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Discrete Applied Mathematics 155 (2007) 2031 – 2038

DISCRETE APPLIED MATHEMATICS

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Note

Note on maximal split-stable subgraphs

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Received 20 October 2005; received in revised form 27 March 2007; accepted 12 April 2007
Available online 18 April 2007

Abstract

A multigraph $G=(V, R\cup B)$ with red and blue edges is an R/B-split graph if V is the union of a red and a blue stable set. Gavril has shown that R/B-split graphs yield a common generalization of split graphs and König–Egerváry graphs. Moreover, R/B-split graphs can be recognized in linear time. In this note, we address the corresponding optimization problem: identify a set of vertices of maximal cardinality that decomposes into a red and a blue stable set. This problem is $\mathcal{NP}$-hard in general. We investigate the complexity of special and related cases (e.g., (anti-)chains in partial orders and stable matroid bases) and exhibit some $\mathcal{NP}$-hard cases as well as polynomial ones.

Keywords: Computational complexity; Matroids; Partitioning; Stable sets

1. Introduction

Consider an undirected (multi-)graph $G=(V,E)$ (with possibly parallel edges between pairs of vertices) whose edge set $E=R\cup B$ consists of "red" and "blue" edges. A subset S of the vertex set V is said to be R-*stable* if no pair of vertices in S is joined by an edge in R. Similarly, a B-stable set is defined. $S\subseteq V$ is *split-stable* if it is of the form $S=S_R\cup S_B$, where S_R is a red and S_B a blue stable set.

The problem of finding a maximal (with respect to cardinality) split-stable set in a graph includes the usual maximal stable set problem. To see this, assume that the graph $G=(V,E)$ given. Color the edges in E "red" and then add all possible "blue" edges (so that $G_B=(V,B)$ is the complete graph on V). Obviously, a maximal split-stable set S in $\bar{G}=(V,E\cup B)$ yields a maximal stable set in G. The problem of finding a stable set of maximal cardinality is known to be $\mathcal{NP}$-hard on the class of all finite graphs. *A fortiori*, also the problem to construct a maximal split-stable set is $\mathcal{NP}$-hard in general.

For many subclasses of graphs (e.g., perfect graphs), the Max Stable Set problem has been shown to be polynomially solvable (see, e.g., [14, Chapter 67]). The present note wants to explore classes of combinatorial models relative to which the Max S-Stable Set problem can be solved in polynomial time.

The problem to decide whether the whole set V of vertices is split-stable in $G=(V,R\cup B)$, i.e., whether G is a so-called *R/B-split graph*, was solved by Gavril [6], who was able to show the equivalence of this decision problem with a 2-Satisfiability Problem. Hence R/B-split graphs can be recognized in linear time. Gavril moreover observed that the model of R/B-split graphs provides a natural common generalization of classical split graphs (see [3]) and

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doi:10.1016/j.dam.2007.04.009

König–Egerváry graphs (see [13]). Recall that a (classical) *split graph* is an uncolored graph whose vertex set can be split into a stable set and a clique, while a *König–Egerváry graph* is a graph in which the size of a maximal matching equals the size of a minimal vertex cover. Korach et al. [12] characterized R/B-split graphs by forbidden subgraphs, which allows in particular to easily regain known characterizations of König–Egerváry graphs and split graphs.

Focussing on the optimization problem to determine a maximal split-stable subgraph, we discuss some polynomially solvable instances. For example, if the subgraphs $G_R = (V, R)$ and $G_B = (V, B)$ are the comparability (resp. co-comparability) graph of a partial order, the problem amounts to determine a maximal subset that can be expressed as the union of two antichains (resp. chains) in the partial order. These maximization problems are well-known to be solvable in polynomial time even for the union of k antichains (resp. chains) (see, e.g., [4]). We show that a maximal union of a chain and an antichain can be found in polynomial time as well.

Interestingly, the problem of determining a maximal union of a red and a blue antichain turns out to be $\mathcal{NP}$-hard already for the class of series-parallel orders, yielding a non-approximability result.

We furthermore investigate a *maximal independent stable set problem* that is related to the Max S-Stable Set Problem, where one asks for a maximal stable set that is independent relative to a matroid on the vertex set of the graph. We find that the Max I-Stable Set Problem is $\mathcal{NP}$-hard for partition matroids while it is polynomial for the dual partition matroids.

2. Maximal split-stable subgraphs

We now define the Max S(plit)-Stable Set Problem formally as follows.

