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Hill Equations with Coexisting Periodic Solutions*

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1. In this note we consider inequalities related to Hill equations

$$x'' + \lambda p(t)x = 0$$

where $p(t)$ is a non-negative, periodic, integrable function of the real variable t and λ is a parameter. The period of p is denoted by ω.

Liapounoff's theory [*1*] of the equation (1) is reviewed in [*2*], Section 4.2, and [*3*], Chapter 1. Liapounoff proves that there exists an increasing sequence of real numbers

$$0 < \lambda_1 \leqslant \Lambda_1 < \lambda_2 \leqslant \Lambda_2 < \cdots \leqslant \Lambda_{n-1} < \lambda_n \leqslant \Lambda_n < \cdots$$

such that the nth eigenvalues of (1) for all possible intervals $[a, a + \omega]$ of length ω are exactly the numbers $\lambda \in [\lambda_n, \Lambda_n]$. This is a byproduct of the theory of stability since it turns out that (1) has a solution which is bounded for all t if and only if $\lambda \in [\lambda_n, \Lambda_n]$ for some n and that all solutions are bounded if $\lambda \in (\lambda_n, \Lambda_n)$. For $n = 2k - 1$, λ_n and Λ_n are eigenvalues of the problem

$$x(0) = -x(\omega); \qquad x'(0) = -x'(\omega).$$

For $n = 2k$, λ_n and Λ_n are eigenvalues of

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This means that for $\lambda = \lambda_{2k-1}$ or Λ_{2k-1}, the equation (1) admits a semi-periodic solution of period ω, whereas for $\lambda = \lambda_{2k}$ or Λ_{2k} there exists a periodic solution of period ω. If $\lambda_n < \Lambda_n$ then only one solution is (semi-)-periodic at a time and the prolongation by continuous variation of the (semi-) periodic solution for $\lambda = \lambda_n$ is not even bounded for $\lambda = \Lambda_n$. Therefore, the condition for the existence of two linearly independent

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solutions of (1) with n zeros in $[0, \omega]$ and which are either periodic ($n = 2k$) or semi-periodic ($n = 2k - 1$) of period ω, is

$$\lambda_n = \Lambda_n . \tag{2}$$

In the theory of stability, the expression

$$I(\lambda) = \lambda\omega \int_0^\omega p(t)\,dt$$

plays an important role. Following Liapounoff, Žukovskiĭ [4] proved that $\lambda \in [\lambda_n, \Lambda_n]$ implies

$$I(\lambda) \geqslant 4n^2. \tag{3}$$

The inequality is best possible; the limit is attained for p, a distribution corresponding to equally spaced identical point masses on a vibrating string. Under special assumptions, a great number of sharper inequalities have been derived; see [2], Section 4.3, and [3]. In this note, we prove that (2) implies

$$I(\lambda_n) \geqslant 4(n+1)^2 \cos^2 \frac{\pi}{2(n+1)} \tag{4}$$

where again equality is obtained for a certain distribution of point masses.

The proof uses unimodular centro-affine differential geometry ([5], [6], [7]). In fact, we reduce the problem of estimating $I(\lambda_n)$ to a minimal problem concerning locally convex curves which is of considerable independent interest. This problem is formulated in Section 4 and solved in Sections 5 to 7.

The author wants to thank Professor Harry Hochstadt who suggested that Liapounoff's theory might be the right setting for the author's previous result on $I(\lambda)$ and properties of the solutions of (1) for $\lambda_n = \Lambda_n$ ([7], Section 17, [8], Section 4). He also was informed by Professor O. Borůvka that centro-affine differential geometry is a major tool in a forthcoming book on the Hill equation by the latter.

2. From now on we suppose that (2) holds and denote by $x_1(t), x_2(t)$ a pair of linearly independent solutions of unit Wronskian of (1) for $\lambda = \lambda_n$. For each value t, we obtain a point $x = (x_1(t), x_2(t))$ in an (x_1, x_2)-plane. Since $\lambda_n\, p(t) \geqslant 0$, the curve obtained in this way is non-convex towards the origin at all points. For n even, we obtain a closed curve as image of $0 \leqslant t \leqslant \omega$. This curve winds n times about the origin and the radius vector r is a univalent function of the polar angle φ, $0 \leqslant \varphi \leqslant 2\pi n$. The polar angle itself is a monotone function $\varphi = \varphi(t) = \arctan x_2(t)/x_1(t)$ of the parameter t. For n odd, the curve $x(t)$, $0 \leqslant t \leqslant 2\omega$, is symmetric with respect to the origin,

$x(t + \omega) = -x(t)$. Here again the polar angle is a monotone function of t and the curve is non-convex towards the origin. The rotation number of the origin is n.

