,\xi) - a(0,\xi)\big| \leqslant \varepsilon|\xi|^{p-1}, \tag{4.20}$$

then there exists a function h in $B_{1/2}$, solution to

$$-\nabla \cdot \bar{a}(Dh) = 0, \quad \text{in } B_{1/2}, \tag{4.21}$$

for some constant coefficient vector field $\bar{a}$ satisfying (1.6) with the same ellipticity constants λ and Λ, such that

$$⨍_{B_{1/2}} \big|u(X) - h(X)\big|^p\, dX < \delta^p.$$

Proof. As before, let us assume, searching for a contradiction, that the thesis of the lemma fails. If so, there would exist a $\delta_0 > 0$ and sequences

$$u_k \in W^{1,p}(B_1), \quad \text{and} \quad f_k \in L^q_{\mathrm{weak}}(B_1),$$

with

$$\fint_{B_1} \left|u_k(X)\right|^p dX \leqslant 1, \tag{4.22}$$

for all $k \geqslant 1$,

$$-\nabla \cdot a_k(X, Du_k) = f_k \quad \text{in } B_1, \tag{4.23}$$

where a_k satisfies (1.6), and

$$\|f_k\|_{L^q(B_1)} + |\xi|^{1-p}\left|a_k(X,\xi) - a_k(0,\xi)\right| = \mathrm{o}(1), \tag{4.24}$$

as $k \to 0$; however

$$\fint_{B_{1/2}} \left|u_k(X) - h(X)\right|^p dX \geqslant \delta_0, \tag{4.25}$$

for any solution h to a homogeneous, constant coefficient equation (4.21), in $B_{1/2}$ and all $k \geqslant 1$. Reasoning as indicated in the proof of Lemma 2.1, we have

$$\int_{B_{1/2}} |\nabla u_k|^p \, dX \leqslant C,$$

for all $k \geqslant 1$. Thus, up to a subsequence, there exists a function $u \in W^{1,p}(B_{1/2})$ for which

$$u_k \rightharpoonup u \quad \text{in } W^{1,p}(B_{1/2}), \qquad u_k \to u \quad \text{in } L^p(B_{1/2}), \quad \text{and} \quad \nabla u_k(X) \to \nabla u(X) \text{ a.e. in } B_{1/2}. \tag{4.26}$$

Also, by the Ascoli theorem, there exists a subsequence under which $a_{k_j}(0,\cdot) \to \bar{a}(0,\cdot)$ locally uniformly. Thus, for any $X \in B_{1/2}$,

$$\left|a_{k_j}(X,\xi) - \bar{a}(0,\xi)\right| \leqslant \left|a_{k_j}(X,\xi) - a_{k_j}(0,\xi)\right| + \left|a_{k_j}(0,\xi) - \bar{a}(0,\xi)\right| = \mathrm{o}(1), \tag{4.27}$$

that is, $a_{k_j}(X,\cdot) \to \bar{a}(0,\cdot)$ locally uniformly. Finally, given a test function $\phi \in W^{1,p}_0(B_{1/2})$, in view of (4.24), (4.26) and (4.27) we have

$$\begin{aligned}
\int_{B_{1/2}} a_k(X, \nabla u_k) \cdot \nabla \phi \, dX &= \int_{B_{1/2}} f_k \varphi \, dX \\
&= \int_{B_{1/2}} \bar{a}(0, \nabla u) \cdot \nabla \phi \, dX + \mathrm{o}(1),
\end{aligned}$$

as $k \to \infty$. Since ϕ was arbitrary, we conclude that u is a solution to a constant coefficient equation in $B_{1/2}$. Finally we reach a contradiction in (4.25) for $k \gg 1$. □

The main difference between Lemma 2.1 and Lemma 4.3 is the fact that the former provides existence of a C^{α_0} function close to u under smallness assumptions on the data. The latter gives a C^1 function near u under smallness assumptions that also involve continuity of the medium. Thus, the following version of Lemma 3.2 can be proven by similar arguments used to establish estimate (4.11).

Lemma 4.4. *Let $u \in W^{1,p}(B_1)$ be a weak solution to (1.5), with $⨍_{B_1} |u|^p \, dX \leqslant 1$. Given $\alpha < 1$, there exist constants $0 < \varepsilon_0 \ll 1$, $0 < \lambda_0 \ll 1/2$ and $c_0 \in \mathbb{R}$ such that if*

$$\|f\|_{L^q_{\text{weak}}(B_1)} \leqslant \varepsilon_0 \quad \textit{and} \quad \bigl|a(X, \xi) - a(0, \xi)\bigr| \leqslant \varepsilon_0 |\xi|^{p-1}, \tag{4.28}$$

then

$$⨍_{B_{\lambda_0}} \bigl|u(X) - c_0\bigr|^p \, dX \leqslant \lambda_0^{p\alpha}. \tag{4.29}$$

Proof. For $\delta > 0$ to be regulated *a posteriori*, let h be a solution to a constant coefficient equation assured by Lemma 4.3, that is δ-close to u in the L^p-norm. From $C^{1,\epsilon}$ regularity theory for constant coefficient equations, (4.6), there exists a constant C depending only on n, p, λ and Λ such that

$$\bigl|h(X) - h(0)\bigr| \leqslant C|X|.$$

Since $\|h\|_{L^p} \leqslant C$, by L^∞ bounds,

$$\bigl|h(0)\bigr| \leqslant C.$$

We now estimate,

$$\begin{aligned} ⨍_{B_{\lambda_0}} \bigl|u(X) - h(0)\bigr|^p \, dX &\leqslant 2^{p-1} \Biggl(⨍_{B_{\lambda_0}} \bigl|h(X) - h(0)\bigr|^p \, dX + ⨍_{B_{\lambda_0}} \bigl|u(X) - h(X)\bigr|^p \, dX \Biggr) \\ &\leqslant 2^{p-1} \delta^p \lambda_0^{-n} + 2^{p-1} C \lambda_0^p. \end{aligned}$$

Since $0 < \alpha < 1$, it is possible to select λ_0 small enough as to assure

$$2^{p-1} C \lambda_0^p \leqslant \frac{1}{2} \lambda_0^{\alpha p}.$$

Once selected λ_0, we set

$$\delta := \frac{1}{2} \lambda_0^{\frac{n}{p} + \alpha},$$

which determines the smallness condition ε_0 through the compactness Lemma 4.3. □

Finally, the proof of Theorem 4.2 follows by the induction argument from Section 4.1, having Lemma 4.4 as its starting basis. We omit the details here.

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