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Subspace gaps and range-kernel orthogonality of an elementary operator

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Abstract

Range-kernel orthogonality is established for certain elementary operators.

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1. Introduction

If M and N are subspaces of a Banach space X, then ([14] IV (2.8), (2.9); [12]) the gap between M and N is defined by

$$\mathrm{gap}(M, N) = \max(\delta(M, N), \delta(N, M)),$$

where

$$\delta(M, N) = \|K_N J_M\| = \sup\{\mathrm{dist}(x, N) : x \in M, \|x\| \leqslant 1\}.$$

Here $J_M : M \to X$ and $K_N : X \to X/N$ are the *natural injection* and the *quotient map* induced by M and N. Dually [13]

$$\gamma(M, N) = \gamma(K_M J_N) = \inf\{\|x\|_M : \mathrm{dist}(x, N) \geqslant 1\}$$

is the conorm or "reduced minimum modulus" of the operator $K_M J_N$. Evidently

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$$\gamma(M, N) \geqslant 1;$$

we say that M *is orthogonal to* N when this holds with equality. In the following we focus on the *range* and the *null space* of an operator.

Definition. If $T : X \to Y$ and $S : Y \to Z$ are bounded linear operators between Banach spaces, and $k > 0$, we shall say that (S, T) has a k-gap, and write $S^{-1}(0) \perp_k TX$, provided

$$y \in S^{-1}(0) \Longrightarrow \|y\| \leqslant k\,\mathrm{dist}(y, T(X)). \tag{1}$$

If we let $M = \mathrm{cl}\,T(X)$ and $N = S^{-1}(0)$, then implication (1) says that the operator $K_M J_N$ is bounded below. The presence of such a gap in the pair (S, T) is sandwiched between various kinds of "skew exactness" [11,13]. If we call (S, T) *left skew exact* when

$$S^{-1}(0) \cap T(X) = \{0\}, \tag{2}$$

strongly left skew exact if there is $k > 0$ for which

$$\|T(.)\| \leqslant k\|ST(.)\| \quad \text{on } X \tag{3}$$

and *splitting left skew exact* when there is a bounded linear $R : Z \to Y$ for which

$$T = RST, \tag{4}$$

then there is implication

$$(4) \Longrightarrow (3) \Longrightarrow (1) \Longrightarrow (2). \tag{5}$$

If we go on to call (S, T) *linearly left skew exact* when

$$S^{-1}(0) \cap \mathrm{cl}\,T(X) = \{0\}, \tag{6}$$

then also

$$(1) \Longrightarrow (6) \Longrightarrow (2). \tag{7}$$

If for example the operator S has a bounded left inverse R then the condition (4) holds, hence also (1). If we let $Y = Z$, then a sufficient condition for this to hold is $\|I - S\| < 1$. This compares with, but does not correspond to (see [13]), a result of Turnšek [21, Theorem 1.1] who notices that if $X = Y = Z = \mathscr{A}$ is a Banach algebra, $S = I - \phi$ and $\phi : \mathscr{A} \to \mathscr{A}$ is a linear transformation such that $\|\phi\| \leqslant 1$, then $S^{-1}(0) \perp_1 S\mathscr{A}$. More generally if $\phi : \mathscr{A} \to \mathscr{A}$ is power bounded (i.e., there exists a number $k > 0$ such that $\sup_n \|\phi^n\| \leqslant k$), $S = I - \phi$ and $T = \phi - \phi^2$, then $S^{-1}(0) \perp_k T\mathscr{A}$. This follows from the argument below upon letting $n \to \infty$. If $y \in (I - \phi)^{-1}(0)$ and $x \in \mathscr{A}$, then

$$\left(\phi - \phi^{n+1}\right)(x) = \sum_{i=1}^{n} \phi^i\{(I - \phi)x + y\} - ny$$

$$\Longrightarrow \|y\| \leqslant \left(\frac{2k}{n}\right)\|x\| + k\|Tx + y\|.$$

Let $B(H)$ denote the algebra of operators (= bounded linear transformations) on a separable complex infinite dimensional Hilbert space H, and let $\mathscr{I}$ denote $B(H)$ (with its usual operator norm $\|\cdot\|$) or one of the von Neumann–Schatten p-classes $\mathscr{C}_p$, $1 \leqslant p < \infty$ (with norm $\|\cdot\|_p$). Particularly interesting examples of the operator ϕ, ϕ not necessarily a contraction, which have attracted a lot of attention are the elementary operator $\varDelta_{\mathbf{ab}} : B(H) \to \mathscr{I}$, $\varDelta_{\mathbf{ab}}(x) = a_1 x b_1 - a_2 x b_2$, where $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ are pairs of mutually commuting normal operators in $B(H)$, and the generalized derivation $\delta_{ab} : B(H) \to \mathscr{I}$, $\delta_{ab}(x) = ax - xb$, where $a, b \in B(H)$ are normal operators (see [1,4,8–10,15,21–23] for further references). In this paper we consider the elementary operator $\varDelta_{\mathbf{ab}}$ for the case in which the operators a_1, b_1^* are hyponormal, the operators a_2, b_2 are normal, a_1 commutes with a_2 and b_1 commutes with b_2. It will be shown that $(\varDelta_{\mathbf{ab}}, \varDelta_{\mathbf{ab}})$ has a k-gap (so that $(\varDelta_{\mathbf{ab}}, \varDelta_{\mathbf{ab}})$ is both left skew exact and linearly left skew exact).