3. In a twodimensional real vectorspace we consider a curve $y = y(u)$, $u_0 \leqslant u \leqslant u_1$. We assume that y is twice continuously differentiable and that y and dy/du are linearly independent for all u (in particular, this implies $y \neq 0$ and the absence of inflection points on the curve). We are interested in the properties of the curve invariant under area preserving linear transformations, i.e., under the action of the matrix group $SL(2, \mathbb{R})$ on the space. According to E. Cartan's method of moving frames [9] we have to find an invariant parameter. An obvious choice is the area spanned by vector and tangent to the curve. The determinant of two vectors a, b will be denoted by $[a, b]$. We define the unimodular parameter t of y by the condition

$$\left[y, \frac{dy}{dt}\right] = 1 \tag{5}$$

or

$$t = t_0 + \int_{u_0}^{u} \left[y, \frac{dy}{du}\right] du.$$

Differentiation with respect to t will be denoted by a prime. By hypothesis, the two vectors $y(t)$ and $y'(t)$ form a basis for every t, $t_0 \leqslant t \leqslant t_1$; hence, they form an invariant moving frame. The Lie algebra of a unimodular group is formed by matrices of trace zero. Therefore, we have the Frenet equation

$$\frac{d}{dt}\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -p(t) & 0 \end{pmatrix}\begin{pmatrix} y \\ y' \end{pmatrix} \tag{6}$$

and two curves are images of one another in a unimodular transformation of the vector space if and only if they define the same curvature $p = p(t)$ referred to the same interval $t_0 \leqslant t \leqslant t_1$. We see also that every C^2 arc without inflection points and not passing through the origin is the solution of an equation

$$y'' + p(t)y = 0.$$

It is easy to find the geometric meaning of p and t. By (5), $t_1 - t_0$ is twice the area covered by the radius vector y for values $\epsilon[t_0, t_1]$. In particular, if $y(t)$ describes a closed curve for $t_0 \leqslant t \leqslant t_0 + \tilde{\omega}$, then $\tilde{\omega}$ is twice the area S covered by the radius vector describing the curve.

For the moment we introduce a cartesian system of coordinates based on two orthonormal vectors (e_1, e_2) relative to some positive definite quadratic form. The angle of the $+x_1$ axis and the oriented tanget to the curve at $y(t)$ is denoted by θ. The distance from the origin of this tangent (measured on the negative euclidean normal) is the support distance h. The euclidean arclength is s and the radius of curvature is $\rho = ds/d\theta$. From (5) it follows that

$$dt = h\,ds = h\rho\,d\theta. \tag{7}$$

We denote by (t, n) the euclidean tangent and normal to the curve at $y(t)$. Then ([9], p. 55)

$$y(t) = -h(t)\vec{n}(t) + \frac{dh}{d\theta}\vec{t}(t).$$

From (7) it follows, with $t = dy/ds$, that

$$\frac{d^2y}{dt^2} = -\frac{1}{\rho h^3}\left(-h(t)\vec{n} + \frac{dh}{d\theta}\vec{t}\right).$$

Hence

$$p(t) = \frac{1}{\rho h^3} \tag{8}$$

and

$$\int_{t_0}^{t_1} p(t)\,dt = \int_{\theta_0}^{\theta_1} \frac{d\theta}{h^2}.$$

The curve whose polar equation is

$$r(\varphi) = \frac{1}{h(\pi/2 - \varphi)} \tag{9}$$

is the polar reciprocal in the euclidean unit circle of the given curve. The integral (9), therefore, is twice the area covered by the radius vector of the reciprocal. The reciprocal of a locally convex curve that winds n times around the origin is a curve of the same type. If $y(t)$ describes a closed curve for $t_0 \leqslant t \leqslant t_0 + \bar{\omega}$, then $\int_0^{\bar{\omega}} p\,dt$ is twice the area S^* covered by the radius vector of the polar reciprocal curve.