Definition 1. Given a finite multigraph $G = (V, E)$ and an edge coloring $c : E \to \{r(ed), b(lue)\}$, determine a maximal cardinality vertex set $S = S_R \cup S_B$ such that the induced subgraph on S_R contains no red and the induced subgraph on S_B contains no blue edge.

Remark. Passing to complement graphs, we observe that it is a generally equivalent task to determine a maximal subset that can be represented as the union of a red and a blue clique, resp. of a red clique and a blue stable set.

It was already observed in the Introduction that Max S-Stable Set is $\mathcal{NP}$-hard relative to the class of all finite multigraphs and all edge bicolorings as the classical Max Stable Set Problem can be reduced to it. Conversely, an instance of Max S-Stable Set can be reduced to an instance of the standard Max Stable Set by a simple construction.

Reduction to Max Stable Set. Given $G = (V, E)$ and $c : E \to \{r, b\}$, let V' be a disjoint copy of V and define the graph $H(G) = (V \cup V', \tilde{E})$ such that

$$(i, j) \in \tilde{E} \quad \Longleftrightarrow \quad \begin{cases} i, j \in V \text{ and } c(i, j) = r, \\ i, j \in V' \text{ and } c(i, j) = b, \\ i \in V,\ j \in V' \text{ and } j = i' \text{ is the copy of } i. \end{cases}$$

The (uncolored) graph $H(G)$ is obtained by joining the "red" and the "blue" subgraphs of G by a (special) perfect matching. A stable set S of $H(G)$ corresponds to a disjoint union $S = S_R \cup S_B$ of a red stable set $S_R = S \cap V$ and a blue stable set $S_B = S \cap V'$.

We now want to identify classes of red and blue graphs that allow polynomial solutions for the associated maximal s-stable set problem. For ease of notation, we write relative to a given coloring $c : E \to \{r, b\}$,

$$R = \{e \in E | c(e) = r\} \quad \text{and} \quad B = \{e \in E | c(e) = b\}.$$

One example of such a tractable class is formed by perfect graphs and complements of chordal graphs. Recall that a graph is said to be *chordal* if each cycle with more than four edges possesses a chord. Since a stable set in the red graph G_R corresponds to a clique in the complement graph $\bar{G}_R$, we find:

Lemma 2. Max S-Stable Set *is polynomially solvable on the class of bicolored graphs* $G = (V, R \cup B)$ *such that* $G_R = (V, R)$ *is the complement of a chordal graph* $\bar{G}_R$ *and* $G_B = (V, B)$ *is a perfect graph.*

Proof. We have to find a maximal subset of the vertex set V that splits into a clique of the chordal graph $\bar{G}_R$ and a stable set of the perfect graph $G_B=(V,B)$. Given a disjoint union of a red clique and a blue stable set, we may assume the red clique to be (inclusionwise) maximal. Fulkerson and Gross [5] showed that all (inclusionwise) maximal cliques C of a chordal graph $\bar{G}_R=(V,\bar{R})$ can be listed in time $\mathrm{O}(|V|+|\bar{R}|)$. To solve the problem, it suffices to determine a maximal stable set S_C in the induced blue (perfect) subgraph $G_B(V\backslash C)$ for each (inclusionwise) maximal clique C, which is a polynomial task (see Grötschel et al. [9]). A largest of the sets $C\cup S_C$ is a largest split-stable set in G. □

Remark. Well-known examples of perfect graphs are comparability and co-comparability graphs of partial orders. However, the approach of the preceding proof works, of course, for any class of bicolored graphs where the (inclusionwise) maximal stable sets of the red graph can be listed in polynomial time and a maximal stable set of the blue graph can be computed efficiently for any induced subgraph.

We next consider the cases where the blue graph is an identical copy of the red graph, which we denote by $G_R\equiv G_B$.

Max S-Stable Set with $G_R\equiv G_B$. If the red and blue graph are identical, the Max S-Split becomes the problem to determine a largest union of two stable sets. Relative to the class of all finite graphs, this problem is still $\mathcal{NP}$-hard as it is a special instance Max Induced Subgraph with Property Π problem, which is known to be $\mathcal{NP}$-hard (see [1, p. 381]).

Restricted to the class of partial orders, however, the problem is polynomial:

Max S-Stable Set with comparability graphs $G_R\equiv G_B$. If G_R and G_B are the comparability graph of a partial order $P=(P,\leqslant)$, the Max S-Stable Set is the problem to determine a largest union of two antichains in P. Greene and Kleitman [8] have shown that this problem is efficiently solvable even for the maximal union of k antichains.