2. Results

Recall that an element $h \in B(X)$, X a Banach space, is said to be *hermitian* if the spatial numerical range $W(h) = \{f(hx) : x \in X, f \in X', \|x\| = \|f\| = f(x) = 1\}$ of h is a subset of the reals, and the operator $a = h + \mathrm{i}g \in B(X)$, h and g hermitian, is said to be *normal* if the *commutator* $[h, g] = hg - gh$ is 0. Recall also that if the operator $\phi : X \to X$ is normal, then $\phi^{-1}(0) \perp_1 \phi X$ [10, Theorem A]. The normality of $a, b \in B(H)$ guarantees the normality of δ_{ab}. (Indeed, the condition is both necessary and sufficient; see [20, Theorem 2.2].) Also, if $\mathscr{I}$ is the Hilbert–Schmidt class $\mathscr{C}_2$ with its Hilbert space structure and $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ are n-tuples of mutually commuting normal operators, then $\mathscr{E}_{\mathbf{ab}}(x) = \sum_{i=1}^n a_i x b_i$ is a normal operator on $\mathscr{I} \to \mathscr{I}$ such that $\mathscr{E}^*_{\mathbf{ab}}(x) = \mathscr{E}_{\mathbf{a}^*\mathbf{b}^*}(x)$ and $\|\mathscr{E}_{\mathbf{ab}}(x) + y\|_2^2 = \|\mathscr{E}_{\mathbf{ab}}(x)\|_2^2 + \|y\|_2^2$ for every $y \in \mathscr{E}^{-1}_{\mathbf{ab}}(0) \cap \mathscr{C}_2$ and $x \in \mathscr{C}_2$ [9]. More generally, if $\mathscr{E}_{\mathbf{ab}}$ (or δ_{ab}): $\mathscr{I} \to \mathscr{I}$ is normal, then $(\mathscr{E}_{\mathbf{ab}}, \mathscr{E}_{\mathbf{ab}})$ (resp., $(\delta_{ab}, \delta_{ab})$) has a 1-gap (see [10, Theorem A]). We remark here that the normality of a_i and b_i in the mutually commuting n-tuples a and b is not enough to warrantee the normality of $\mathscr{E}_{\mathbf{ab}}$.

Example 2.1. Recall from [22, Theorem 2.4] that if $a, b \in B(H)$ are commuting normal operators and $\phi : B(H) \to B(H)$ is defined by $\phi(x) = axa^* - bxb^*$, then

$$\|\phi(x) + y\|_{\mathscr{I}} \geqslant \|y\|_{\mathscr{I}} \tag{8}$$

for every $y \in \phi^{-1}(0) \cap \mathscr{I}$ and all $x \in B(H)$ such that $\phi(x) \in \mathscr{I}$ if and only if $a^{-1}(0) \cap b^{-1}(0) = \{0\}$. Thus if we choose the normal a and b to be such that $a^{-1}(0) \cap b^{-1}(0) \neq \{0\}$, then ϕ cannot be normal (for if it were then we would have by [10, Theorem A] that (8) holds). Now choose the operator b in $\phi(x) = axa^* - bxb^*$ to be the identity operator and let the normal a be such that $a^{-1}(0) \neq \{0\}$. Then $a^{-1}(0) \cap b^{-1}(0) = \{0\}$ and if $\phi^{-1}(0) \neq \{0\}$ then 1 is an eigenvalue of ϕ. Suppose that ϕ is normal. Then Φ_a, $\Phi_a(x) = \phi(x) + x = axa^*$, is normal. Hence

$\|\Phi_a(x) + y\|_{\mathscr{I}} \geqslant \|y\|_{\mathscr{I}}$, $\mathscr{I} \neq \mathscr{C}_2$, for every $y \in \Phi_a^{-1}(0) \cap \mathscr{I}$ and all $x \in B(H)$ such that $\Phi_a(x) \in \mathscr{I}$. This however contradicts [22, Proposition 2.1]. Hence ϕ is not normal.

Example 2.1 shows that the normality of the commuting pair of operators a and b is not sufficient for the subspaces $\phi^{-1}(0) \cap \mathscr{I}$ and $\operatorname{cl} \phi\mathscr{I}$, $\mathscr{I} \neq \mathscr{C}_2$, to have a 1-gap: It is however sufficient for the said subspaces to have a k-gap for some $k \geqslant 1$ (see [15, Theorem 2]). The normality of a and b is not a necessary condition for $\phi^{-1}(0) \cap \mathscr{I}$ and $\operatorname{cl} \phi\mathscr{I}$, $\mathscr{I} \neq \mathscr{C}_2$, to have a 1-gap. Thus if ϕ is such that 0 is not in the interior of the numerical range of $\phi \in B(B(H))$, then $\|x + \phi y\| \geqslant \|x| - \sqrt{8\|\phi(x)\|\|y\|}$ for all $x, y \in B(H)$ [4, Theorem 6, p. 20]. In particular, if 0 is an eigen-value which is not in the interior of the numerical range of ϕ, then $\phi^{-1}(0) \perp_1 \phi(B(H))$. (We remark here that this is precisely the situation when $S : \mathscr{A} \to \mathscr{A}$, $\mathscr{A}$ a unital Banach algebra, is a contraction and $\phi = I - S$.) Our main result, Theorem 2.7, shows that the hypothesis a and b are commuting normal operators can be replaced by the hypothesis that a is a hyponormal operator which commutes with the normal operator b. (We note here that a better result holds in the case in which $\mathscr{I} = \mathscr{C}_2$, see Remark 2.6 below.) The following complementary lemmas will be required in the proof of our main result.

Lemma 2.2. *Let t have the block matrix representation $t = (t_{ij})$, $1 \leqslant i, j \leqslant n$.*

(i) *If $t \in B(H)$, then*

$$n^{-2} \sum_{i,j} \|t_{ij}\|^2 \leqslant \|t\|^2 \leqslant \sum_{i,j} \|t_{ij}\|^2.$$

(ii) *If $t \in \mathscr{C}_p$, $1 \leqslant p < \infty$, then*

$$n^{2-p} \|t\|_p^p \leqslant \sum_{i,j} \|t_{ij}\|_p^p \leqslant \|t\|_p^p$$

if $2 \leqslant p < \infty$, and

$$\|t\|_p^p \leqslant \sum_{i,j} \|t_{ij}\|_p^p \leqslant n^{2-p} \|t\|_p^p$$

if $1 \leqslant p \leqslant 2$.