4. A comparison with the developments of Sections 1 and 3 shows that

$$I(\lambda_n) = \begin{matrix} 4SS^* & n \text{ even} \\ SS^* & n \text{ odd} \end{matrix}$$

for the curve $x(t)$ given by the periodic solutions of (1). The inequality (4) will be proved if we show that

$$SS^* \geqslant (2k+1)^2 \cos^2 \frac{\pi}{2(2k+1)} \tag{10a}$$

for any locally convex, closed curve for which the polar angle is a monotone function of the arclength and the rotation number (Kronecker index) of the origin is k, and

$$SS^* \geqslant 4(n+1)^2 \cos^2 \frac{\pi}{2(n+1)} \tag{10b}$$

for the special case of a centrally symmetric curve of rotation number n.

The cases $n = 1$ of (10b) and $k = 1$ of (10a) are due to K. Mahler [*10*]. The case $n = 1$ of (4) is essentially contained in a result of Petty and Barry [*11*]. The invariant SS^* for convex curves was studied by Santalo [*12*], see also [7], Section 10. By our own construction, SS^* is invariant for the action of unimodular central affinities. But it is also easily seen to be invariant for homotheties of center 0. Hence, SS^* is invariant for all affine transformations for which the origin 0 is a fixed point. Santaló proves that every oval contains a unique point for which S^* is minimal. His proof carries over to our case of non-convex, locally convex curves. However, we shall not need this fact.

5. We shall consider the inequalities (10a) and (10b) for closed polygonal arcs. If a locally convex curve is bounded away from the origin, then for sufficiently close locally convex polygonal approximation the reciprocal polygon will be an approximation of the reciprocal curve. Therefore, the proof of the inequality for polygons will imply the validity of the same inequality for curves. In fact, we shall see that the case of equality can happen only for polygons and, therefore, equality in (4) can hold only if p is a distribution but not a function. At this occasion we note that the Blaschke selection theorem ([*13*], p. 34) implies that *the support functions of locally convex, closed curves of rotation number k form a locally compact set in the space of continuous functions defined on* $[0, 2k\pi]$ *and provided with the topology of uniform convergence*. In fact, for $k = 1$ this is the Blaschke theorem since Minkowski addition then is equivalent to addition of support functions and the Blaschke metric becomes the metric $\max_\varphi | h_1(\varphi) - h_2(\varphi)|$. For $k > 1$ we consider the convex sets $S_i (i = 1,..., 2k)$ defined as the convex hull of the arc of support function $h(\varphi)$, $(i-1)\,\pi \leqslant \varphi \leqslant i\pi$. A Cauchy sequence of support functions $h^{(s)}(\varphi)$ in the metric $d(h^{(s)}, h^{(t)}) = \max | h^{(s)}(\varphi) - h^{(t)}(\varphi)|$, $0 \leqslant \varphi \leqslant 2k\pi$, induces a set of convex sets $S_i^{(s)}$. By a diagonal process based on the Blaschke theorem we may find a sequence (s') such that all $\lim S_i^{(s')}$ and $\lim S_i^{(s')} \cap S_{i+1}^{(s')}$ exist. But this means that $\lim h^{(s')}$ exists and is the support function of a locally convex curve of rotation number k. This implies also the existence of a minimum for our functional since it is constant on sets of similar figures and, hence, is continuous on a compact space.

6. Next, we consider a polygon P of vertices $X_i (i = 1,..., N)$ numbered in cyclic order. We assume that the polygon is locally convex, closed, and