Dually, Greene [7] showed that the maximal union of k chains can be found in polynomial time. The question arises whether the maximal union of a chain and an antichain of the same poset can be calculated efficiently. We show in the following section that the maximal union of a chain and an antichain is easy to determine and characterize those posets that contain a disjoint pair of a maximal chain and a maximal antichain.

2.1. The union of a chain and an antichain

Given a poset $P=(V,\leqslant)$, the parameter $w(P)=\max\{|A|\,|\,A \text{ antichain in } P\}$ is the *width* of P and the parameter $\ell(P)=\max\{|C|\,|\,C \text{ chain in } P\}$ is called the *length* of P.

We consider the Max S-Stable Set Problem on graphs $G=(V,R\cup B)$, where $G_R=(V,R)$ is the co-comparability graph and $G_B=(V,B)$ the comparability graph of some partial order $P=(V,\leqslant)$ on the set V. So the problem asks for a largest union of a chain and an antichain relative to P.

Since a chain and an antichain of $P=(P,\leqslant)$ intersect in at most one element, just taking a largest chain C and a largest antichain A yields a split-stable set $S=C\cup A$ such that

$$\ell(P)+w(P)\geqslant|S|=|C\cup A|\geqslant\ell(P)+w(P)-1.$$

So this special Max S-Stable Set Problem really poses the question whether there exists a disjoint pair of a maximal chain and a maximal antichain in P. Theorem 4 characterizes the posets with a positive answer to this question.

Definition 3. Given $P=(V,\leqslant)$ and an element $a\in V$, we define the *ideal* $a^{\downarrow}$ (resp. the *filter* $a^{\uparrow}$) generated by a as

$$a^{\downarrow}=\{p\in P\,|\,p\leqslant a\}\quad\text{and}\quad a^{\uparrow}=\{p\in P\,|\,p\geqslant a\}.$$

Theorem 4. *The partial order $P=(V,\leqslant)$ admits a disjoint pair of a longest chain and a largest antichain if and only if there exist elements $a<b$ such that*

$$\ell(P)=\ell(a^{\downarrow})+\ell(b^{\uparrow})\quad\text{and}\quad w(P)=w(P\backslash(a^{\downarrow}\cup b^{\uparrow})).\tag{1}$$

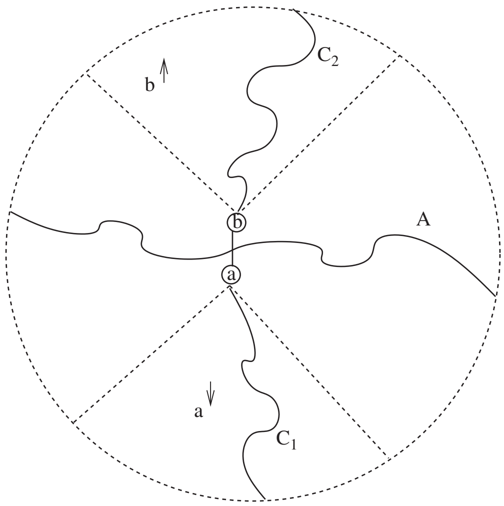

Fig. 1. A maximal chain and antichain.

Proof. The condition (1) is sufficient as it allows us to select a longest chain $C = C_a \cup C_b$ with $C_a \subseteq a^{\downarrow}$ and $C_b \subseteq b^{\uparrow}$. If now A is a largest antichain in $P \backslash (a^{\downarrow} \cup b^{\uparrow})$, we have $C \cap A = \emptyset$, as claimed.

To prove that condition (1) is necessary, let $C = \{c_1 < \cdots < c_{\ell}\}$ be a maximal chain and A a maximal antichain with $C \cap A = \emptyset$ (Fig. 1). The maximality of C and A implies

$$A \cap c_1^{\downarrow} = \emptyset = A \cap c_{\ell}^{\uparrow} \quad \text{and} \quad A \cap (c^{\downarrow} \cup c^{\uparrow}) \neq \emptyset \quad \text{for all } c \in C.$$

Let $b = c_i$ be the minimal member of C such that $c_i^{\downarrow} \cap A \neq \emptyset$. Then $i > 1$ holds. Consider now $a = c_{i-1} < c_i = b$. The maximality of C implies

$$i - 1 = \ell(a^{\downarrow}) \quad \text{and} \quad |C| - (i - 1) = \ell(b^{\uparrow}) \quad \text{i.e. } \ell(P) = \ell(a^{\downarrow}) + \ell(b^{\uparrow}).$$