Proof. See [3], Theorems 1 and 2. □

Lemma 2.3. *Suppose that the eigen-space corresponding to the eigen-value 0 of $a \in B(H)$ is reducing. If $\Phi_a : B(H) \to B(H)$ is defined by $\Phi_a(x) = axa^*$, then ($y \in \Phi_a^{-1}(0)$ implies $a^*ya = 0$ and) there exists a constant k such that $(\Phi_a^{-1}(0) \cap \mathscr{I}) \perp_k (\Phi_a(B(H)) \cap \mathscr{I})$.*

Proof. The inequality being trivially true if $a^{-1}(0) = \{0\}$, we assume that $a^{-1}(0) \neq \{0\}$. Then, with respect to the decomposition $H = \ker^{\perp} a \oplus \ker a$, a has a direct sum decomposition $a = a_1 \oplus 0$, where $a_1 = a|_{\ker^{\perp} a}$ is an injection. A simple calculation shows that if $y \in \Phi_a^{-1}(0)$, then y has a matrix representation $y = \begin{bmatrix} 0 & y_1 \\ y_2 & y_3 \end{bmatrix}$, where the operators y_i are arbitrary. (Clearly, $a^* y a = 0$.) Letting $x \in B(H)$ have the matrix representation $x = [x_{ij}]_{i,j=1}^{2}$, it follows that

$$\Phi_a(x) + y = \begin{bmatrix} a_1 x_{11} a_1^* & y_1 \\ y_2 & y_3 \end{bmatrix}.$$

We consider the cases $\Phi(x) + y \in B(H)$ and $\Phi(x) + y \in \mathscr{C}_p$ separately.

The operator norm case. Applying Lemma 2.2(i),

$$\begin{aligned}\|\Phi_a(x) + y\|^2 &\geqslant \frac{1}{4}\left\{\|a_1 x_{11} a_1^*\|^2 + \sum_{i=1}^{3}\|y_i\|^2\right\}\\ &\geqslant \frac{1}{4}\sum_{i=1}^{3}\|y_i\|^2 \geqslant \frac{1}{4}\|y\|^2,\end{aligned}$$

and we may choose k to be 2.

$1 \leqslant p \leqslant 2$. Applying Lemma 2.2(ii),

$$\begin{aligned}\|\Phi_a(x) + y\|_p^p &\geqslant 2^{p-2}\left\{\|a_1 x_{11} a_1^*\|_p^p + \sum_{i=1}^{3}\|y_i\|_p^p\right\}\\ &\geqslant 2^{p-2}\sum_{i=1}^{3}\|y_i\|_p^p \geqslant 2^{p-2}\|y\|_p^p,\end{aligned}$$

and we may choose k to be $2^{2/p-1}$.

$2 < p < \infty$. Again applying Lemma 2.2(ii),

$$\begin{aligned}\|\Phi_a(x) + y\|_p^p &\geqslant \|a_1 x_{11} a_1^*\|_p^p + \sum_{i=1}^{3}\|y_i\|_p^p\\ &\geqslant \sum_{i=1}^{3}\|y_i\|_p^p \geqslant 2^{2-p}\|y\|_p^p,\end{aligned}$$

and we may choose k to be $2^{1-2/p}$. □

Recall that the eigen-spaces of a hyponormal operator are reducing. Let a, b^* be hyponormal operators such that $ayb = 0$. Define a_1 and t $(\in B(H \oplus H))$ by $a_1 =$

$a \oplus b^*$ and $t = \begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}$. Then $a_1 t a_1^* = 0$, and it follows from Lemma 2.3 that there exists a constant k such that $\|t\| \leqslant k\|a_1 x a_1^* + t\|$ for all $x \in B(H \oplus H)$ satisfying $a_1 x a_1^* \in \mathscr{I}$. In particular, there exists a constant k, $k \geqslant 1$, such that $k\|axb + y\|_{\mathscr{I}} \geqslant \|y\|_{\mathscr{I}}$ for every $y \in \mathscr{I}$ satisfying $ayb = 0$ and all $x \in B(H)$ satisfying $axb \in \mathscr{I}$.

In the following the (elementary) operator $\Delta_{a,b} : B(H) \to B(H)$ shall be defined by $\Delta_{a,b}(x) = axb - x$. (Thus, if $\mathbf{a}$ and $\mathbf{b}$ are the pairs $\mathbf{a} = (a, I)$ and $\mathbf{b} = (b, I)$, then $\Delta_{a,b}(x) = \Delta_{\mathbf{ab}}(x)$.)

Lemma 2.4. *Let $a, b \in B(H)$ and $1 \leqslant p < \infty$. Then*:

(i) $y \in \{\Delta_{a,b}^{-1}(0) \cap \Delta_{a^*,b^*}^{-1}(0)\} \cap \mathscr{C}_p$ *and* $x \in \mathscr{C}_p \Longrightarrow$
$\|y\|_p \leqslant \|\Delta_{a,b}(x) + y\|_p$.

(ii) $\Delta_{a,b}^{-1}(0) \subseteq \Delta_{a^*,b^*}^{-1}(0) \Longrightarrow \|y\|_{\mathscr{I}} \leqslant \|\Delta_{a,b}(x) + y\|_{\mathscr{I}}$,
for all $y \in \Delta_{a,b}^{-1}(0)$ *and* $x \in B(H)$ *such that* $\Delta_{a,b}(x) \in \mathscr{I}$.

Proof. (i) If $y \in \{\Delta_{a,b}^{-1}(0) \cap \Delta_{a^*,b^*}^{-1}(0)\} \cap \mathscr{C}_p$ and $1 < p < \infty$, then a proof of (i) follows from an application of [8, Theorem 3.6]. Thus let $p = 1$. For $y \in \Delta_{a,b}^{-1}(0) \cap \Delta_{a^*,b^*}^{-1}(0)$, let $H_1 = \ker^\perp y$, $H_2 = \overline{\operatorname{ran} y}$, and let the quasi-affinity $y_1 : H_1 \to H_2$ be defined by setting $y_1 h = yh$ for each $h \in H_1$. Let $a_1 = a|_{H_2}$ and $b_1 = b|_{H_1}$. Then $\Delta_{a_1,b_1}(y_1) = 0 = \Delta_{a_1^*,b_1^*}(y_1)$, and it follows from a consideration of the equations $b_1 y_1^* \Delta_{a_1,b_1}(y_1) = 0 = \Delta_{a_1,b_1}(y_1) y_1^* a_1$ that $[a_1, |y_1^*|] = 0 = [b_1, |y_1|]$. Let y_1 have the polar decomposition $y_1 = u_1|y_1|$, u_1 unitary. Then $a_1 u_1 |y_1| b_1 = a_1 u_1 b_1 |y_1| = u_1 |y_1|$, which implies that $a_1 u_1 b_1 = u_1$, b_1 is invertible and b_1^{-1} is unitarily equivalent to a_1. In particular, $u_1 \in \Delta_{a_1,b_1}^{-1}(0) \cap \Delta_{a_1^*,b_1^*}^{-1}(0)$. Theorem 3.7 of [8] applies and we conclude that