such that the radius vector from the origin O is a univalent function of the polar angle. In addition, we assume that O is not an interior point of any triangle formed by three consecutive vertices. (Hence, $N \geqslant 4$). We denote by g_i the segment $X_i X_{i+1}$ and indicate the polar reciprocal of any datum by an asterisc. First, we study the minimum of $S^* = S(P^*)$ if all vertices of P are fixed except X_i and X_i is varied in such a way that $S = S(P)$ is constant. This means that X_i may vary on a segment g parallel $X_{i-1}X_{i+1}$ and contained in the interior of the angle formed by the lines which carry g_{i-1} and g_{i+1}. In the reciprocal figure, the lines X_{i-1}^* and X_{i+1}^* pass through the points g_{i-2}^* and g_{i+1}^*, respectively. Our problem is to find the line X_i^* through the point g^* in the interior of the triangle formed by X_{i-1}^* and X_{i+1}^* over the base $b = g_{i-2}^* g_{i+1}^*$ such that the area of the convex figure bounded by g_{i-2}^*, X_{i-1}^*, X_i^*, X_{i+1}^*, g_{i+1}^* be minimal. Let ℓ be a line through g^* and let us assume for definiteness that the segment of ℓ between g^* and X_{i+1}^* be not less than the segment intercepted by g^* and X_{i-1}^*. Also, let k be the line $g^* g_{i+1}^*$. A reflection in g^* shows that the area of the triangle of edges b, k, X_{i-1}^* is smaller than the area of the quadrilateral bounded by b, X_{i+1}^*, ℓ, X_{i-1}^*. Hence, the minimum of S^* is obtained if either $X_i^* = k$ or $X_i^* = g_{i-2}^* g^*$, i.e., either $X_i = g \cap g_{i+1}$ or $X_i = g \cap g_{i-1}$ and the N-gon is a $(N - 1)$gon.

We assume first that P is a convex polygon. Then the preceding construction shows that we can find a triangle $\triangle$ such that

$$S(P)\, S(P^*) \geqslant S(\triangle)\, S(\triangle^*).$$

Since all triangles are affine to one another, the minimum of SS^* can be computed, e.g., on the right isosceles triangle. On finds easily that other minimum is obtained for 0 the centroid and

$$\min S(\triangle) S(\triangle)^* = \frac{27}{4} = 3^2 \cos^2 \frac{\pi}{6}.$$

Similarly, for a convex symmetric polygon P of $2N$-vertices we obtain by symmetric variation of X_i and X_{N+i} an $(N - 2)$gon of smaller area $S(P^*)$. The process ends with the symmetric 4-gon, the parallelogram, for which

$$S(P_4) S(P_4^*) = 8 = 4 \cdot 2^2 \cdot \cos^2 \frac{\pi}{2 \cdot 2}.$$

7. In the general case of a polygon of rotation number $k > 1$ we see that for a given P we can find a P', homotopic to P in the pointed plane, such that O is in the interior of every triangle formed by three consecutive vertices and $S(P)S(P^*) \geqslant S(P')S(P'^*)$. We change the name P' into P and again

consider three consecutive vertices X_{i-1}, X_i, X_{i+1}. This time we vary X_i so as to preserve the local convexity, the condition $O \epsilon$ interior $X_{i-1}X_iX_{i+1}$, and the sum of the areas of the triangles $X_{i-1}OX_i$ and X_iOX_{i+1}. This again means that the locus of X_i is a segment $g \parallel X_{i-1}X_{i+1}$ and bounded by the lines $X_{i-1}O$ and $X_{i+1}O$. In the dual configuration, we are looking for a line X_i^* through a fixed point g^* such that the points of intersection $A = X_i^* \cap X_{i-1}^*$ and $B = X_i^* \cap X_{i+1}^*$ define a triangle ABO of minimum area. By a very elementary reflection argument ([*14*], p. 115, [*15*] problem 5-1, No. 12) one sees that the minimal area is obtained for the line for which the two segments defined by g^* are equal: $Ag^* = g^*B$.

Now we use the fact that SS^* is invariant under centro-affine transformations. This means that we may assume that we are dealing with an affine image of P for which X_{i-1}, X_{i+1} and g^* are on the unit circle of center O. Then X_i^* is the tangent to that circle at g^* and $\triangle(X_{i-1}OX_i) \cong \triangle(X_iOX_{i+1})$. By a repetition of this argument we see that to P we can construct P' such that

$$S(P)\,S(P^*) \geqslant S(P')\,S(P'^*)$$

and any two adjacent triangles of P' are affine images of a pair of isosceles triangles symmetric with respect to their common edge, i.e., P' is affine to a regular starpolygon. By the argument of Section 6, P' will be minimal if the number of its vertices is minimal. The right hand sides of the inequalities (10*a*) and (10*b*) simply express SS^* for these starpolygons.

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