A contains no element of $a^{\downarrow}$ by our choice of a. Moreover, $A \cap b^{\downarrow} \neq \emptyset$ and $b \notin A$ imply $A \cap b^{\uparrow} = \emptyset$. So we also conclude

$$A \subseteq V \backslash (a^{\downarrow} \cup b^{\uparrow}) \quad \text{i.e. } w(P \backslash (a^{\downarrow} \cup b^{\uparrow})) = |A| = w(P). \qquad \Box$$

Corollary 5. *A maximal union of a chain and an antichain in a partially ordered set P can be found in polynomial time*.

Proof. For each $a < b$ in P, one computes a maximal chain $C_{ab} \subseteq a^{\downarrow} \cup b^{\uparrow}$ and a maximal antichain A_{ab} in $P \backslash (a^{\downarrow} \cup b^{\uparrow})$. The largest set S of the form $S = C_{ab} \cup A_{ab}$ then yields the desired maximal union. $\Box$

So far, we have seen that it is easy to either decide whether we can cover all vertices of a graph with two colored stable sets or to find the maximal union of two chains, two antichains or one chain and one antichain relative to one partial order. However, if we ask for the maximal union of a red and a blue antichain, resp. chain, the problem becomes $\mathscr{NP}$-hard, as we prove in the following section.

2.2. Unions of two colored antichains

Assume to be given a red partial order $P_R = (V, \leqslant_R)$ and a blue partial order $P_B = (V, \leqslant_B)$ on the same ground set V. Let $w_R = w(P_R)$ resp. $w_B = w(P_B)$ denote the size of a maximal red resp. blue antichain. In case $w_R + w_B < |V|$, it is obviously impossible to cover all elements with a red and a blue antichain. However, we might still wonder if we can find a red and a blue antichain that cover $w_R + w_B$ elements, i.e., if we can find a disjoint pair of a maximal red and a maximal blue antichain.

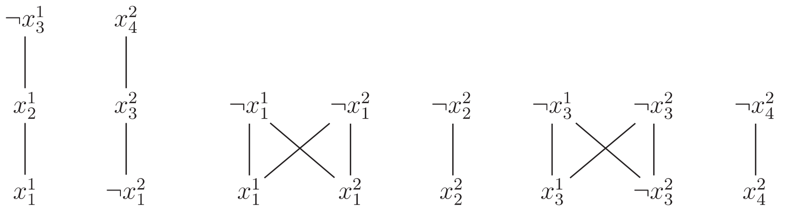

Fig. 2. The red and the blue order of $(x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_3 \vee \neg x_4)$.

It turns out that the problem of deciding whether there exist two disjoint differently colored maximal antichains is $\mathcal{NP}$-complete already on the class of series-parallel orders. This fact directly implies $\mathcal{NP}$-hardness of the associated Max S-Stable Set Problem.

Theorem 6. *Given two partial orders $P_R = (V, \leqslant_R)$ and $P_B = (V, \leqslant_B)$ on the same ground set V, it is $\mathcal{NP}$-hard to decide whether there exist maximal antichains A_R in P_R and A_B in P_B with $A_R \cap A_B = \emptyset$.*

Proof. We show $\mathcal{NP}$-hardness by a reduction from 3-SAT. Consider a 3-SAT instance with k clauses on n variables x_i,

$$\bigwedge_{j=1}^{k} (\ell_1^j \vee \ell_2^j \vee \ell_3^j),$$

i.e. $\ell_p^j \in \{x_1, \ldots, x_n, \neg x_1, \ldots, \neg x_n\}$ for $p = 1, 2, 3$. The ground set V contains all literals and their negations, where occurrences of the same literal in different clauses are distinguished:

$$V = \{x_i^j, \neg x_i^j \mid \exists p \in \{1, 2, 3\} : (x_i^j = \ell_p^j) \vee (\neg x_i^j = \ell_p^j)\}.$$

In the following, when referring to a literal ℓ_p^j, we mean its occurrence in clause j, i.e. $\ell_p^j = (\neg)x_i^j$ for the appropriate i. The red and blue orders are defined as follows:

$$\forall j : \ell_1^j < \ell_2^j < \ell_3^j,$$

$$\forall i, j, j' : x_i^j < \neg x_i^{j'}.$$

Fig. 2shows the Hasse diagram (without the incomparable items) of the orders associated with the 3-Sat instance $(x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_3 \vee \neg x_4)$.