$$\|y_1\|_1 \leqslant \|\Delta_{a_1,b_1}(z) + y_1\|_1$$

for all $z \in \mathscr{C}_1(H_1 \to H_2)$. Now let $x \in \mathscr{C}_1$. Then $x : H_1 \oplus (H \ominus H_1) \to H_2 \oplus (H \ominus H_2)$ has a representation $x = [x_{ij}]_{i,j=1}^2$, where $x_{11} = z$. Letting $a = a_1 \oplus a_2$ and $b = b_1 \oplus b_2$ we have:

$$\begin{aligned} \|y\|_1 = \|y_1\|_1 &\leqslant \|\Delta_{a_1,b_1}(x_{11}) + y_1\|_1 \\ &\leqslant \left\| \begin{bmatrix} \Delta_{a_1,b_1}(x_{11}) + y_1 & \Delta_{a_1,b_2}(x_{12}) \\ \Delta_{a_2,b_1}(x_{21}) & \Delta_{a_2,b_2}(x_{22}) \end{bmatrix} \right\|_1 \\ &= \|\Delta_{a,b}(x) + y\|_1. \end{aligned}$$

This completes the proof of (i).

(ii) Let $y \in \Delta_{ab}^{-1}(0) \subseteq \Delta_{a^*,b^*}^{-1}(0)$. Defining a_1, b_1 and y_1 as in the proof of (i) above it follows that a_1 and b_1^{-1} are unitarily equivalent operators, and both y_1 and

a_1y_1 are in $\Delta^{-1}_{a_1,b_1}(0) \cap \Delta^{-1}_{a_1^*,b_1^*}(0)$. Hence $\Delta_{a_1^*,b_1^*}(a_1y_1) = 0 = a_1\Delta_{a_1^*,b_1^*}(y_1)$, which implies that a_1 and b_1 are normal operators. Thus

$$\|y_1\|_{\mathscr{I}} \leqslant \|\Delta_{a_1,b_1}(z) + y_1\|_{\mathscr{I}}$$

for every $y \in \Delta^{-1}_{a,b}(0) \cap \mathscr{I}$ and all $z \in B(H_1 \to H_2)$ such that $\Delta_{a_1,b_1}(z) \in \mathscr{I}$. The proof is now completed upon arguing as in the proof of (i) above. □

Remark 2.5. The hypothesis $\Delta^{-1}_{a,b}(0) \subseteq \Delta^{-1}_{a^*,b^*}(0)$ is satisfied for a and b^* belonging to a number of the commonly considered classes of Hilbert space operators (see [7,19]), amongst them the class of hyponormal operators. Thus Lemma 2.4(ii) holds, in particular, for hyponormal a and b^*. Indeed more is true: The following (Putnam–Fuglede type) commutativity result holds. *If $a \in B(H)$ is a hyponormal operator which commutes with the normal operator $b \in B(H)$, then $\phi^{-1}(0) \subseteq \phi^{*-1}(0)$, where (as before, $\phi(x) = axa^* - bxb^*$ and) $\phi^*(x) = a^*xa - b^*xb$.* This is seen as follows. Recall from the Berberian extension theorem that given an $a \in B(H)$ there exists a Hilbert space $K \supset H$ and an isometric $*$-isomorphism $a \to a^o$ preserving order such that $\sigma(a) = \sigma(a^o)$ and $\sigma_\pi(a) = \sigma_\pi(a^o) = \sigma_p(a^o)$. (Here $\sigma_p(a)$ and $\sigma_\pi(a)$ denote the point spectrum and the approximate point spectrum, respectively, of a.) Let $y \in \phi^{-1}(0)$, and let $a^o = c$, $b^o = d$ and $y^o = z$; then $czc^* - dzd^* = 0$, where the operator c is hyponormal, the operator d is normal and $[c, d] = 0$. Since $\ker(d)$ reduces d, d has a decomposition $d = 0 \oplus d_{22}$ (on $K = \ker(d) \oplus \ker^\perp(d)$). Letting $c : \ker(d) \oplus \ker^\perp(d) \to \ker(d) \oplus \ker^\perp(d)$ have the matrix representation $c = [c_{ij}]^2_{i,j=1}$, it follows from the commutativity of c and d (combined with the injectivity of d_{22}) that $c_{12} = c_{21} = 0$. Now let $z : \ker(d) \oplus \ker^\perp(d) \to \ker(d) \oplus \ker^\perp(d)$ have the representation $[z_{ij}]^2_{i,j=1}$. Then $0 = c_{11}z_{11}c^*_{11} = c_{11}z_{12}c^*_{22} = c_{22}z_{21}c^*_{11}$ and $c_{22}z_{22}c^*_{22} - d_{22}z_{22}d^*_{22} = 0$. The operators c_{11} and c_{22} being hyponormal, it follows from an application of Lemma 2.3 that $0 = c^*_{11}z_{11}c_{11} = c^*_{11}z_{12}c_{22} = c^*_{22}z_{21}c_{11}$. The operator d_{22} is invertible and $[d^{-1}_{22}, c_{22}] = 0$. Hence $r = d^{-1}_{22}c_{22} = c_{22}d^{-1}_{22}$ is hyponormal, and $rz_{22}r^* - z_{22} = 0$. Recall now that if r is a hyponormal operator and $rt - tr^* = 0$ for some operator $t \in B(K)$, then $r^*t - tr = 0$ [7,19]. This in view of [7, Theorem 2] implies that $r^*z_{22}r - z_{22} = 0$, or, $c^*_{22}z_{22}c_{22} - d^*_{22}z_{22}d_{22} = 0$. Combining, we now have that $c^*zc - d^*zd = 0$, which implies that $a^*ya - b^*yb = 0$.

