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Mixed Artin–Tate motives over number rings

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ABSTRACT

This paper studies Artin–Tate motives over bases $S \subset \operatorname{Spec} \mathcal{O}_F$, for a number field F. As a subcategory of motives over S, the triangulated category of Artin–Tate motives $\mathbf{DATM}(S)$ is generated by motives $\phi_* \mathbf{1}(n)$, where ϕ is any finite map. After establishing the stability of these subcategories under pullback and pushforward along open and closed immersions, a motivic t-structure is constructed. Exactness properties of these functors familiar from perverse sheaves are shown to hold in this context. The cohomological dimension of mixed Artin–Tate motives ($\mathbf{MATM}(S)$) is two, and there is an equivalence $\mathbf{DATM}(S) \cong \mathbf{D}^b(\mathbf{MATM}(S))$.

Geometric motives, as developed by Hanamura [5], Levine [8] and Voevodsky [14], are established as a valuable tool in understanding geometric and arithmetic aspects of algebraic varieties over fields. However, the stupefying ambiance inherent to motives, exemplified by Grothendieck's motivic proof idea of the Weil conjectures, remains largely conjectural—especially what concerns the existence of mixed motives $\mathbf{MM}(K)$ over some field K. That category should be the heart of the so-called motivic t-structure on $\mathbf{DM}_{\mathrm{gm}}(K)$, the category of geometric motives. Much the same way as the cohomology groups of a variety X over K, e.g. $\mathrm{H}^n_{\text{ét}}(X \times_K \overline{K}, \mathbb{Q}_\ell)$, ℓ-adic cohomology for $\ell \neq \operatorname{char} K$ are commonly realized as cohomology groups of a complex, e.g. $\mathrm{R}\Gamma_\ell(X, \mathbb{Q}_\ell)$, there should be mixed motives $\mathrm{h}^n(X)$ that are obtained by applying truncation functors belonging to the t-structure to $\mathrm{M}(X)$, the motive of X. However, progress on mixed motives has proved hard to come by. To date, such a formalism has been developed for motives of zero- and one-dimensional varieties, only. This is due to Levine [7], Voevodsky [14], Orgogozo [9] and Wildeshaus [16].

Building upon Voevodsky's work, Ivorra [6] and recently Cisinski and Déglise [3] developed a theory of geometric motives $\mathbf{DM}_{\mathrm{gm}}(S)$ over more general bases. The purpose of this work is to join the ideas of Beilinson et al. on perverse sheaves [2] with the ones on Artin–Tate motives over fields to obtain a workable category of mixed Tate and Artin–Tate motives over bases S which are open subschemes of $\operatorname{Spec} \mathcal{O}_F$, the ring of integers in a number field F. As over a field, this provides some evidence for the existence and properties of the conjectural category of mixed motives over S.

The triangulated category $\mathbf{DTM}(S)$ ($\mathbf{DATM}(S)$) of Tate (Artin–Tate) motives is defined 2.2 to be the triangulated subcategory of $\mathbf{DM}_{\mathrm{gm}}(S)$ (with rational coefficients) generated by direct summands of $\mathbf{1}(n)$ and $i_*\mathbf{1}(n)$ ($\phi_*\mathbf{1}(n)$, respectively). Here, $\mathbf{1}$ is a shorthand for the motive of the base scheme, (n) denotes the Tate twist, $i : \operatorname{Spec} \mathbb{F}_p \to S$ is a closed point, $\phi : V \to S$ is any finite map and $\phi_* : \mathbf{DM}_{\mathrm{gm}}(V) \to \mathbf{DM}_{\mathrm{gm}}(S)$ etc. denotes the pushforward functor on geometric motives. In case S is a finite disjoint union of $\operatorname{Spec} \mathbb{F}_p$, the usual definition of (Artin–)Tate motives over S is recalled in Definition 2.1.

The following theorem and its "proof" is an overview of the paper.

Theorem 0.1. *The categories* $\mathbf{DTM}(S)$ *and* $\mathbf{DATM}(S)$ *are stable under standard functoriality operations such as* $i^!$, j_* *etc. for open and closed embeddings* j *and* i, *respectively.*

Both categories enjoy a non-degenerate t*-structure called* motivic t-structure. *Its heart is denoted* $\mathbf{MTM}(S)$ *or* $\mathbf{MATM}(S)$, *respectively and called category of* mixed (Artin–)Tate motives.

The functors i^*, j_* *etc. feature exactness properties familiar from the corresponding situation of perverse sheaves. For example,* $i^!$ *is left-exact, and* j_* *is exact with respect to the motivic* t*-structure.*

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The cohomological dimension of **MTM**(S) *and* **MATM**(S) *is one and two, respectively. We have an equivalence of categories*

$$\mathbf{D}^{\mathrm{b}}(\mathbf{MATM}(S)) \cong \mathbf{DATM}(S)$$

and likewise for Tate motives.

The "site" of mixed Artin–Tate motives over S has enough points in the sense that a mixed Artin–Tate motive over S is zero if and only if its restrictions to all closed points of S vanish.

Proof. The first statement is Theorem 2.4. It is proven using the localization, purity and base-change properties of geometric motives.

We will write $T(S)$ for either **DTM**(S) or **DATM**(S). The existence of the motivic t-structure on $T(S)$ is proven in three steps. The first ingredient is the well-known motivic t-structure on Artin–Tate motives over finite fields (Lemma 3.6). The second step is the study of a subcategory $\tilde{T}(S) \subset T(S)$ generated by $\phi_* \mathbf{1}(n)$, where ϕ is finite and étale (Artin–Tate motives), or just by $\mathbf{1}(n)$ (Tate motives). This category is first equipped with an auxiliary t-structure. Using the cohomology functor for the auxiliary t-structure, a motivic t-structure on $\tilde{T}(S)$ is defined in Section 3. This statement uses (and its proof imitates) the corresponding situation for Artin–Tate motives over number fields due to Levine and Wildeshaus. Thirdly, the t-structure on $\tilde{T}(S)$ is glued with the one over finite fields, using the general gluing procedure of t-structures of [2], see Theorem 3.8. Much the same way as with perverse sheaves, there are shifts accounting for $\dim S = 1$, that is to say, $i_* \mathbf{1}(n)$ and $\mathbf{1}(n)[1]$ are mixed Tate motives. Beyond the formalism of geometric motives, the only non-formal ingredient of the motivic t-structure are vanishing properties of the algebraic K-theory of number rings, number fields and finite fields due to Quillen, Borel and Soulé.

The exactness statements are shown in Theorem 4.2. This theorem gives some content to the exactness axioms for general mixed motives over S [11, Section 4]. The key step stone is the following: for any immersion of a closed point $i : \operatorname{Spec} \mathbb{F}_p \to S$, the functor i^* maps the heart $T^0(S)$ of $T(S)$ to $T^{[-1,0]}(\operatorname{Spec} \mathbb{F}_p)$, that is, the category of (Artin–)Tate motives over $\mathbb{F}_p$ whose only nonzero cohomology terms are in degrees -1 and 0. The proof is a careful reduction to basic calculations relying on facts gathered in Section 3 about the heart of $\tilde{T}(S)$.

The cohomological dimensions are calculated in Proposition 4.4. The Artin–Tate case is a special (but non-conjectural) case of a similar fact for general mixed motives over S. The difference in the Tate case is because the generators of **DTM**(S) have a good reduction at all places.

By an argument of Wildeshaus, the identity on $T^0(S)$ extends to a functor $\mathbf{D}^{\mathrm{b}}(T^0(S)) \to T(S)$ (Theorem 4.5). While it is an equivalence in the case of Tate motives for formal reasons, the Artin–Tate case requires some localization arguments.

The last statement is Proposition 4.6. It might be seen as a first step into motivic sheaves. □

Deligne and Goncharov define a category of mixed Tate motives over rings $\mathcal{O}_S$ of S-integers of a number field F [4, 1.4., 1.7.]. Unlike the mixed Tate motives we study, their category is a *sub*category of mixed Tate motives over F, consisting of motives subject to certain non-ramification constraints, akin to Scholl's notion of mixed motives over $\mathcal{O}_F$ [12].

This paper is an outgrowth of part of my thesis. I owe many thanks to Annette Huber for her advice during that time. I am also grateful to Denis-Charles Cisinski and Frédéric Déglise for teaching me their work on motives over general bases.

1. Geometric motives

This section briefly recalls some properties of the triangulated categories of geometric motives $\mathbf{DM}_{\mathrm{gm}}(X)$, where X is either a number field F or an open or closed subscheme of $\operatorname{Spec} \mathcal{O}_F$. All of this is due to Cisinski and Déglise [3]. In this section, all references in brackets refer to *op. cit.*, e.g. [Section 14.1].

Let X be any of the afore-mentioned bases. There is the triangulated category **DM**(X) of Beilinson motives and its subcategory $\mathbf{DM}_{\mathrm{gm}}(X)$ of compact objects.[1] Objects of the latter category will be referred to as *geometric motives*. The categories are related by adjoint functors

$$f^* : \mathbf{DM}(X) \leftrightarrows \mathbf{DM}(Y) : f_*, \tag{1}$$

where $f : Y \to X$ is any map [13.2.11, 1.1.11]. If f is separated and of finite type this adjunction restricts to an adjunction between the subcategories of compact objects [14.1.5, 14.1.26] and there is an adjunction [13.2.11, 2.4.2]

$$f_! : \mathbf{DM}_{\mathrm{gm}}(Y) \leftrightarrows \mathbf{DM}_{\mathrm{gm}}(X) : f^!. \tag{2}$$

If f is smooth in addition, $f^* : \mathbf{DM}_{\mathrm{gm}}(X) \to \mathbf{DM}_{\mathrm{gm}}(Y)$ also has a left adjoint $f_\sharp$ [13.2.11, 1.1.2]. These five functors respect composition of morphisms in the sense that there are natural isomorphisms

$$f_* \circ g_* = (f \circ g)_*, \qquad f^* \circ g^* = (g \circ f)^* \text{ etc.} \tag{3}$$

for any two composable maps f and g [Section 1.1, 2.4.21]. The category $\mathbf{DM}_{\mathrm{gm}}(X)$ enjoys inner Hom's, denoted $\underline{\mathrm{Hom}}$, and a tensor structure such that pullback functors f^* are monoidal [13.2.11, 1.1.28]. The unit of the tensor structure is denoted **1**.