Obviously, a maximal red antichain covers exactly one literal per clause, whereas a maximal blue antichain corresponds to a consistent assignment of the variables. Note that a maximal red and a maximal blue antichain are disjoint if and only if the literals covered by the red antichain are *false* in the variable assignment corresponding to the blue antichain. Therefore, if we can find two maximal disjoint antichains, negating the variable assignment corresponding to the blue antichain produces a satisfying variable assignment for the original 3-Sat instance. On the other hand, if there are no such two antichains, there also is no variable assignment satisfying all clauses. As this reduction is obviously polynomial, $\mathcal{NP}$-hardness follows. □

Recall that a partial order P is said to be *series-parallel* if its comparability graph does not contain a path on four elements as an induced subgraph or, equivalently, four elements a, b, c, d with the induced order relations

$$a < c, \ \ a < d, \ \ b < d \quad \text{(but not } b < c\text{)}.$$

A *cograph* is the comparability graph of a series-parallel order. (For more about cographs, the reader is referred to, e.g., [2].)

Corollary 7. Max S-Stable Set *is $\mathcal{NP}$-hard on the class of bicolored graphs $G = (V, R \cup B)$ such that $G_R = (V, R)$ and $G_B = (V, B)$ are cographs.*

Proof. It is straightforward to check that the red and the blue order of a 3-Sat instance are series-parallel orders. □

Corollary 8. *Given two series-parallel orders P_1 and P_2 on the same ground set, it is $\mathscr{NP}$-hard to decide whether there exists a maximal chain C_1 relative to P_1 and a maximal antichain A_2 relative to P_2 such that $C_1 \cap A_2 = \emptyset$.*

Proof. It is well known that the complement of a cograph (i.e., the co-comparability graph of a series-parallel order) is again a cograph. □

Remark. Given a red and a blue partial order on the same ground set V, it follows directly that the problem to determine the maximal union of a red and a blue chain is an $\mathscr{NP}$-hard problem. We have seen that the question, whether V can be covered by a red and a blue chain, can be answered by solving a 2-satisfiability problem. However, given three different posets on V, we do not know the complexity status of the decision problem whether V can be covered by three chains of different colors.

The construction of Theorem 6 also produces a reduction from Max 3-Sat. This provides direct insight into the approximability problem associated with Max S-Stable Set. (For the basic notions of approximation theory, the reader is referred to, e.g., [1].)

Corollary 9. *For $\varepsilon > 0$, there cannot exist a $(\frac{31}{32} + \varepsilon)$-approximative algorithm for* Max S-Stable Set *on pairs of cographs unless $\mathscr{P} = \mathscr{NP}$.*

Proof. Note that it does not make any difference if an element is covered by the red, blue or both antichains (relative to the underlying series-parallel orders). Hence we may assume without loss of generality that the blue antichain is of maximal cardinality $3k$ and thus corresponds to a consistent assignment of all variables. Then the size of the red antichain associated with a 3-Sat instance is exactly the number of clauses satisfied by the negated variable assignment. As it is $\mathscr{NP}$-hard to approximate Max 3-Sat better than $7/8$ (see [10]), it is easy to calculate that approximating Max S-Stable Set better than $31/32$ is $\mathscr{NP}$-hard as well. □

Remark. Taking the union of a maximal red and a maximal blue stable set yields a simple 2-approximation algorithm, which altogether places Max S-Stable Set on pairs of perfect graphs into the class of so-called $\mathscr{APX}$-complete problems [1].

3. Independent stable sets

In case the red graph of the graph $G = (V, R \cup B)$ consists of a union of disjoint cliques, i.e.,

$$G_R = (V, R) = C_1 \dot{\cup} \cdots \dot{\cup} C_k,$$

G is an R/B-split graph if and only if V is the union of a stable set in $G_B = (V, B)$ and an independent set in the partition matroid $\mathscr{M} = (V, \mathscr{I})$ with independent sets $\mathscr{I}$, where

$$I \in \mathscr{I} \iff |I \cap C_i| \leqslant 1 \quad \forall i = 1, \ldots, k.$$

As we may assume the red stable set to be (inclusionwise) maximal (and hence a basis of $\mathscr{M}$), the equivalent problem is

$$\max\{|I^*| \| I^* \text{ is independent in } \mathscr{M}^* \text{ and stable in } G_B\},$$

where $\mathscr{M}^*$ is the matroid dual of $\mathscr{M}$, whose independent sets are the subsets $I^* \subseteq V$ whose complements $V \backslash I^*$ contain a basis of $\mathscr{M}$. This suggests to investigate and formally define the *maximal independent stable set problem*.