Remark 2.6. Choosing $\mathscr{I}$ to be the Hilbert space $\mathscr{C}_2$ with the inner product $\langle x, y\rangle = tr(y^*x)$ it is seen that if the operators a and b are as in Remark 2.5, then adjoint of $\phi : \mathscr{C}_2 \to \mathscr{C}_2$ is defined by $\phi^*(x) = a^*xa - b^*xb$,

$$\begin{aligned}\|\phi(x) + y\|^2_2 &= \|\phi(x)\|^2_2 + \|y\|^2_2 + 2\mathrm{Re}\langle\phi(x), y\rangle \\ &= \|\phi(x)\|^2_2 + \|y\|^2_2 + 2\mathrm{Re}\langle x, \phi^*(y)\rangle\end{aligned}$$

and

$$\|\phi^*(x) + y\|^2_2 = \|\phi^*(x)\|^2_2 + \|y\|^2_2 + 2\mathrm{Re}\langle x, \phi(y)\rangle.$$

Hence (by the commutativity property proved in Remark 2.5)

$$\|\phi(x)+y\|_2^2=\|\phi(x)\|_2^2+\|y\|_2^2$$

and

$$\|\phi^*(x)+y\|_2^2=\|\phi^*(x)\|_2^2+\|y\|_2^2$$

for every $y\in\phi_{ab}^{-1}(0)\cap\mathscr{C}_2$ and all $x\in\mathscr{C}_2$.

The following is our main result.

Theorem 2.7. *Let $\Delta_{\mathbf{ab}}\in B(B(H))$ be the elementary operator $\Delta_{\mathbf{ab}}(x)=a_1xb_1-a_2xb_2$, where a_1 and $b_1^*\in B(H)$ are hyponormal operators, a_2 and $b_2\in B(H)$ are normal operators, a_1 commutes with a_2 and b_1 commutes with b_2. Then there exists a constant k such that $(\Delta_{\mathbf{ab}}^{-1}(0)\cap\mathscr{I})\perp_k(\Delta_{\mathbf{ab}}(B(H))\cap\mathscr{I})$.*

Proof. If we set $\widehat{H}=H\oplus H$, and define the operators $a, b, s, z\in B(\widehat{H})$ and the operator $\phi\in B(B(H))$ by $a=a_1\oplus b_1^*$, $b=a_2\oplus b_2^*$, $s=\begin{bmatrix}0&y\\0&0\end{bmatrix}$, $z=\begin{bmatrix}0&x\\0&0\end{bmatrix}$ and $\phi(z)=aza^*-bzb^*$, then

$$\phi(z)+s=\begin{bmatrix}0&\Delta_{\mathbf{ab}}(x)+y\\0&0\end{bmatrix},\quad \|s\|=\|y\|,\quad \|\Delta_{\mathbf{ab}}(x)+y\|=\|\phi(z)+s\|,$$

and

$$\|s\|_p=\|y\|_p,\quad \|\Delta_{\mathbf{ab}}(x)+y\|_p=\|\phi(z)+s\|_p$$

for every $y\in\mathscr{C}_p$ and all x such that $\Delta_{\mathbf{ab}}(x)\in\mathscr{C}_p$. Thus to prove the theorem it will suffice to prove the following: *If $a\in B(H)$ is a hyponormal operator which commutes with the normal operator $b\in B(H)$, then there exists a constant k such that* $(\phi^{-1}(0)\cap\mathscr{I})\perp_k(\phi(B(H))\cap\mathscr{I})$. We divide the proof into the cases: (i) b is injective; (ii) $a^{-1}(0)\cap b^{-1}(0)=\{0\}$; (iii) $a^{-1}(0)=b^{-1}(0)$; and (iv) $a^{-1}(0)\neq b^{-1}(0)$, where cases (i) and (ii) will be used in proving cases (iii) and (iv).

Case (i). We start by noticing that if b is invertible, then $r=ab^{-1}=b^{-1}a$ is hyponormal and $\Delta_{r,r^*}^{-1}(0)\subseteq\Delta_{r^*,r}^{-1}(0)$ (by Remark 2.5). Applying Lemma 2.4(ii) it follows that

$$\begin{aligned}\|\phi(x)+y\|_{\mathscr{I}}&=\|(ab^{-1})(bxb^*)(ab^{-1})^*-(bxb^*)+y\|_{\mathscr{I}}\\&=\|\Delta_{r,r^*}(bxb^*)+y\|_{\mathscr{I}}\geqslant\|y\|_{\mathscr{I}}\end{aligned}$$

for all $y\in\phi^{-1}(0)\cap\mathscr{I}$ and $x\in B(H)$ such that $\phi(x)\in\mathscr{I}$. Suppose now that $b^{-1}(0)=\{0\}$. Let δ_n denote the set of complex numbers λ such that $|\lambda|\leqslant\frac{1}{n}$ for some natural number n, and let $E_b(\delta_n)$ denote the corresponding spectral projection. Set $P_n=I-E_b(\delta_n)$; then P_n converges strongly to I. Since $[a,b]=0=[a,b^*]$, P_nH reduces both a and b. This, upon decomposing H by $H=(I-P_n)H\oplus P_nH$, implies that a and b have the direct sum decompositions $a=a_{1(n)}\oplus a_{2(n)}$ and $b=$

$b_{1(n)} \oplus b_{2(n)}$, where $a_{i(n)}$, $i = 1, 2$, are hyponormal and $b_{2(n)}$ is invertible. Let $y \in \phi^{-1}(0) \cap \mathscr{I}$ have the representation $[y_{ij}]_{i,j=1}^{2}$ and let $x \in B(H)$ have the representation $x = [x_{ij}]_{i,j=1}^{2}$. Then, since

$$\begin{aligned} P_n(\phi(x) + y)P_n &= P_n(axa^* - bxb^* + y)P_n \\ &= P_n a P_n x P_n a^* P_n - P_n b P_n x P_n b^* P_n + P_n y P_n \\ &= a_{2(n)} P_n x P_n a^*_{2(n)} - b_{2(n)} P_n x P_n b^*_{2(n)} + P_n y P_n \end{aligned}$$

and the operator $b_{2(n)}$ is invertible, we have that

$$\begin{aligned} \|\phi(x) + y\|_{\mathscr{I}} &= \left\| \begin{bmatrix} * & * \\ * & P_n(\phi(x) + y)P_n \end{bmatrix} \right\|_{\mathscr{I}} \\ &\geqslant \|P_n(\phi(x) + y)P_n\|_{\mathscr{I}} \geqslant \|P_n y P_n\|_{\mathscr{I}}. \end{aligned}$$