[1] **DM**(X) is denoted $\mathbf{DM}_{\mathrm{B}}(X)$ in [3, Sections 13.2, 14.1].

In particular $f^*\mathbf{1}_X = \mathbf{1}_Y$ for $f : Y \to X$. The *motive* of any separated scheme $f : Y \to X$ of finite type is defined as $f_! f^! \mathbf{1}$ and denoted $\mathrm{M}(Y)$. (For f smooth, [Section 1.1.] puts $\mathrm{M}(Y) := f_\sharp f^*\mathbf{1}$. The two agree, see Lemma 1.2.) The tensor structure in $\mathbf{DM}_{\mathrm{gm}}(X)$ is such that

$$\mathrm{M}(Y) \otimes \mathrm{M}(Y') = \mathrm{M}(Y \times_X Y') \tag{4}$$

for any two smooth schemes Y and Y' over X [1.1.35]. There is a distinguished object $\mathbf{1}(1)$ such that $\mathrm{M}(\mathbb{P}^1_X) = \mathbf{1} \oplus \mathbf{1}(1)[2]$. Tensoring with $\mathbf{1}(1)$ is an equivalence on $\mathbf{DM}_{\mathrm{gm}}(X)$ [2.1.5], and $\mathbf{1}(n)$ is defined in the usual way in terms of tensor powers of $\mathbf{1}(1)$. We exclusively work with rational coefficients, i.e., all morphism groups are $\mathbb{Q}$-vector spaces. If X is regular, morphisms in $\mathbf{DM}_{\mathrm{gm}}(X)$ are given by

$$\mathrm{Hom}_{\mathbf{DM}_{\mathrm{gm}}(X)}(\mathbf{1}, \mathbf{1}(q)[p]) \cong K_{2q-p}(X)^{(q)}_{\mathbb{Q}}, \tag{5}$$

the q-th Adams eigenspace in algebraic K-theory of X, tensored with $\mathbb{Q}$ [Section 13.2]. Having rational coefficients (or coefficients in a bigger number field) is vital when it comes to vanishing properties of Hom-groups in $\mathbf{DM}_{\mathrm{gm}}(X)$. (With integral coefficients, the existence of a t-structure is unclear even in the case of Artin motives over a field.)

For any closed immersion $i : Z \to X$ with open complement j we have the following functorial distinguished *localization* triangles in $\mathbf{DM}_{\mathrm{gm}}(X)$ [2.2.14, 2.3.3]:

$$j_! j^* \to \mathrm{id} \to i_* i^*. \tag{6}$$

Moreover $i^* i_* = \mathrm{id}$ [2.3.1], so that

$$i^* j_! = 0, \tag{7}$$

and i_* is fully faithful. There is an isomorphism of functors

$$f_! \xrightarrow{\cong} f_* \tag{8}$$

for any proper map f [2.2.14, 2.2.16]. For smooth and quasi-projective maps f of constant relative dimension d there is a *relative purity* isomorphism [Theorem 1, p. 5]

$$f^! \cong f^*(d)[2d]. \tag{9}$$

Moreover, when $i : Z \to X$ is a closed immersion of constant relative codimension c and Z and X are regular, we have an isomorphism

$$i^! \mathbf{1} \cong i^* \mathbf{1}(-c)[-2c]. \tag{10}$$

This is called *absolute purity* [Sections 2.4, 13.4]. Finally, for $f : Y \to X$, $g : X' \to X$, $f' : Y' := X' \times_X Y \to X'$ and $g' : Y' \to Y$, there is a natural *base-change* isomorphism of functors [Section 2.2]

$$f^* g_! \cong g'_! f'^*. \tag{11}$$

The *Verdier dual* functor $D_X : \mathbf{DM}_{\mathrm{gm}}(X)^{\mathrm{op}} \to \mathbf{DM}_{\mathrm{gm}}(X)$ is defined by $D_X(M) := \underline{\mathrm{Hom}}(M, \pi^! \mathbf{1}(1)[2])$ for any $M \in \mathbf{DM}_{\mathrm{gm}}(X)$, where $\pi : X \to \mathrm{Spec}\, \mathbb{Z}$ denotes the structural map.

Lemma 1.1. *For an open subscheme X of* $\mathrm{Spec}\, \mathcal{O}_F$ *we have*

$$D_X(-) = \underline{\mathrm{Hom}}(-, \mathbf{1}(1)[2]).$$

Secondly, we have $D_{\mathrm{Spec}\, \mathbb{F}_q}(-) = \underline{\mathrm{Hom}}(-, \mathbf{1})$.

Proof. The structural map $\pi : X \to \mathrm{Spec}\, \mathbb{Z}$ factors as

$$X \xrightarrow{j} \mathrm{Spec}\, \mathcal{O}_F \xrightarrow{i} \mathbb{A}^n_{\mathbb{Z}} \xrightarrow{p} \mathrm{Spec}\, \mathbb{Z},$$

where j is an open immersion, i is a closed immersion and p is the projection. Thus we have $\pi^! \mathbf{1} = \pi^* \mathbf{1}$ by absolute purity (10), applied to i, and relative purity (9), applied to j and p. Using (10) we get the second statement. □

The Verdier dual functor exchanges "!" and "∗", that is, there are natural isomorphisms [Section 14.3]

$$D(f^! M) \cong f^* D(M), \qquad f_! D(M) \cong D(f_* M). \tag{12}$$

For example, the Verdier dual of (6) yields a distinguished triangle

$$i_* i^! \to \mathrm{id} \to j_* j^*. \tag{13}$$

Lemma 1.2. *For $f : X \to Y$ smooth, we have a natural isomorphism $f_! f^! \mathbf{1} = f_\sharp f^* \mathbf{1}$.*

Proof. This is well known. We can assume f is of constant relative dimension d. Then the claim follows from the adjunctions $f_\sharp \leftrightarrows f^* \overset{(9)}{=} f^!(-d)[-2d]$ and $f_!(d)[2d] \leftrightarrows f^!(-d)[-2d]$. □

Let $X = \mathrm{Spec}\, \mathcal{O}_F$. The colimit over the triangles (13) over increasingly small open subschemes $j : U \subset X$ is still a distinguished triangle. For any geometric motive M over X we get the following distinguished triangle in $\mathbf{DM}(X)$:

$$\oplus_{\mathfrak{p}} i_{\mathfrak{p}*} i^!_{\mathfrak{p}} M \to M \to \eta_* \eta^* M, \tag{14}$$

where $\eta : \mathrm{Spec}\, F \to \mathrm{Spec}\, \mathcal{O}_F$ is the generic point, the sum runs over all closed points $\mathfrak{p} \in X$, $i_{\mathfrak{p}}$ is the closed immersion. Indeed $\mathrm{colim} j_* j^* M = \eta_* \eta^* M$ for any $M \in \mathbf{DM}_{\mathrm{gm}}(X)$ [Section 14.2].

2. Triangulated Artin–Tate motives

Recall the following classical definition. We apply it to a number field or a finite field:

Definition 2.1. Let K be a field. The category of Tate motives $\mathbf{DTM}(K)$ over K is by definition the triangulated subcategory of $\mathbf{DM}_{\mathrm{gm}}(K)$ generated by $\mathbf{1}(n)$ where $n \in \mathbb{Z}$. The smallest full triangulated subcategory $\mathbf{DATM}(K)$ stable under tensoring with $\mathbf{1}(n)$ and containing direct summands of motives $f_*\mathbf{1}$, where $f : K' \to K$ is any finite map, is called a category of Artin–Tate motives over K. For a scheme S of the form $S = \sqcup \operatorname{Spec} K_i$, a finite disjoint union of spectra of fields, we put $\mathbf{DATM}(S) := \oplus_i \mathbf{DATM}(K_i)$ and likewise for $\mathbf{DTM}$.

This section gives a generalization of that definition to bases S which are open subschemes of $\operatorname{Spec} \mathcal{O}_F$ based on the idea that Artin–Tate motives over S should be compatible with the ones over F and $\mathbb{F}_p$ under standard functoriality.

Definition 2.2. The categories $\mathbf{DTM}(S) \subset \mathbf{DM}_{\mathrm{gm}}(S)$ of *Tate motives* and $\mathbf{DATM}(S) \subset \mathbf{DM}_{\mathrm{gm}}(S)$ of *Artin–Tate motives* over S are the triangulated subcategories generated by the direct summands of

$$\mathbf{1}(n),\ i_*\mathbf{1}(n) \quad \text{(Tate motives)}$$

and

$$\phi_*\mathbf{1}(n), \quad \text{(Artin–Tate motives)}$$

respectively, where $n \in \mathbb{Z}$, $\phi : V \to S$ is any finite map (including those that factor over a closed point) and $i : \operatorname{Spec} \mathbb{F}_p \to S$ is the immersion of any closed point of S.

Remark 2.3.
- We can assume by localization (see (6), (13)) that the domain of ϕ is a reduced scheme.
- The category of Tate motives $\mathbf{DTM}(S)$ agrees with the triangulated category generated by the above generators (without taking direct summands). Indeed, by (5), the endomorphism rings $\operatorname{End}(\mathbf{1}(n))$, $\operatorname{End}(i_*\mathbf{1}(n))$ identify with $K_0(S)^{(0)}_{\mathbb{Q}}$ and $K_0(\mathbb{F}_p)^{(0)}_{\mathbb{Q}}$, respectively, which are both one-dimensional over $\mathbb{Q}$. Hence these objects do not have any proper direct summands.

For brevity, we write $T(S)$ or T for $\mathbf{DATM}(S)$ or $\mathbf{DTM}(S)$ in the sequel. In most proofs, we will only spell out the case of Artin–Tate motives.

Theorem 2.4. *Let $j : S' \to S$ be any open immersion, $i : Z \to S$ be any closed immersion and $f : V \to S$ any finite map such that V is regular. Let $\eta : \operatorname{Spec} F \to S$ be the generic point. Then the functors $f_* \overset{(8)}{=} f_!$, f^* and $f^!$ preserve Artin–Tate motives. Similar statements hold for Artin–Tate and Tate motives for j and i. Moreover, η^*, the Verdier dual functor D and the tensor product on $\mathbf{DM}_{\mathrm{gm}}(S)$ respect the subcategories of (Artin–)Tate motives.*

The functor η_* does not respect Artin–Tate motives: we will see in Proposition 4.6 that any Artin–Tate motive M of the form $M = \eta_* M_\eta$, where M_η is a geometric motive over F, necessarily satisfies $M = 0$.