Definition 10. Given a matroid $\mathscr{M} = (V, \mathscr{I})$ with system $\mathscr{I}$ of independent sets and a graph $G = (V, E)$, Max I-Stable Set is the problem to find a maximal stable set in the graph G that is independent in $\mathscr{M}$ as well.

Lemma 11. *Let $\mathscr{M}$ be given as the dual of a partition matroid $\mathscr{M}^*$ on the set V. Then the* Max I-Stable Set *Problem relative to the graph $G = (V, E)$ and $\mathscr{M}$ can be efficiently reduced to a* Max S-Stable Set problem.

Proof. Let $\mathscr{M}^*$ be defined via the partition $V = V_i \dot\cup \cdots \dot\cup V_k$ and consider the red-blue graph $\hat{G} = (V, R \cup B)$ such that $B = E$ and G_R consists of the disjoint cliques corresponding to the V_i. As at the beginning of this section, we now find that the maximal s-stable sets of $\hat{G}$ correspond to the maximal independent stable sets relative to G and $\mathscr{M}$. □

On special classes of graphs the maximal independent stable set problem may turn out to be polynomial for general matroids (which we assume to be available via independence oracles). To exhibit such an example, recall that a partially ordered set $P = (V, \leqslant)$ is a *tree order* if its Hasse diagram is a rooted tree (we assume the root to be at the bottom). Since a stable set in a co-comparability graph is a chain in the corresponding order and vice versa, the following lemma directly implies that the Max I-Stable Set is polynomially solvable when the underlying graph $G = (V, E)$ is the co-comparability graph of a tree order.

Lemma 12. *If $P = (V, \leqslant)$ is a tree order, a maximal independent chain can be computed in polynomial time.*

Proof. In the tree order P, each leaf i of the Hasse diagram is a maximal element of a unique (inclusionwise) maximal chain C_i. For each leaf i determine a maximal independent (for $\mathscr{M}$) subchain $\hat{C}_i$ of C_i, which is easily accomplished with the matroid greedy algorithm.

Any one of maximal cardinality among the chains $\hat{C}_i$ is a maximal independent chain with respect to order P and matroid $\mathscr{M}$. □

The general problem of finding a maximal independent chain (or antichain) turns out to be $\mathscr{NP}$-hard even for a partition matroids and series-parallel orders.

Theorem 13. Max I-Stable Set *is $\mathscr{NP}$-hard for partition matroids on cographs.*

Proof. It suffices to amend the proof of Theorem 6 only slightly: Given a 3-Sat formula, we take just the literals as they appear in the clauses (and not their negations):

$$V = \{x_i^j \mid \exists p \in \{1, 2, 3\} : x_i^j = \ell_p^j\} \cup \{\neg x_i^j \mid \exists p \in \{1, 2, 3\} : \neg x_i^j = \ell_p^j\}.$$

The crucial observation is now that the red order in the proof of Theorem 6 can be substituted by the partition matroid on V where V is partitioned into the k clauses of the 3-Sat formula:

$$V = \bigcup_{j=1}^{k} \{\ell_1^j, \ell_2^j, \ell_3^j\}.$$

The definition of the (blue) order remains the same, but now on only half as many elements. A stable basis here corresponds to a consistent variable assignment satisfying (at least) one literal in each clause. □

Corollary 14. *For $\varepsilon > 0$, there cannot exist a $(\frac{7}{8} + \varepsilon)$-approximative algorithm for a longest independent (anti-)chain in a series-parallel order unless $\mathscr{P} = \mathscr{NP}$.*

Proof. In the construction above, the associated variable assignment of an independent antichain of length l satisfies (at least) l clauses. □

It is remarkable that the Max I-Stable Set is polynomial in case $\mathscr{M}$ is the dual of a partition matroid, whereas the problem is $\mathscr{NP}$-hard (in fact, even $\mathscr{APX}$-hard) if $\mathscr{M}$ is a partition matroid.

Acknowledgments

We thank anonymous referees on an earlier version of this paper for pointing out F. Gavril's work [6], and Gyula Pap for fruitful conversations.

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