Thus

$$\sup_n \|P_n y P_n\|_{\mathscr{I}} \leqslant \|\phi(x) + y\|_{\mathscr{I}} < \infty$$

for all $s \in \phi^{-1}(0) \cap \mathscr{I}$ and $x \in B(H)$ such that $\phi(x) \in \mathscr{I}$. Since $P_n y P_n \to y$ weakly (even strongly), it follows from an application of [15, Lemma 3] that

$$\|y\|_{\mathscr{I}} \leqslant \sup_n \|P_n y P_n\|_{\mathscr{I}} \leqslant \|\phi(x) + y\|_{\mathscr{I}}.$$

Case (ii). Decompose H by $H = (\ker(b) \oplus \ker(a)) \oplus (\ker^{\perp}(b) \ominus \ker(a))$. Then it is seen (from a straightforward argument using the fact that $[a, b] = 0 = [a, b^*]$) that $a = a' \oplus a_2$ and $b = b' \oplus b_2$, where a_2, b_2 are injective, $b' = 0$ and a' is injective in the case in which $\ker(a) = \{0\}$, and $a' = a_1 \oplus 0$ and $b' = 0 \oplus b_1$, with a_1 and b_1 injective, in the case in which $\ker(a) \neq \{0\}$. Let $y \in \phi^{-1}(0) \cap \mathscr{I}$. Then (this is easily seen) $y = y' \oplus y_{33}$, where $y' = 0$ in the case in which $\ker(a) = \{0\}$ and $y' = \begin{bmatrix} 0 & y_{22} \\ y_{11} & 0 \end{bmatrix}$ in the case in which $\ker(a) \neq \{0\}$. We consider the cases $\ker(a) \neq \{0\}$ and $\ker(a) = \{0\}$ separately. It is convenient at this point for us to define the operator ϕ_{rs} by $\phi_{rs}(x) = rxr^* - sxs^*$.

Case $\ker(a) \neq \{0\}$. Let $(0 \neq) y_{ii}$, $i = 1, 2$, have the polar decomposition $u_{ii}|y_{ii}|$, and let $x = [x_{ij}]_{i,j=1}^{3} \in B(H)$ be such that $\phi(x) \in \mathscr{I}$. Define $v \in B(H)$ by $v = \begin{bmatrix} 0 & u^*_{11} \\ u^*_{22} & 0 \end{bmatrix} \oplus I_{\ker^{\perp}(b) \ominus \ker(a)}$. Then

$$\begin{aligned} \|\phi(x) + y\|_{\mathscr{I}} \geqslant \|v(\phi(x) + y)\|_{\mathscr{I}} &= \left\| \begin{bmatrix} |y_{11}| & * & * \\ * & |y_{22}| & * \\ * & * & \phi_{a_2 b_2}(x_{33}) + y_{33} \end{bmatrix} \right\|_{\mathscr{I}} \\ &\geqslant \left\| \begin{bmatrix} t & 0 \\ 0 & \phi_{a_2 b_2}(x_{33}) + y_{33} \end{bmatrix} \right\|_{\mathscr{I}}, \end{aligned}$$

where $t = \begin{bmatrix} |y_{11}| & 0 \\ 0 & |y_{22}| \end{bmatrix}$. The normal operator b_2 being injective, it follows from Case (i) that $\|\phi_{a_2b_2}(x_{33}) + y_{33}\|_{\mathscr{I}} \geqslant \|y_{33}\|_{\mathscr{I}}$. Hence

$$\|\phi(x) + y\| \geqslant \max\{\|t\|, \|y_{33}\|\} = \|y\|$$

and

$$\|\phi(x) + y\|_p \geqslant (\|t\|_p^p + \|y_{33}\|_p^p)^{\frac{1}{p}} = \|y\|_p.$$

Case $\ker(a) = \{0\}$. In this case if we let x, $x \in B(\ker(b) \oplus \ker^{\perp}(b))$ such that $\phi(x) \in \mathscr{I}$, have the matrix representation $\begin{bmatrix} x_0 & x_1 \\ x_2 & x_{33} \end{bmatrix}$, then

$$\|\phi(x) + y\|_{\mathscr{I}} = \left\| \begin{bmatrix} * & * \\ * & \phi_{a_2b_2}(x_{33}) + y_{33} \end{bmatrix} \right\|_{\mathscr{I}} \geqslant \|y_{33}\|_{\mathscr{I}} = \|y\|_{\mathscr{I}}.$$

Case (iii). In the case in which $a^{-1}(0) = b^{-1}(0)$, let H have the direct sum decomposition $H = \ker^{\perp} b \oplus \ker b$. Then $a = a_1 \oplus 0$, $b = b_1 \oplus 0$, $a_1^{-1}(0) \cap b_1^{-1}(0) = \{0\}$ and every $y \in \phi^{-1}(0)$ has the representation $y = [y_{ij}]_{i,j=1}^2$, where $\phi_{a_1b_1}(y_{11}) = 0$ and the other y_{ij} are arbitrary. Letting $x = [x_{ij}]_{i,j=1}^2 \in B(H)$, it follows that

$$\begin{aligned} \|\phi(x) + y\|^2 &= \left\| \begin{bmatrix} \phi_{a_1b_1}(x_{11}) + y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \right\|^2 \\ &\geqslant \frac{1}{4} \left\{ \|\phi_{a_1b_1}(x_{11}) + y_{11}\|^2 + \sum_{i,j:i,j\neq 1} \|y_{ij}\|^2 \right\} \\ &\geqslant \frac{1}{4} \sum_{i,j} \|y_{ij}\|^2 \geqslant \frac{1}{4}\|y\|^2 \end{aligned}$$

(where we use Case (ii) to conclude that $\|\phi_{a_1b_1}(x_{11}) + y_{11}\| \geqslant \|y_{11}\|$). A similar argument gives that

$$\|\phi(x) + y\|_p \geqslant k\|y\|_p,$$

where $k = 2^{1-2/p}$ if $1 \leqslant p < 2$ and $k = 2^{2/p-1}$ if $2 < p < \infty$.