Proof. The stability of (Artin–)Tate motives under j^*, η^*, i_* and i^*, f^* and — for Artin–Tate motives, under f_* — is immediate from the definition, (8), and (11). For example, $i^*\phi_*\mathbf{1}(n) = \phi''_*\mathbf{1}(n)$. Here $\phi : S' \to S$ is any finite map and $\phi'' : Z' \to Z$ is its pullback along i. Let $i' : Z' \to S'$ be the pullback of i. For the stability under $i^!$ we use $i^!\phi_*\mathbf{1} \overset{(11)}{=} \phi''_* i'^! \mathbf{1}$. We can assume S' is reduced and, since the zero-dimensional case is easy, one-dimensional. Let $n : S'' \to S'$ be the normalization map; let $v : Y' \subset S'$ be the "exceptional divisor", i.e., the smallest (zero-dimensional) closed reduced subscheme such that $n^{-1}(S' \backslash Y') \to S' \backslash Y'$ is an isomorphism. Moreover, put $z : Y'' := Y' \times_{S'} S'' \to S'' \to S'$. Consider the the distinguished triangle

$$\mathbf{1}_{S'} \to v_*\mathbf{1}_{Y'} \oplus n_*\mathbf{1}_{S''} \to z_*\mathbf{1}_{Y''}.$$

It is a special case of [3, Theorem 4, p. 5] or can alternatively be derived from localization. Note that $i^! n_* \mathbf{1}_{S''} \overset{(11)}{=} n'_* i''^! \mathbf{1}_{S''} \overset{(10)}{=} n'_*\mathbf{1}(-1)[-2]$ by the regularity of S''. Here, again, n' and i'' denote the pullback maps. Similar considerations for $i^! v_* \mathbf{1}_{Y'}$ and $i^! z_* \mathbf{1}_{Y''}$ show that $i^! \mathbf{1}_{S'}$ is an Artin–Tate motive.

For the stability under j_* it is sufficient to show $j_*\phi'_*\mathbf{1}$ is an Artin–Tate motive over S for any finite flat map $\phi' : V' \to S'$. Choose some finite flat (possibly non-regular) model $\phi : V \to S$ of ϕ', i.e., $V \times_S S' = V'$, so that $j^*\phi_*\mathbf{1} = \phi'_*\mathbf{1}$ is an Artin–Tate motive over S'. The localization triangle (13)

$$i_* i^! \phi_* \mathbf{1} \to \phi_* \mathbf{1} \to j_* j^* \phi_* \mathbf{1}$$

and the above steps show that $j_*\phi'_*\mathbf{1}$ is an Artin–Tate motive over S.

To see the stability under the Verdier dual functor D, it is enough to see that

$$D(\phi_*\phi^*\mathbf{1}) \overset{(12)}{=} \phi_!\phi^! D(\mathbf{1}) \overset{1.1}{=} \phi_*\phi^!\mathbf{1}(1)[2]$$

is an Artin–Tate motive for any finite map $\phi : V \to S$ with reduced domain (Remark 2.3). If V is zero-dimensional, this follows from purity (10), (9) and the regularity of S. If not, there is an open (non-empty) immersion $j : S' \to S$ such that $V' := V \times_S S'$ is regular (for example, take S' such that V'/S' is étale). Let i be the complement of j. We apply the localization triangle (13) to $\phi_* \phi^! \mathbf{1}$. By base-change (11) we obtain

$$i_* \phi''_* \phi''^! i^! \mathbf{1} \to \phi_* \phi^! \mathbf{1} \to j_* \phi'_* \phi'^! j^* \mathbf{1}.$$

Here ϕ'' and ϕ' is the pullback of ϕ along i and j, respectively. By the regularity of S and purity we have $i^! \mathbf{1} = \mathbf{1}(-1)[-2]$, so the left hand term is an Artin–Tate motive. The right one also is by purity. This shows the claim for D.

The stability under $f^!$, $i^!$, and $j_!$ now follow for duality reasons.

As for the stability under tensor products we note that $\phi_* \mathbf{1} \otimes \phi'_* \mathbf{1} \overset{(4)}{=} (\phi \times \phi')_* \mathbf{1}$ if ϕ and ϕ' are (finite and) smooth, cf. (4). Using the localization triangle (6), it is easy to reduce the general case of merely finite maps ϕ, ϕ' to this case. □

Remark 2.5. Theorem 2.4 also holds for a similarly defined category of Artin–Tate motives over open subschemes S of a smooth curve over a field.

Proposition 2.6. *Let $M \in \mathbf{DATM}(S)$ be any Artin–Tate motive. Then there is a finite map $f : V \to S$ such that $f^* M \in \mathbf{DTM}(S) \subset \mathbf{DATM}(S)$. We describe this by saying that f* splits *M.*

Proof. As f^* is triangulated, this statement is stable under triangles (with respect to M), and also under direct sums and summands. Therefore, we only have to check the generators, i.e., $M = \phi_* \mathbf{1}(n)$ with $\phi : S' \to S$ a finite map with reduced domain. The corresponding splitting statement for Artin–Tate motives over finite fields is well-known. Therefore, by localization (6), (13), it is sufficient to find a splitting map f after replacing S by a suitable small open subscheme, so we may assume ϕ étale. We first assume that ϕ is moreover Galois of degree d, i.e., $S' \times_S S' \cong S'^{\sqcup d}$, a disjoint union of d copies of S'. In that case one has $\phi^* \phi_* \mathbf{1} = \mathbf{1}^{\oplus d}$ by base-change (11), so the claim is clear. In general ϕ need not be Galois, so let S'' be the normalization of S in some normal closure of the function field extension $k(S')/k(S)$. Both $\mu : S'' \to S$ and $\psi : S'' \to S'$ are generically Galois. By shrinking S we may assume both are Galois. From $\mathrm{Hom}(\mathbf{1}_{S'}, \psi_* \mathbf{1}_{S''}) = \mathrm{Hom}(\mathbf{1}_{S''}, \mathbf{1}_{S''}) = \mathbb{Q}$ and $\mathrm{Hom}(\psi_* \mathbf{1}_{S''}, \mathbf{1}_{S'}) = \mathrm{Hom}(\mathbf{1}_{S''}, \psi^! \mathbf{1}_{S'}) = \mathrm{Hom}(\mathbf{1}_{S''}, \mathbf{1}_{S''}) = \mathbb{Q}$ we see that $\mathbf{1}_{S'}$ is a direct summand of $\psi_* \mathbf{1}_{S''}$. Therefore $\mu^* \phi_* \mathbf{1}_{S'}$ is a summand of $\mu^* \phi_* \psi_* \mathbf{1}_{S''} = \mu^* \mu_* \mathbf{1}_{S''} = \mathbf{1}^{\oplus \deg S''/S}$, a Tate motive. □

3. The motivic *t*-structure

In this section, we establish the motivic t-structure on the category of Artin–Tate motives over S (Theorem 3.8). It is obtained by the standard gluing procedure, applied to the t-structures on Artin–Tate motives over finite fields and on a subcategory $\tilde{T}(S') \subset T(S')$ for open subschemes $S' \subset S$. Under the analogy of mixed (Artin–Tate) motives with perverse sheaves, the objects in the heart of the t-structure on $\tilde{T}(S')$ correspond to sheaves that are locally constant, i.e., have good reduction. We refer to [2, Section 1.3.] for generalities on t-structures.

Definition 3.1 (*Compare [7, Def. 1.1]*)**.** For $-\infty \leq a \leq b \leq \infty$, let $\tilde{T}_{[a,b]}$ denote the smallest triangulated subcategory of $T(S)$ containing direct factors of $\phi_* \mathbf{1}(n)$, $a \leq -2n \leq b$, where $\phi : S' \to S$ is a finite *étale* map. For Tate motives, ϕ is required to be the identity map. (We will not specify this restriction *expressis verbis* in the sequel.) Furthermore, $\tilde{T}_{[a,a]}$ and $\tilde{T}_{[-\infty,\infty]}$ are denoted $\tilde{T}_a$ and $\tilde{T}$. If it is necessary to specify the base, we write $\tilde{T}_{[a,b]}(S)$ etc.

We need the following vanishing properties of the K-theory of number fields, related Dedekind rings and finite fields up to torsion. In order to weigh the material appropriately, it should be said that the content of the theorem below is the only non-formal part of the proofs in this paper, and all complexity occurring with Artin–Tate motives ultimately lies in these computations.