Case (iv). As before, let $\phi_{rs}(x) = rxr^* - sxs^*$. Letting H have the direct sum decomposition $H = \ker^{\perp} b \oplus \ker b$, it follows from the commutativity of a and b that $b = b_1 \oplus 0$ and $a = a_1 \oplus a_{22}$, where $b_1 = b|_{\ker^{\perp} b}$ is injective. Now decompose $\ker b$ by $\ker b = (\ker b \ominus \ker a_{22}) \oplus \ker a_{22}$. Then $b = b_1 \oplus 0 \oplus 0$ and $a = a_1 \oplus a_2 \oplus 0$, where a_2 is injective. A simple calculation shows that if $y \in \phi^{-1}(0)$, then (with respect to the decomposition $H = \ker^{\perp} b \oplus (\ker b \ominus \ker a_{22}) \oplus \ker a_{22}$ of H)

$$y = \begin{bmatrix} y_1 & y_2 & y_3 \\ y_4 & 0 & y_5 \\ y_6 & y_7 & y_8 \end{bmatrix},$$

where $\phi_{a_1b_1}(y_1) = 0$, $\Phi_{a_1a_2^*}(y_2) = 0 = \Phi_{a_2a_1^*}(y_3)$ and the remaining y_i are arbitrary. (Recall that $\Phi_{mt}(z) = mzt$ and $\Phi_m(z) = mzm^*$.) Let $x \in B(H)$ have the representation $x = [x_{ij}]_{i,j=1}^3$. Then

$$\|\phi(x) + y\|^2 = \left\| \begin{bmatrix} \phi_{a_1b_1}(x_{11}) + y_1 & \Phi_{a_1a_2^*}(x_{12}) + y_2 & y_4 \\ \Phi_{a_2a_1^*}(x_{21}) + y_3 & \Phi_{a_2}(x_{22}) & y_5 \\ y_6 & y_7 & y_8 \end{bmatrix} \right\|^2$$
$$\geqslant \frac{1}{9}\Bigg\{ \|\phi_{a_1b_1}(x_{11}) + y_1\|^2 + \|\Phi_{a_1a_2^*}(x_{12}) + y_2\|^2$$
$$+ \|\Phi_{a_2a_1^*}(x_{21}) + y_3\|^2 + \|\Phi_{a_2}(x_{22})\|^2 + \sum_{i=4}^{8} \|y_i\|^2 \Bigg\}$$

(by Lemma 2.2(i)). Applying Case (i) to the first entry, Lemma 2.3 to the second and third entries, and ignoring the fourth entry in the sum above, it follows that

$$\|\phi(x) + y\|^2 \geqslant \frac{1}{9}\left\{ \|y_1\|^2 + \frac{1}{4}(\|y_2\|^2 + \|y_3\|^2) + \sum_{i=4}^{8} \|y_i\|^2 \right\}$$
$$\geqslant \frac{1}{36}\sum_{i=1}^{8} \|y_i\|^2 \geqslant \frac{1}{36}\|y\|^2$$

for all $x \in B(H)$. Arguing similarly it is seen that

$$\|\phi(x) + y\|_p \geqslant 6^{1-2/p}\|y\|_p$$

if $1 \leqslant p < 2$, and

$$\|\phi(x) + y\|_p \geqslant 6^{2/p-1}\|y\|_p$$

if $2 < p < \infty$, for every $y \in \phi^{-1}(0) \cap \mathscr{C}_p$ and all $x \in B(H)$ such that $\phi(x) \in \mathscr{C}_p$. This completes the proof. □

The operator $\Delta_{\mathbf{ab}}$ of Theorem 2.7 has ascent $\leqslant 1$. (The ascent of the Banach space operator t, asc(t), is the least non-negative integer n such that $t^{-n}(0) = t^{-(n+1)}(0)$.) The constant k in Theorem 2.7 is in general greater than 1. The theorem fails if $k = 1$: This is proved in [22] for the case in which a and b are normal. We do not know if the hypotheses that a_2 and b_2^* are normal can be replaced by the hypothesis that they are subnormal. The following corollary is immediate from Case (ii) of the proof of Theorem 2.7.

Corollary 2.8. *If a_1 and $b_1^* \in B(H)$ are hyponormal operators, a_2 and $b_2 \in B(H)$ are normal operators, $[a_1, a_2] = 0 = [b_1, b_2]$ and $a_1^{-1}(0) \cap a_2^{-1}(0) = \{0\} = b_1^{*-1}(0) \cap b_2^{*-1}(0)$, then $(\Delta_{\mathbf{ab}}^{-1}(0) \cap \mathscr{I}) \perp_1 (\Delta_{\mathbf{ab}}(B(H)) \cap \mathscr{I})$.*

Recall from Remark 2.5 that if a, b^* are hyponormal operators, then $\Delta_{a,b}^{-1}(0) \subseteq \Delta_{a^*,b^*}^{-1}(0)$. Let $y \in \Delta_{a,b}^{-1}(0)$, and let a and b have the decompositions $a = a_1 \oplus a_2$ and $b = b_1 \oplus b_2$ with respect to the decomposition $H = \ker^{\perp} y \oplus \ker y$ of H. Defining the quasi-affinity $y_1 : \ker^{\perp} y \to \overline{\operatorname{ran} y}$ by setting $y_1 h = yh$ for each $h \in \ker^{\perp} y$, it then follows from the argument of the proof of Lemma 2.4(ii) that a_1 and b_1 are normal operators. We have:

$$\begin{aligned}\|y\|_{\mathscr{I}} = \|y_1\|_{\mathscr{I}} &\leqslant \|\Delta_{a_1^*,b_1^*}(x_{11}) + y_1\|_{\mathscr{I}} \\ &\leqslant \left\| \begin{bmatrix} \Delta_{a_1^*,b_1^*}(x_{11}) + y_1 & \Delta_{a_1^*,b_2^*}(x_{12}) \\ \Delta_{a_2^*,b_1^*}(x_{21}) & \Delta_{a_2^*,b_2^*}(x_{22}) \end{bmatrix} \right\|_{\mathscr{I}} = \|\Delta_{a^*,b^*}(x) + y\|_{\mathscr{I}}\end{aligned}$$

for each $x = [x_{ij}]_{i,j=1}^2$ such that $\Delta_{a,b}(x) \in \mathscr{I}$. Using this version of Lemma 2.4(ii) in the proof of Case (i) of the proof of Theorem 2.7 it is seen that the following holds.

Theorem 2.9. *Let a be a hyponormal operator which commutes with the normal operator b. If $a^{-1}(0) \cap b^{-1}(0) = \{0\}$ and $\phi^* : B(H) \to B(H)$ is defined by $\phi^*(x) = a^*xa - b^*xb$, then $(\phi^{-1}(0) \cap \mathscr{I}) \perp_1 (\phi^*(B(H)) \cap \mathscr{I})$.*

We end this paper with the following two remarks, stated below as propositions. The first of these propositions concerns the operator $\Delta_{a,b}$, where a and b are C_ρ-contractions [18]. Then $\Delta_{a,b}^{-1}(0) \perp_k \Delta_{a,b}(B(H))$. By requiring more of the operators a and b we prove that it is possible to choose $k = 1$. Our second proposition gives a sufficient condition for the operator ϕ of Theorem 2.2 to be splitting left skew exact.

The operator $a \in B(H)$ is said to be of the class $\mathbf{C}_\rho$, $\rho > 0$, if there exists a Hilbert space $K \supset H$ such that $a^n = \rho P_H u^n|_H$, $n = 1, 2, \ldots$, where P_H denotes the orthogonal projection of K onto H [18, p. 45]. Operators $a \in \mathbf{C}_\rho$ are power bounded; indeed, if $a \in \mathbf{C}_\rho$ then there exists a positive invertible operator p and a contraction c such that $a = pcp^{-1}$ [18, p. 92]. Clearly, if $a \in \mathbf{C}_{\rho_1}$ and $b \in \mathbf{C}_{\rho_2}$ for some $\rho_1, \rho_2 > 0$, then Δ_{ab} satisfies Theorem 2.7: The following theorem says that if $y \in \Delta_{a,b}^{-1}(0)$ satisfies certain additional properties, then the constant k can be chosen to be 1. (Recall that the implication $\phi^{-1}(0) \perp_k \phi X \Longrightarrow \phi^{-1}(0) \perp_1 \phi X$ is generally false.) We state the theorem for the case $\mathscr{I} = B(H)$; the minor modifications required in the statement of the theorem for the case $\mathscr{I} = \mathscr{C}_p$ is obvious.

Proposition 2.10. *If $a \in \mathbf{C}_{\rho_1}$ and $b \in \mathbf{C}_{\rho_2}$ for some $\rho_1, \rho_2 > 0$, and if $\Delta_{a,b}(y) = 0$ for some compact quasi-affinity $y \in B(H)$, then $\Delta_{a,b}^{-1}(0) \perp_1 \Delta_{a,b}(B(H))$.*

Proof. We prove that a and b are unitary in such a case; this will then imply the theorem.

If $a \in \mathbf{C}_{\rho_1}$ and $b \in \mathbf{C}_{\rho_2}$, then there exist positive invertible operator p, q and contractions c_1, c_2 such that $a = p^{-1}c_1 p$ and $b = qc_2q^{-1}$. Set $pyq = t$; then t is a compact quasi-affinity, $\Delta_{c_1,c_2}(t) = 0$, $|t^*|^2 \leqslant c_1|t^*|^2c_1^*$ and $|t|^2 \leqslant c_2^*|t|^2c_2$. Applying [6, Theorem 8] to $|t^*|^2 \leqslant c_1|t^*|^2c_1^*$ and $|t|^2 \leqslant c_2^*|t|^2c_2$ it follows that c_1 and c_2 are unitaries. Since $\Delta_{a,b}(y) = 0$ implies $\Delta_{c_1,b}(py) = 0$, $\Delta_{b^*,b}(|py|^2) = 0$, where $|py|$ is a compact quasi-affinity. The equality $\Delta_{b^*,b}(|py|^2) = 0$ when taken alongwith the invertibilty of b implies the existence of a unitary u such that $u|py| = |py|b$. Applying [2, Theorem 2] to $u|py| = |py|b$ to conclude that u is singular unitary followed by an application of [2, Theorem 1(ii)] we conclude that b is unitary. The equation $\Delta_{a,b}(y) = 0$ now implies that $\Delta_{a,a^*}(|y^*|^2) = 0$, and hence that there exists a unitary w such that $w|y| = |y|a^*$. The operator a^* being $\mathbf{C}_{\rho_1}$, it follows (from an application of [2, Theorem 2] followed by an application of [2, Theorem 1(ii)]) that a is unitary. This completes the proof. □

The *descent* of a Banach space operator $T \in B(X)$, $\mathrm{dsc}(T)$, is defined to be the least non-negative integer n such that $T^n(X) = T^{n+1}(X)$ [11,16]. Recall that if both $\mathrm{asc}(T)$ and $\mathrm{dsc}(T)$ are finite, then $\mathrm{asc}(T) = \mathrm{dsc}(T)$ [16, Proposition 4.10.6]. In the following we call the pair (S, T), S and $T \in B(X)$, *splitting skew exact* if there exists a bounded linear $R : X \to X$ such that $T = RST$ and $STR = S$. The following proposition says that a necessary and sufficient condition for the pair (ϕ, ϕ) to be splitting skew exact is that $\mathrm{dsc}(\phi)$ be finite.

Proposition 2.11. *Let Δ be the operator $\Delta_{\mathbf{ab}}$ of the statement of Theorem* 2.7. *Then Δ is splitting skew exact if and only if* $\mathrm{dsc}(\Delta)$ *is finite*.

Proof. It is clear from the implications

$$\begin{aligned}\Delta^{-1}(0) \perp_k \Delta(B(H)) &\Longrightarrow \Delta^{-1}(0) \cap \mathrm{cl}\, \Delta(B(H)) = \{0\}\\ &\Longrightarrow \Delta^{-1}(0) \cap \Delta(B(H)) = \{0\} \Longleftrightarrow \mathrm{asc}(\Delta) = 1\end{aligned}$$

and the hypothesis $\mathrm{dsc}(\Delta) < \infty$ that $\mathrm{asc}(\Delta) = \mathrm{dsc}(\Delta) = 1$ and $B(H) = \Delta^{-1}(0) \oplus \Delta(B(H))$ [16, Proposition 4.10.6]. In particular, Δ is Drazin invertible [5,17]. Thus there exists an operator $\Delta^D \in B(B(H))$ such that $\Delta^D\Delta\Delta = \Delta = \Delta\Delta\Delta^D$, which implies that (Δ, Δ) is splitting skew exact. If, on the other hand, (Δ, Δ) is splitting skew exact, then the existence of an operator $r \in B(B(H))$ such that $\Delta\Delta r = \Delta$ implies that $\Delta^{-1}(0) + \Delta(B(H)) = B(H)$ [13, Theorem 2]. Hence Δ has finite descent. □

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