Theorem 3.2 (*Borel, Quillen, Soulé*)**.** *Let $\phi : S' \to S$ and $\psi : V \to S$ be two finite maps with zero-dimensional domains.*

$$\mathrm{Hom}_S(\phi_* \mathbf{1}, \psi_* \mathbf{1}(n)[m]) = \begin{cases} \textit{finite-dimensional} & n = m = 0 \\ 0 & \textit{else.} \end{cases}$$

Now let $\phi : S' \to S$ and $\psi : V \to S$ be two finite étale maps over S. Then

$$\mathrm{Hom}_S(\phi_* \mathbf{1}, \psi_* \mathbf{1}(n)[m]) = \begin{cases} \textit{finite-dimensional} & n = m = 0 \\ \textit{finite-dimensional} & m = 1,\ n \textit{ odd and positive} \\ 0 & \textit{else.} \end{cases}$$

Proof. By (5)

$$\mathrm{Hom}_V(\mathbf{1}, \mathbf{1}(q)[p]) \cong K_{2q-p}(V)^{(q)}_{\mathbb{Q}},$$

for a regular scheme V. For the first statement, we may assume that S' and V are finite fields. Then the statement follows from adjunction, base-change, purity and

$$K_n(\mathbb{F}_q) = \begin{cases} \mu_{q^i-1} & n = 2i-1, i > 0 \\ 0 & n = 2i, i > 0 \\ \mathbb{Z} & n = 0 \end{cases}$$

[10]. K-theory of Dedekind rings R whose quotient field is a number field is known (up to torsion) by Borel's work. The relation to the K-theory of number fields is given by an exact sequence (due to Soulé [13, Th. 3]; up to two-torsion) for $n > 1$

$$0 \to K_n(R) \xrightarrow{\eta^*} K_n(F) \to \oplus_{\mathfrak{p}} K_{n-1}(\mathbb{F}_{\mathfrak{p}}) \to 0.$$

Here $\eta : \operatorname{Spec} F \to \operatorname{Spec} R$ is the generic point and the direct sum runs over all (finite) primes in R. Also, $K_0(R) = \mathbb{Z} \oplus Pic(R)$ and $K_1(R) = R^\times$. In particular, for all n and m, $K_n(R)_{\mathbb{Q}}^{(m)}$ vanishes when $K_n(F)_{\mathbb{Q}}^{(m)}$ vanishes, since η^* respects the Adams grading. One has the following list (see e.g. [15])

$$K_{2q-p}(F)_{\mathbb{Q}}^{(q)} = \begin{cases} 0 & q < 0 \\ 0 & q = 0, p \neq 0 \\ \mathbb{Q} & q = p = 0 \\ 0^{BS} & q > 0, p \leq 0 \\ 0 & q > 0, \text{even}, p = 1 \\ F^\times \otimes_{\mathbb{Z}} \mathbb{Q} & q = p = 1 \\ \mathbb{Q}^{r_1+r_2} & q > 1, q \equiv 1 \,(\mathrm{mod}\, 4), p = 1 \\ \mathbb{Q}^{r_2} & q > 0, q \equiv 3 \,(\mathrm{mod}\, 4), p = 1 \\ 0 & q > 0, p > 1. \end{cases}$$

As usual, r_1 and r_2 are the numbers of real and pairs of complex embeddings of F, respectively. (The agreement of $K_{2q-1}(F)$ and $K_{2q-1}(F)^{(q)}$ for odd positive q is not mentioned in [15].) The spot marked 0^{BS} is referred to as *Beĭlinson–Soulé vanishing* (see e.g. [7]). As first realized by Levine [7], this translates into the non-existence of morphisms in the "wrong" direction with respect to the motivic t-structure.

For the last claim, put $V' = V \times_S S'$:

$$\begin{tikzcd} V' \arrow[r, "\phi'"] \arrow[d, "\psi'"] & V \arrow[d, "\psi"] \\ S' \arrow[r, "\phi"] & S. \end{tikzcd}$$

To save space, we omit the twist and the shift in writing the Hom-groups. By (2), (11), and (1) we have

$$\mathrm{Hom}_S(\phi_* \mathbf{1}, \psi_* \mathbf{1}) = \mathrm{Hom}_{S'}(\mathbf{1}, \phi^! \psi_* \mathbf{1}) = \mathrm{Hom}_{S'}(\mathbf{1}, \psi'_* \phi'^! \mathbf{1}) = \mathrm{Hom}_{V'}(\mathbf{1}, \phi'^! \mathbf{1}).$$

Now, V' is (affine and) étale over V, so $\phi'^! \mathbf{1} \overset{(9)}{=} \phi'^* \mathbf{1} = \mathbf{1}$ by (9) and we are done in that case by the above vanishings of the K-theory up to torsion. □

The following lemma is a variant of [7, Lemma 1.2], [16, Lemma 1.9] and can be proven by faithfully imitating the technique in *loc. cit.*

Lemma 3.3. *For any $-\infty \leq a < b \leq c \leq \infty$, $(\tilde{T}_{[a,b-1]}, \tilde{T}_{[b,c]})$ is a t-structure on $\tilde{T}_{[a,c]}$.*

Definition 3.4. The resulting truncation and cohomology functors are denoted $F_{\leq b}$ and $F_{>b}$ and gr_b^F, respectively.

The following definition is modeled on [7, Def. 1.4]. We also refer to [1, Section 2.1.3] for a general way (due to Morel) of constructing a t-structure starting from a given set of generators. For any odd integer n set $\mathbf{1}(n/2) := 0$, for notational convenience.

Definition 3.5. Let S be an open subscheme of $\operatorname{Spec} \mathcal{O}_F$. Let $\tilde{T}_a^{\geq 0}(S)$ $(\tilde{T}_a^{\leq 0}(S))$ be the full subcategory of $\tilde{T}_a(S)$ (Definition 3.1) generated by direct summands of

$$\phi_* \mathbf{1}\left(-\frac{a}{2}\right)[n+1]$$

for any $n \leq 0$ ($n \geq 0$, respectively), and any finite étale map ϕ. "Generated" means the smallest subcategory containing the given generators stable under isomorphism, finite direct sums, and $\mathrm{cone}(f)[-1]$ ($\mathrm{cone}(f)$, resp.) for any morphism f in $\tilde{T}_a^{\geq 0}(S)$ $(\tilde{T}_a^{\leq 0}(S)$, respectively).

For any $-\infty \le a \le b \le \infty$, let $\tilde{T}^{\ge 0}_{[a,b]}(S)$ be the triangulated subcategory generated by objects X, such that for all $a \le c \le b$, $\mathrm{gr}^F_c(X) \in \tilde{T}^{\ge 0}_c(S)$ and similarly for $\tilde{T}^{\le 0}_{[a,b]}(S)$. For $a = -\infty$ and $b = \infty$ we simply write $\tilde{T}^{\le 0}(S)$, $\tilde{T}^{\ge 0}(S)$. We may omit S in the notation, if no confusion arises.

In particular $\mathbf{1}(-a/2)[1] \in \tilde{T}^0_a(S)$. This shift is as in the situation of perverse sheaves [2], [11, Section 3]. Before stating and proving the existence of the motivic t-structure, we need some preparatory steps. Levine has established the existence of the motivic t-structure on Tate motives over number fields and finite fields [7, Theorem 1.4.]. This has been generalized to Artin–Tate motives by Wildeshaus [16, Theorem 3.1]. We briefly recall these precursor statements. Let K be either a finite field or a number field. For any $-\infty \le a \le b \le \infty$, let $T_{[a,b]}(K)$ be the triangulated subcategory of $T(K)$ generated by $\mathbf{1}(n)$ with $a \le -2n \le b$ (Tate motives) and direct summands of $\phi_* \mathbf{1}(n)$, $\phi : \operatorname{Spec} K' \to \operatorname{Spec} K$ a finite map (Artin–Tate motives, respectively). For any $a \le c < b$, the datum $\left(T_{[a,c]}, T_{[c+1,b]}\right)$ forms a t-structure on $T_{[a,b]}$. Let gr^F_* be the cohomology functor corresponding to that t-structure. Write $T_a(K)$ for $T_{[a,a]}(K)$ and let $T^{\ge 0}_a(K)$ and $T^{\le 0}_a(K)$ be the subcategories of $T_a(K)$ generated by $\mathbf{1}(-a/2)[n]$ with $n \le 0$ and $n \ge 0$, respectively. Here, "generated" has the same meaning as in Definition 3.5. Let $T^{\ge 0}_{[a,b]}$ and $T^{\le 0}_{[a,b]}$ be the subcategories of $T_{[a,b]}$ of objects X such that all $\mathrm{gr}^F_c X \in T^{\ge 0}_c$ ($\mathrm{gr}^F_c X \in T^{\le 0}_c$, respectively) for all $a \le c \le b$. Then, $\left(T^{\le 0}_{[a,b]}(K), T^{\ge 0}_{[a,b]}(K)\right)$ is a non-degenerate t-structure on $T_{[a,b]}$.

The following well-known fact is a consequence of vanishing of all K-theory groups of finite fields except for $K_0(\mathbb{F}_\mathfrak{p})^{(0)}_\mathbb{Q}$, see Theorem 3.2.

Lemma 3.6. *Let $\mathfrak{p}$ be a closed point in S with residue field $\mathbb{F}_\mathfrak{p}$. The inclusions $T_a(\mathbb{F}_\mathfrak{p}) \subset T(\mathbb{F}_\mathfrak{p})$ induce an equivalence of categories*

$$\bigoplus_{a \in \mathbb{Z}} T_a(\mathbb{F}_\mathfrak{p}) = T(\mathbb{F}_\mathfrak{p}).$$

There are canonical equivalences of categories

$$T(Z) := \bigoplus_{\mathfrak{p} \in Z, a \in \mathbb{Z}} T_a(\mathbb{F}_\mathfrak{p}) = \bigoplus_{\mathfrak{p}, a} \mathbf{D}^{\mathrm{b}}(\underline{\mathbb{Q}}[\mathrm{Perm}, \mathrm{Gal}(\mathbb{F}_\mathfrak{p})]) = \bigoplus_{\mathfrak{p}, a} \underline{\mathbb{Q}}[\mathrm{Perm}, \mathrm{Gal}(\mathbb{F}_\mathfrak{p})]^{\mathbb{Z}-\text{graded}}.$$

Here and in the sequel, $\underline{\mathbb{Q}}[\mathrm{Perm}, \mathrm{Gal}(\mathbb{F}_\mathfrak{p})]$ denotes finite-dimensional rational permutation representations of the absolute Galois group. By means of that equivalence, $T(Z)$ is endowed with the obvious t-structure. The heart $T^0_a(\mathbb{F}_\mathfrak{p}) = T^{\le 0}_a(\mathbb{F}_\mathfrak{p}) \cap T^{\ge 0}_a(\mathbb{F}_\mathfrak{p})$ is semisimple and consists of direct sums of summands of $\phi_* \mathbf{1}(a)$, ϕ finite.

We now provide the motivic t-structure on $\tilde{T}(S)$, which stems from the one on $T(F)$. The two together will then be glued to give the t-structure on $T(S)$. Recognizably, the following is again an adaptation of Levine's proof of the t-structure on Tate motives over number fields.

Proposition 3.7. *For any $-\infty \le a \le b \le \infty$, $\left(\tilde{T}^{\le 0}_{[a,b]}, \tilde{T}^{\ge 0}_{[a,b]}\right)$ is a non-degenerate t-structure on $\tilde{T}_{[a,b]}(S)$ (Definitions 3.1 and 3.5). The cohomology functors associated to it are denoted ${}^{\mathrm{p}}\mathrm{H}^*$. The functor $\eta^*[-1] : \tilde{T}_{[a,b]}(S) \to T_{[a,b]}(F)$ is t-exact.*

Any motive in $\tilde{T}^0_a(S)$ is a finite direct sum of summands of motives $\phi_* \mathbf{1}(-a/2)[1]$ with ϕ finite étale. The closure of the direct sum of the $\tilde{T}^0_a(S)$, $a \in \mathbb{Z}$, under extensions (in the abelian category $\tilde{T}^0(S)$) is $\tilde{T}^0(S)$.

Proof. We may assume that a and b are finite, since

$$\tilde{T}(S) = \bigcup_{-\infty < a \le b < \infty} \tilde{T}_{[a,b]}(S)$$

and the inclusion functors given by the identity between the various $T_{[-,-]}$ are exact.

The proof proceeds by induction on $b - a$. The case $b = a$ is treated as follows: the category $\tilde{T}_a := \tilde{T}_a(S)$ is generated by $\phi_* \mathbf{1}(-a/2)[n]$, $n \in \mathbb{Z}$, ϕ étale and finite. The functor $\eta^*[-1](a/2) : \tilde{T}_a(S) \to T_0(F)$ is fully faithful. To see this it suffices to remark $\mathrm{Hom}_S(\phi_* \mathbf{1}(-a/2)[n+1], \psi_* \mathbf{1}(-a/2)[n'+1]) = \mathrm{Hom}_F(\phi_{\eta *} \mathbf{1}[n], \psi_{\eta *} \mathbf{1}[n'])$, for any finite étale maps ϕ and ψ with generic fiber ϕ_η and ψ_η. This equality follows from the K-theory computations, see the proof of Theorem 3.2. Therefore, the image of $\eta^*[-1](a/2)$ is a triangulated subcategory of $T_0(F)$ which contains the generators of $T_0(F)$, so the functor establishes an equivalence between $\tilde{T}_a(S)$ with the derived category of finite-dimensional rational permutation representations of $\mathrm{Gal}(F)$ by [14, 3.4.1]. Hence $\tilde{T}_a(S)$ carries a non-degenerate t-structure.

The remainder of the proof is done as in Levine's proof. One shows

$$\mathrm{Hom}\left(\tilde{T}^{\le 0}_{[a+1,b]}, \tilde{T}^{\ge 0}_c\right) = 0 \tag{15}$$

for any $c \le a$. This reduces to the Beĭlinson–Soulé vanishing. Then the t-structure axioms follow for formal reasons.

The exactness of $\eta^*[-1]$ is obvious from the definitions. The statement about the heart $\tilde{T}^0_a$ is done as follows: the exact functor $\eta^*[-1](a/2)$ identifies $\tilde{T}^0_a(S) = \tilde{T}^{\ge 0}_a(S) \cap \tilde{T}^{\le 0}_a(S)$ with the semi-simple category $T^0_0(F) = \underline{\mathbb{Q}}[\mathrm{Perm}, \mathrm{Gal}(F)]$. We claim that for any object $X \in \tilde{T}_a(S)$, all ${}^{\mathrm{p}}\mathrm{H}^n(X)$ are direct summands of sums of motives $\phi_* \mathbf{1}(-a/2)[1]$, ϕ finite and étale. This claim does hold for the generators of $\tilde{T}_a(S)$. We now show that the condition is stable under triangles, which accomplishes the proof of the claim and thus the proof of the statement. Let $A \to X \to B$ be a triangle in $\tilde{T}_a(S)$ such that A and B satisfy the claim. The long exact cohomology sequence

$$\cdots \to {}^{\mathrm{p}}\mathrm{H}^{n-1}B \xrightarrow{\delta^{n-1}} {}^{\mathrm{p}}\mathrm{H}^n A \to {}^{\mathrm{p}}\mathrm{H}^n X \to {}^{\mathrm{p}}\mathrm{H}^n B \xrightarrow{\delta^n} {}^{\mathrm{p}}\mathrm{H}^{n+1}A \to \cdots$$

yields the short exact sequence in $\tilde{T}^0_a(S)$

$$0 \to \operatorname{coker}\delta^{n-1} \to {}^{\mathrm{p}}\mathrm{H}^n X \to \ker\delta^n \to 0.$$

By the semi-simplicity of $\tilde{T}^0_a(S)$ (this is the key point!), the sequence splits and there is a non-canonical isomorphism ${}^{\mathrm{p}}\mathrm{H}^n X \cong \operatorname{coker}\delta^{n-1} \oplus \ker\delta^n$ and $\operatorname{coker}\delta^{n-1}$ and $\ker\delta^n$ are direct summands of ${}^{\mathrm{p}}\mathrm{H}^n A$ and ${}^{\mathrm{p}}\mathrm{H}^n B$, respectively.

For the statement concerning $\tilde{T}^0(S)$ one uses the finite exhaustive F-filtration of any $X \in \tilde{T}^0(S)$:

$$0 = F_a X \subset F_{[a,a+1]}X \subset \cdots \subset F_{[a,b]}X = X.$$

The successive quotients $\mathrm{gr}^F_* X$ of that chain are in $\tilde{T}^0_*(S)$, since truncations with respect to the t-structure related to F are exact with respect to the motivic t-structure, by definition. Thus the claim about $\tilde{T}^0(S)$ follows. □

Theorem 3.8. *The motivic t-structures on $T(Z)$ and $\tilde{T}(S')$ glue to a non-degenerate t-structure on the category $T(S)$ of (Artin–)Tate motives over S (Definition 2.2). It is called* motivic t-structure. *Here S' runs through open subschemes of S and $Z := S \backslash S'$.*

Proof. We apply the gluing procedure of t-structures of [2, Theorem 1.4.10]: for any open subscheme $j : S' \subset S$, we write $T_{S'}(S)$ for the full triangulated subcategory of objects $X \in T(S)$ such that $j^*X \in \tilde{T}(S') \subset T(S')$. Let $i : Z' \to S$ be the closed complement of j. Put

$$T^{\leq 0}_{S'}(S) := \{X \in T_{S'}(S), j^*X \in \tilde{T}^{\leq 0}(S'), i^*X \in T^{\leq 0}(Z')\},$$
$$T^{\geq 0}_{S'}(S) := \{X \in T_{S'}(S), j^*X \in \tilde{T}^{\geq 0}(S'), i^!X \in T^{\geq 0}(Z')\}.$$

The assumptions of the gluing theorem, [2, 1.4.3], namely the existence of $i_*, i^*, i^!, j_*, j_!, j^*$ satisfying the usual adjointness properties, $j^*i_* = 0$, localization sequences and full faithfulness of $i_*, j_!$ and j_* are met, since they are in the surrounding categories of geometric motives, cf. Section 1, and the stability of the subcategories of Artin–Tate motives under these functors (Theorem 2.4). Thus, the above defines a t-structure on $T_{S'}(S)$.

The field F is of characteristic zero, so any finite map $\phi : V \to S$ with V reduced and one-dimensional is generically étale. This implies $T(S) = \cup_{S' \subset S} T_{S'}(S)$. We set

$$T^{\geq 0}(S) := \bigcup_{S' \subset S} T^{\geq 0}_{S'}(S)$$

and dually for $T^{\leq 0}(S)$. The t-structure axioms on $T(S)$ and the non-degeneracy are implied by the exactness of the identical inclusion $T_{S'}(S) \to T_{S''}(S)$ for any $S'' \subset S'$.

To see the exactness of the identity, let $j'' : S'' \subset S$ and $i'' : Z'' \subset S$ be its complement. Let $X \in T^{\leq 0}_{S'}(S)$. It is clear that $j''^*X \in \tilde{T}^{\leq 0}(S'')$. Let us check $i''^*X \in T^{\leq 0}(Z'')$. The pullback i''^*X decomposes as a direct sum parametrized by the points of Z'' and we only have to deal with the points that are not contained in Z'. Let $p : \operatorname{Spec}\mathbb{F}_\mathfrak{p} \to S$ be such a point; it factors over S': $p = j \circ q$, where $q : \operatorname{Spec}\mathbb{F}_\mathfrak{p} \to S'$ is the same point as p. Thus $p^*X \overset{(3)}{=} q^*j^*X \in q^*\tilde{T}^{\leq 0}(S')$. The containment $q^*\tilde{T}^{\leq 0}(S') \subset T^{\leq 0}(\operatorname{Spec}\mathbb{F}_\mathfrak{p})$ follows from $q^*\tilde{T}^{\leq 0}_a(S') \subset T^{\leq 0}_a(\operatorname{Spec}\mathbb{F}_\mathfrak{p})$, since q^* clearly commutes with the F-truncation functors belonging to the auxiliary t-structure. To see the latter containment, it suffices to check the generators (in the sense of Definition 3.5) of $\tilde{T}^{\leq 0}_a(S')$, that is, it is sufficient to remark

$$q^*\phi_*\mathbf{1}(-a/2)[n+1] \overset{(11)}{=} \phi'_*\mathbf{1}(-a/2)[n+1] \in T^{\leq -1}_a(\operatorname{Spec}\mathbb{F}_\mathfrak{p}) \subset T^{\leq 0}_a(\operatorname{Spec}\mathbb{F}_\mathfrak{p}),$$

where $n \geq 0$ and ϕ is a finite étale map with pullback ϕ'. This shows that the identity is left-exact. The right-exactness is done dually. □

4. Mixed Artin–Tate motives

Definition 4.1. The heart $T^0(S)$ of the motivic t-structure is called the category of *mixed (Artin–)Tate motives* over S, denoted $\mathbf{MTM}(S)$ and $\mathbf{MATM}(S)$, respectively. The cohomology functors belonging to the motivic t-structure are denoted ${}^{\mathrm{p}}\mathrm{H}^*$.

We now study the categories of mixed Tate motives over S in some detail. The key is Theorem 4.2 below, establishing exactness properties of pullback and pushforward functors along closed and open immersions. The exactness axioms for mixed motives over number rings (see [11, Section 4]) are modeled on this theorem. Of course, the theorem is an Artin–Tate motivic analog of a similar fact about perverse sheaves [2, Prop. 1.4.16, 4.2.4.], suggesting that the theory of perverse sheaves is to some extent quite formal. Proposition 4.4 calculates the cohomological dimension of mixed (Artin–)Tate motives. We obtain an equivalence $\mathbf{DTM}(S) \cong \mathbf{D}^{\mathrm{b}}(\mathbf{MTM}(S))$, using a result of Wildeshaus, and likewise for Artin–Tate motives. Finally, we do a first step into (Artin–Tate) motivic sheaves, in Proposition 4.6.

All exactness statements below are with respect to the motivic t-structure of Theorem 3.8. Recall from Theorem 2.4 that the functors discussed below do preserve (Artin–)Tate motives. For brevity, we write $T^{[a,b]}$ for the full subcategory of objects M satisfying ${}^{\mathrm{p}}\mathrm{H}^n M = 0$ for all $n < a$ and $n > b$. We say that a triangulated functor F between categories of Artin–Tate motives has *cohomological amplitude* $[a, b]$ if $F(T^0)$ is contained in $T^{[a,b]}$. Note that F is right exact iff $b \leq 0$ and left exact iff $a \geq 0$.

Theorem 4.2. *Let $j : S' \to S$ be an open immersion, $i : Z \to S$ a closed immersion with $\dim Z = 0$. Finally, let $f : V \to S$ be a finite map with regular one-dimensional domain.*

(i) *The Verdier duality functor D is exact in the sense that it maps $T^{\geq 0}$ to $T^{\leq 0}$ and vice versa. Therefore, it induces an endofunctor on $T^0(S)$.*
(ii) *The functors $j_*, j_!, j^*$, as well as $i_* = i_!$ are exact.*
(iii) *The functor i^* has cohomological amplitude $[-1, 0]$. Dually, $i^!$ has cohomological amplitude $[0, 1]$.*
(iv) *The functor $f_* = f_!$ is exact. The cohomological amplitude of f^* and $f^!$ is $[-1, 0]$ and $[0, 1]$, respectively. If f is also étale, $f^* = f^!$ is exact.*
(v) *The functor $\eta^*[-1] : T(S) \to T(\operatorname{Spec} F)$ is exact.*

Proof. (i) This is clear from (12) and the definitions of the t-structures on $T(S)$, $\tilde{T}(S')$ and $T(Z)$, for open and closed subschemes S' and Z of S, respectively. Notice that this requires putting $\mathbf{1}[1]$ in degree 0.

(ii) The following exactness properties are immediate from the definition: j^* and i_* are exact, j_* and $i^!$ are left-exact and $j_!$ and i^* are right-exact. For example, let us show the left-exactness of j_*. Given some motive $M \in T^{\geq 0}(S')$, we have to show $j_* M \in T^{\geq 0}(S)$. Let $j_1 : S_1 \subset S'$ be an open immersion such that $j_1^* M \in \tilde{T}^{\geq 0}(S_1)$. Let i_1 be the immersion of $Z_1 := S' \backslash S_1$ into S', then $i_1^! M \in T^{\geq 0}(Z_1)$. The situation is as follows:

$$\begin{tikzcd}
 & & Z_1 \arrow[dl, "i_1"'] \arrow[dr] & \\
S_1 \arrow[r, "j_1"] \arrow[drr] & S' \arrow[dr, "j"] & & S \backslash S_1 \arrow[dl, "i"] \\
 & & S &
\end{tikzcd}$$

Now $(j \circ j_1)^* j_* M = j_1^* M \in T^{\geq 0}(S_1)$. Let $i : S \backslash S_1 \to S$ be the complement of $j \circ j_1$. By (7), $i^! j_* M$ is supported only in Z_1, where it agrees with $i_1^! M$. This shows $j_* M \in T^{\geq 0}(S)$.

To prove (iii) we first show

$$i^* j_* \tilde{T}^0(S') \subset T^{[-1,0]}(Z) \tag{16}$$

for any two complementary immersions $i : Z \to S$ (closed) and $j : S' \to S$ (open). By Proposition 3.7, $\tilde{T}^0(S)$ is generated by means of direct sums and extensions by summands of $\phi_* \mathbf{1}(n)[1]$, where $n \in \mathbb{Z}$ is arbitrary and ϕ is finite and étale. For any short exact sequence

$$0 \to A \to X \to B \to 0$$

in $\tilde{T}^0(S)$, such that $i^* j_* A \in T^{[-1,0]}(Z)$ and $i^* j_* B \in T^{[-1,0]}(Z)$, it follows $i^* j_* X \in T^{[-1,0]}(Z)$. This uses the non-degeneracy of the motivic t-structure on Z. A similar remark applies to direct summands and sums. Therefore we only have to check that the generators $X = \phi_* \mathbf{1}(n)[1]$ of $\tilde{T}^0(S')$ are mapped to $T^{[-1,0]}(Z)$ under $i^* j_*$. By (13), there is a distinguished triangle in $T(Z)$

$$i^* \phi_* \mathbf{1}(n)[1] \to i^* j_* j^* \phi_* \mathbf{1}(n)[1] \overset{(11)}{=} i^* j_* \phi'_* \mathbf{1}(n)[1] \to i^! \phi_* \mathbf{1}(n)[2] \to i^* \phi_* \mathbf{1}(n)[2].$$

Here ϕ' is the pullback of ϕ along j. The first term is in degree -1. The third term is in degree 0 by absolute purity (10), using the regularity of S. The claim (16) is shown.

We now show $i^* T^0(S) \subset T^{[-1,0]}(Z)$. Any $X \in T^0(S)$ is in some $T^0_{S'}(S)$ for sufficiently small S'. We shrink S' if necessary to ensure that $S' \cap Z = \emptyset$. Let $j : S' \to S$ be the open immersion and let $p : W \to S$ be its closed complement. There is a triangle

$$p^! X \to p^* X \to p^* j_* j^* X \to p^! X[1].$$

By the above, $p^!$ (p^*) is left-exact (right-exact), that is to say, the first (second) term is in degrees ≥ 0 (≤ 0, respectively). By assumption $j^* X \in \tilde{T}^0(S')$, so $p^* j_* j^* X \in T^{[-1,0]}(W)$ by (16). As the t-structure on W is non-degenerate $p^* X$ is in degrees $[-1, 0]$. As W is the disjoint union of Z and some more (finitely many) closed points, this also shows $i^* X \in T^{[-1,0]}(Z)$.

Now let $i : Z \to S$ and $j : S' \to S$ be complementary. We claim $i^* j_* T^0(S') \subset T^{[-1,0]}(Z)$. Given an object $X \in T^0(S')$, there is some open immersion $j' : S'' \to S'$ such that $j'^* X \in \tilde{T}^0(S'')$. We have $i^* j_* X = i^* j_* j'_* j'^* X$. The motive $i_* i^* j_* j'_* j'^* X$ is a direct summand of $p_* p^* (j \circ j')_* j'^* X$, where p is the complement of $j \circ j'$. By the above, $p^*(j \circ j')_* j'^* X \in T^{[-1,0]}(Z)$, so the full faithfulness and exactness of p_* implies the claim. Part (iii) is shown.

The cohomological amplitude of $i^* j_*$ implies the exactness of j_*: given a mixed (Artin–)Tate motive $M \in T^0(S')$, the terms in the localization triangle

$$j_! M \to j_* M \to i_* i^* j_* M$$

are in degrees ≤ 0, ≥ 0 and $[-1, 0]$, respectively, by the above. From the non-degeneracy of the t-structure we see that $j_* M$ is then in degree 0. This implies the exactness of j_* by the non-degeneracy of the t-structure. The exactness of $j_!$ follows by the Verdier duality, as does the cohomological amplitude of $i^!$. Thus, (ii) is shown.

(iv) It is easy to see that $f^* : \tilde{T}(S) \to \tilde{T}(V)$ is exact. Using this and (6), one sees that f^* has cohomological amplitude $[-1, 0]$ and dually for $f^!$. By a general criterion on t-exactness of adjoint functors [2, 1.3.17], the adjunctions $f^* \leftrightarrows f_* \overset{(8)}{=} f_! \leftrightarrows f^!$ imply that f_* is exact. If f is étale then $f^! \overset{(9)}{=} f^*$, so that their exactness is clear in that case, too.

(v) This follows from the exactness of $j^* : T(S) \to T(S')$ and the exactness of $\eta'^*[-1] : \tilde{T}(S') \to T(\operatorname{Spec} F)$ (Proposition 3.7), where η' is the generic point of S'. $\square$

Definition 4.3 (*Compare [2, 1.4.22]*)**.** Let $j : S' \to S$ be an open immersion. For any mixed (Artin–)Tate motive M over S', put

$$j_{!*} M := \operatorname{im} j_! M \to j_* M.$$

This is called the *intermediate extension* of M along j.

The image is taken in the (abelian) category of mixed (Artin–)Tate motives over S, using the exactness of $j_!$ and j_*. Thereby, $j_{!*}$ is a (non-exact) functor $T^0(S') \to T^0(S)$. Given any mixed motive M over S, such that $i^! M$ is concentrated in cohomological degree -1 (as opposed to the general range $[-1, 0]$), and such that $i^* M$ is in degree $+1$, there is a canonical isomorphism

$$j_{!*} j^* M = M. \tag{17}$$

In particular, this applies to $M \in \tilde{T}^0(S)$, such as $M = \mathbf{1}[1]$. Moreover, taking the intermediate extension commutes with compositions of open immersions. These features will be used below, see [11, Section 4] for a proof. The reader may want to check that that proof only uses the motivic t-structure and exactness properties of $i^!$ etc., which are established by Theorems 3.8, 4.2.

Proposition 4.4. *The cohomological dimension of* $\mathbf{DTM}(S)$ *and* $\mathbf{DATM}(S)$ *is one and two, respectively.*

Proof. We have to show $\operatorname{Hom}(M, M'[n]) = 0$ for any mixed motives M, M' over S and $n > 1$ (Tate) and $n > 2$ (Artin–Tate). Let $j : S' \to S$ be an open immersion such that $j^* M, j^* M' \in \tilde{T}^0(S')$. Let i be the complementary closed immersion of j. In the sequel we write $(-, -)^n$ for $\operatorname{Hom}(-, -[n])$ for brevity.

The case $n \ge 3$ is done as follows: the localization triangle (13) for M' and adjunction (1) gives a long exact sequence

$$(\underbrace{i^* M}_{[-1,0]}, \underbrace{i^! M'[n]}_{[-n,-n+1]})^0 \to (M, M')^n \to (M, j_* j^* M')^n \to (\underbrace{i^* M}_{[-1,0]}, \underbrace{i^! M'[n+1]}_{[-n-1,-n]})^0.$$

We have written the cohomological degrees of the motives underneath, using the cohomological range of i^* and $i^!$. The cohomological dimension zero of (Artin–)Tate motives over finite fields makes the outer terms vanish. Similar vanishings will be used below without further discussion. Hence we only have to look at $(j^* M, j^* M')^n$, i.e., we may assume M and $M' \in \tilde{T}^0(S)$. In that case one reduces (exactly as below) to $M = \phi_* \mathbf{1}(a)[1]$ and $M = \phi'_* \mathbf{1}(a')[1]$, where ϕ and ϕ' are finite and étale. In that case the vanishing is given by Theorem 3.2.

The vanishing in the case $n = 2$ for Tate motives needs a more involved localization argument. A similar reasoning for Artin–Tate motives fails—the difference is because the motives $\mathbf{1}(n)[1]$, which generate $\tilde{T}^0(S)$ in the case of Tate motives, have good reduction at all places by absolute purity.

The localization triangle (6) for M' gives an exact sequence

$$(M, j_! j^* M')^2 \to (M, M')^2 \to (M, i_* i^* M')^2 \overset{(1)}{=} (\underbrace{i^* M}_{[-1,0]}, \underbrace{i^* M'[2]}_{[-3,-2]})^0 = 0.$$

Therefore, in order to show that the middle term vanishes, we may replace M' by $j_! j^* M'$. Similarly, we may replace M by $j_* j^* M$. In particular $M \in j_* \tilde{T}^0(S')$, $M' \in j_! \tilde{T}^0(S')$. By Proposition 3.7 and Remark 2.3, $\tilde{T}^0(S')$ is generated by means of extensions by $\mathbf{1}(a)[1]$ where $a \in \mathbb{Z}$. The claim is stable under extensions so that we may assume $M = j_* A$, $A := \mathbf{1}(a)[1]$, $M' = j_! A'$, $A' := \mathbf{1}(a')[1]$. Let $\tilde{A} := \mathbf{1}(a)[1] \in \tilde{T}^0(S)$ and define $\tilde{A}'$ similarly. We have $j^* \tilde{A} = A$ and similarly with A'.

The localization triangle $j_*A' \to i_*i^*j_*A' \to j_!A'[1]$ maps to $j_*A' \to i_*{}^{\mathrm{p}}\mathrm{H}^0 i^*j_*A' \to (j_{!*}A')[1] = \tilde{A}[1]$. We apply $(\tilde{A}, -)^1$ to this map, which gives the last two exact rows in the diagram. The first exact row maps to the second via the map $\tilde{A} = j_{!*}A \to j_*A$.

$$\begin{tikzcd}
(j_*A, j_*A')^1 \arrow[r] \arrow[d, equal] & (j_*A, i_*i^*j_*A')^1 \arrow[r] \arrow[d, equal] & (j_*A, j_!A')^2 \arrow[r] \arrow[d] & 0 \\
(\tilde{A}, j_*A')^1 \arrow[r] \arrow[d, equal] & (\tilde{A}, i_*i^*j_*A')^1 \arrow[r] \arrow[d, equal] & (\tilde{A}, j_!A)^2 \arrow[r] \arrow[d] & 0 \\
(\tilde{A}, j_*A')^1 \arrow[r] & (\tilde{A}, i_*{}^{\mathrm{p}}\mathrm{H}^0 i^*j_*A')^1 \arrow[r] & (\tilde{A}, \tilde{A})^2 \arrow[r] & 0.
\end{tikzcd}$$

The $=$ signs in the leftmost column are by adjunction (1) and $j^*j_*A = j^*\tilde{A} = A$. The $=$ signs in the second column all use the adjunction $i^* \leftrightarrows i_*$ as well as the cohmological dimension zero of Tate motives over finite fields and cohomological amplitude of i^*, which imply

$$(\underbrace{i^*j_*A}_{[-1,0]}, \underbrace{i^*j_*A'[1]}_{[-2,-1]})^0 = ({}^{\mathrm{p}}\mathrm{H}^{-1} i^*j_*A, {}^{\mathrm{p}}\mathrm{H}^0 i^*j_*A')^0.$$

Applying i^* to the triangle $i_*{}^{\mathrm{p}}\mathrm{H}^{-1} i^*j_*A \to j_!A \to j_{!*}A$ and using $i^*j_! \overset{(7)}{=} 0$ we see $({}^{\mathrm{p}}\mathrm{H}^{-1} i^*j_*A, {}^{\mathrm{p}}\mathrm{H}^0 i^*j_*A')^0 = (i^*j_{!*}A, {}^{\mathrm{p}}\mathrm{H}^0 i^*j_*A')^1$. This justifies the upper $=$ in the second column. The lower $=$ in that column follows by the same argument. However, $(\tilde{A}, \tilde{A}')^2 = 0$, by vanishing of the K-theory in the relevant range (see Theorem 3.2). $\square$

Theorem 4.5. *For both Tate and Artin–Tate motives, the inclusion $T^0(S) \subset T(S)$ extends to a triangulated functor*

$$\mathbf{D}^{\mathrm{b}}(T^0(S)) \to T(S). \tag{18}$$

This functor is an equivalence of categories.

Proof. The category $\mathbf{DM}_{\mathrm{gm}}(S)$ and thus the subcategories of (Artin–)Tate motives embed into some unbounded derived category $\mathbf{D}(\mathcal{A})$, where $\mathcal{A}$ is an exact category. This implies the first statement by a general fact in homological algebra [17, Theorem 1.1.]. Indeed, the interpretation of $\mathbf{DM}_{\mathrm{gm}}(S)$ in terms of h-sheaves shows that (using the notation of [3] and abbreviating **Shv** for the category of $\mathbb{Q}$-linear sheaves with respect to the h-topology on the big site of schemes of finite type over S)

$$\mathbf{DM}_{\mathrm{gm}}(S) \cong \mathbf{D}_{\mathbb{A}^1}(\mathbf{Shv}) \subset \mathbf{D}^{\mathrm{eff}}_{\mathbb{A}^1}(\mathbf{Sp}(\mathbf{Shv})) \subset \mathbf{D}(\mathbf{Sp}(\mathbf{Shv})).$$

More precisely, $\mathbf{DM}_{\mathrm{gm}}(S)$ identifies with the subcategory of W_Ω-local objects in the middle category, which identifies with the subcategory of $W_{\mathbb{A}^1}$-local objects in the right hand category [3, Sections 5.2, 5.3].

The t-structure on $T(S)$ is bounded and non-degenerate, so it remains to show the full faithfulness of (18) or equivalently that the map

$$f_n : \mathrm{Ext}^n_{T^0}(M, M') \to \mathrm{Hom}_T(M, M'[n])$$

is an isomorphism for any $M, M' \in T^0(S)$. The general theory (see e.g. [4, 1.1.5]) shows that f_0 and f_1 are isomorphisms and that f_2 is injective for all M and M'. For Tate motives, f_2 is therefore an isomorphism, since the right hand side is zero by Proposition 4.4. We now show that f_2 is an isomorphism for Artin–Tate motives. The motives M and M' are fixed, so there is some open embedding $j : S' \to S$ such that j^*M and j^*M' are in $\tilde{T}^0(S')$. Let i be the complement of j. The following exact sequences are a consequence of (6) and Theorem 4.2:

$$0 \to i_*{}^{\mathrm{p}}\mathrm{H}^{-1} i^*M \overset{a}{\to} j_!j^*M \to K := \operatorname{coker} a \to 0 \tag{19}$$
$$0 \to K \to M \to i_*{}^{\mathrm{p}}\mathrm{H}^0 i^*M \to 0. \tag{20}$$

We write ${}^n(-,-)$ for Ext^n and ${}_n(-,-)$ for $\mathrm{Hom}_T(-,-[n])$. (19) induces a commutative diagram with exact rows

$$\begin{tikzcd}
{}^1(i_*{}^{\mathrm{p}}\mathrm{H}^{-1} i^*M, M') \arrow[r] \arrow[d, equal] & {}^2(K, M') \arrow[r] \arrow[d] & {}^2(j_!j^*M, M') \arrow[d] \\
{}_1(i_*{}^{\mathrm{p}}\mathrm{H}^{-1} i^*M, M') \arrow[r] & {}_2(K, M') \arrow[r] & {}_2(j_!j^*M, M') = {}_2(j^*M, j^*M').
\end{tikzcd}$$

The rightmost lower term is zero by the vanishing of the K-theory (cf. the argument in the proof of Proposition 4.4), so all vertical maps are isomorphisms. This and (20) yields a similar diagram:

$$\begin{tikzcd} {}^2(i_* {}^{\mathrm{p}}\mathrm{H}^0 i^* M, M') \arrow[r] \arrow[d] & {}^2(M, M') \arrow[r] \arrow[d, "r"] & {}^2(K, M') \arrow[r] \arrow[d, equal] & {}^3(i_* {}^{\mathrm{p}}\mathrm{H}^0 i^* M, M') \arrow[d] \\ {}_2(i_* {}^{\mathrm{p}}\mathrm{H}^0 i^* M, M') \arrow[r] & {}_2(M, M') \arrow[r] & {}_2(K, M') \arrow[r] & {}_3(i_* {}^{\mathrm{p}}\mathrm{H}^0 i^* M, M'). \end{tikzcd}$$

The outer terms in the lower row vanish because the cohomological dimension of Artin–Tate motives over $\mathbb{F}_{\mathfrak{p}}$ is zero and $i^!$ has cohomological amplitude $[0, 1]$. We now show that the rightmost upper term is zero. Altogether, this implies that r is also surjective. We write $A := {}^{\mathrm{p}}\mathrm{H}^0 i^* M$; it is a mixed motive over $\mathbb{F}_{\mathfrak{p}}$. Any element of the Yoneda-Ext-group in question is represented by an exact sequence

$$0 \to i_* A \to X_1 \xrightarrow{s} X_2 \to X_3 \to M' \to 0$$

in $\mathbf{MATM}(S)$. This extension is the image under the concatenation mapping

$${}^2(i_* A, \operatorname{coker} s) \times {}^1(\operatorname{coker} s, M') \to {}^3(i_* A, M').$$

The left hand factor is a subgroup of ${}_2(i_* A, \operatorname{coker} s) = {}_2(A, i^! \operatorname{coker} s) = 0$ (see above). Therefore, the extension above splits and we have shown that the second Ext-groups and Hom-groups agree.

This shows that the $\operatorname{Hom}(M, M'[n])$ form an effaceable δ-functor, so they are universal and agree with $\operatorname{Ext}^n(M, M')$ for all $n \geq 0$. Indeed, for $n \leq 2$ the groups are effaceable since they agree with Ext's by the above, for $n > 2$ the groups are zero by Proposition 4.4. □

The functor $\eta_* : \mathbf{DM}(F) \to \mathbf{DM}(S)$ does not preserve Artin–Tate motives:

$$\operatorname{Hom}_{\mathbf{DM}(S)}(\mathbf{1}, \eta_* \mathbf{1}(1)[1]) \overset{(1)}{=} \operatorname{Hom}_{\mathbf{DM}(F)}(\mathbf{1}, \mathbf{1}(1)[1]) \overset{(5)}{=} K^1(F)^{(1)}_{\mathbb{Q}} = F^\times \otimes \mathbb{Q},$$

which is a countably infinite-dimensional $\mathbb{Q}$-vector space. However, the dimensions of all Hom-groups in $T(S)$ are finite (Theorem 3.2). This example is sharpened by the following proposition. It might be paraphrased by saying that the "site" of mixed Artin–Tate motives over S has enough points.

Proposition 4.6. *For any Artin–Tate motive M over $S \subset \operatorname{Spec} \mathcal{O}_F$, the following are equivalent:*

(i) $M = 0$.
(ii) $M = \eta_* M_\eta$, *where M_η is some geometric motive over F.*
(iii) $i^*_{\mathfrak{p}} M = 0$ *for all closed points $\mathfrak{p}$ of S.*
(iv) $i^!_{\mathfrak{p}} M = 0$ *for all closed points $\mathfrak{p}$ of S.*

Proof. The equivalence of (ii), (iii), and (iv) is an easy consequence of Verdier duality (12) and the limiting localization triangle (14). We now show (iii) $\Rightarrow$ (i). Using localization (6), the claim for M is implied by the one for $j^* M$ for any open immersion j. Therefore we may assume $M \in \tilde{T}(S)$. Using the (-1)-exactness of $i^*_{\mathfrak{p}} : \tilde{T}(S) \to T(\mathbb{F}_{\mathfrak{p}})$ we can even assume $M \in \tilde{T}^0(S)$. Given a short exact sequence in the abelian category $\tilde{T}^0(S)$

$$0 \to A \to M \to B \to 0$$

with $\eta_* \eta^* M = M$, it follows that $\eta_* \eta^* A = A$ and likewise for B. This is shown as follows: for all closed points $\mathfrak{p} \in S$, $i_{\mathfrak{p}*} i^!_{\mathfrak{p}} M = 0$ implies $i^!_{\mathfrak{p}} B = i^!_{\mathfrak{p}} A[1]$, by the full faithfulness of $i_{\mathfrak{p}*}$. The long exact ${}^{\mathrm{p}}\mathrm{H}^-$-sequence and the cohomological amplitude of $i^!_{\mathfrak{p}}$ (Theorem 4.2) shows ${}^{\mathrm{p}}\mathrm{H}^0 i^!_{\mathfrak{p}} B = {}^{\mathrm{p}}\mathrm{H}^1 i^!_{\mathfrak{p}} A$ and all other ${}^{\mathrm{p}}\mathrm{H}^* i^!_{\mathfrak{p}} B$, ${}^{\mathrm{p}}\mathrm{H}^* i^!_{\mathfrak{p}} A$ vanish. However, for any $B \in \tilde{T}^0(S)$, $i^!_{\mathfrak{p}} B$ is in cohomological degree 1 (as opposed to the general range $[0, 1]$): this may be checked on generators of $\tilde{T}^0_a(S)$ for all a, where it follows directly from the definitions (see the proof of Theorem 4.2). Thus ${}^{\mathrm{p}}\mathrm{H}^0 i^!_{\mathfrak{p}} B = 0$, whence $i^!_{\mathfrak{p}} B = i^!_{\mathfrak{p}} A[1] = 0$ for all $\mathfrak{p}$.

Thus the statement for M is implied by the one for A and B. By the characterization of $\tilde{T}^0(S)$ of Proposition 3.7, we therefore only need to check the statement for generators of $\tilde{T}^0_{-2n}(S)$.

We first do this in the case of Tate motives. Then $\tilde{T}^0_{-2n}(S)$ consists of direct sums of motives $G := \mathbf{1}(n)[1]$. In that case the claim is clear, since none of the (nonzero) generators G satisfy $\eta_* \eta^* G = G$: we can twist it so that $n = 1$. Then $\mathrm{H}^0(\eta_* \eta^* G)$ is infinite-dimensional, namely the group of units in some number field (tensored with $\mathbb{Q}$), but $\mathrm{H}^0(G)$ is the group of units in some ring of S-integers, which are of finite rank.

In the case of Artin–Tate motives, the category $\tilde{T}^0_{-2n}(S)$ is generated by means of direct sums and summands by motives $G := \phi_* \mathbf{1}(n)[1]$, $\phi : V \to S$ finite and étale. Actually, we may assume ϕ is Galois: by the same argument as in the proof of Proposition 2.6, after shrinking S sufficiently, $\mathbf{1}_V$ is a direct summand of $\tilde{\phi}_* \mathbf{1}$ where $\tilde{\phi} : \tilde{V} \to V$ is the map corresponding to some normal closure of the function field extension $k(V)/k(S)$. Let M be a summand of G satisfying $\eta_* \eta^* M = M$. There is a map $f : S' \to S$ such that f^*M is a Tate motive, Proposition 2.6. By base-change (11) and the preceding step, we get $f^*M = 0$. The map $\mathrm{End}(M) \subset \mathrm{End}(G) \xrightarrow{a} \mathrm{End}(f^*G)$ factors over $\mathrm{End}(f^*M) = 0$, so we have to show that a is injective. This is done with the same argument as in the proof of Proposition 2.6: we may shrink S so that f is étale. Since ϕ is Galois, we have

$$\mathrm{End}(G) \overset{(1),(9)}{=} \mathrm{Hom}(\mathbf{1}_V, \phi^*\phi_*\mathbf{1}_V) \overset{(11)}{=} \mathrm{Hom}(\mathbf{1}_V, \mathbf{1}_V^{\oplus \deg \phi})$$

and

$$\mathrm{End}(f^*G) = \mathrm{Hom}(\mathbf{1}_{V'}, \phi'^*\phi'_*\mathbf{1}_{V'}) = \mathrm{Hom}(\mathbf{1}_{V'}, \mathbf{1}_{V'}^{\oplus \deg \phi'}),$$

where $\phi' : V' := V \times_S S' \to S'$ is the pullback of ϕ along f. It is also Galois and $\deg \phi = \deg \phi'$. □

References

[1] Joseph Ayoub, Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique. I, in: Astérisque, vol. 314, 2008, p. x+466. 2007.

[2] A.A. Beĭlinson, J. Bernstein, P. Deligne, Faisceaux pervers, in: Analysis and Topology on Singular Spaces, I (Luminy, 1981), in: Astérisque, vol. 100, Soc. Math., France, Paris, 1982, pp. 5–171.

[3] Denis-Charles Cisinski, Frédéric Déglise, Triangulated categories of motives, Preprint, 2010.

[4] Pierre Deligne, Alexander B. Goncharov, Groupes fondamentaux motiviques de Tate mixte, Ann. Sci. École Norm. Sup. (4) 38 (1) (2005) 1–56.

[5] Masaki Hanamura, Mixed motives and algebraic cycles. I, Math. Res. Lett. 2 (6) (1995) 811–821.

[6] Florian Ivorra, Réalisation l-adique des motifs triangulés géométriques. I, Doc. Math. 12 (2007) 607–671.

[7] Marc Levine, Tate motives and the vanishing conjectures for algebraic K-theory, in: Algebraic K-Theory and Algebraic Topology (Lake Louise, AB, 1991), in: NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 407, Kluwer Acad. Publ., Dordrecht, 1993, pp. 167–188.

[8] Marc Levine, Mixed Motives, in: Mathematical Surveys and Monographs, vol. 57, American Mathematical Society, Providence, RI, 1998.

[9] Fabrice Orgogozo, Isomotifs de dimension inférieure ou égale à un, Manuscripta Math. 115 (3) (2004) 339–360.

[10] Daniel Quillen, On the cohomology and K-Theory of the general linear groups over a finite field, Ann. of Math. (2) 96 (1972) 552–586.

[11] Jakob Scholbach, f-cohomology and motives over $\mathbb{Z}$. Preprint, Feb. 2010, http://arxiv.org/abs/1003.1219, 2010.

[12] Anthony J. Scholl, Remarks on special values of L-functions, in: L-Functions and Arithmetic (Durham, 1989), in: London Math. Soc. Lecture Note Ser., vol. 153, Cambridge Univ. Press, Cambridge, 1991, pp. 373–392.

[13] C. Soulé, K-théorie des anneaux d'entiers de corps de nombres et cohomologie étale, Invent. Math. 55 (3) (1979) 251–295.

[14] Vladimir Voevodsky, Triangulated categories of motives over a field, in: Cycles, Transfers, and Motivic Homology Theories, in: Ann. of Math. Stud., vol. 143, Princeton Univ. Press, Princeton, NJ, 2000, pp. 188–238.

[15] Charles Weibel, Algebraic K-theory of rings of integers in local and global fields, in: Handbook of K-Theory. Vol. 1, 2, Springer, Berlin, 2005, pp. 139–190.

[16] J. Wildeshaus, Notes on Artin–Tate motives. Preprint, Nov 2008, arXiv, http://arxiv.org/abs/0811.4551v1.

[17] Jörg Wildeshaus, f-catégories, tours et motifs de Tate, C. R. Math. Acad. Sci. Paris 347 (23–24) (2009) 1337–1342.