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# Modified Jarratt method with sixth-order convergence

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ABSTRACT

In this paper, we present a variant of Jarratt method with order of convergence six for solving non-linear equations. Per iteration the method requires two evaluations of the function and two of its first derivatives. The new multistep iteration scheme, based on the new method, is developed and numerical tests verifying the theory are also given.



## 1. Introduction

Solving non-linear equations is a common and important problem in science and engineering. In this paper, we consider iterative methods to find a simple root of a non-linear equation $f(x) = 0$, where $f : D \subset \mathbb{R} \to \mathbb{R}$ for an open interval $D$ is a scalar function.

Newton method for a single non-linear equation is written as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1}$$

This is an important and basic method [1], which converges quadratically.

To improve the local order of convergence, many modified methods have been proposed. The Jarratt method [2], which has fourth-order convergence, is defined by

$$x_{n+1} = x_n - J_f(x_n)\frac{f(x_n)}{f'(x_n)}, \tag{2}$$

where $y_n = x_n - \frac{2}{3}f(x_n)/f'(x_n)$ and

$$J_f(x_n) = \frac{3f'(y_n) + f'(x_n)}{6f'(y_n) - 2f'(x_n)}.$$

The Jarratt method is widely considered and applied for the local order of convergence four.

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Recently, a variant of Jarratt method with sixth-order convergence is developed in [3,4], which improves the local order of convergence of Jarratt method by an additional evaluation of the function. The similar methods have also been studied in [5–9] and the orders of convergence of many classical methods have been improved. From a practical point of view, it is interesting to improve the order of convergence of the known methods.

In this paper, we present a new variant of Jarratt method, based on the composition of Jarratt method and Newton method instead of two-step Newton method. This variant consists in adding the evaluation of the function at another point in the procedure iterated by Jarratt method. As a consequence, the local order of convergence is improved from four for Jarratt method to six for the new method. Per iteration the new method requires two evaluations of the function and two of its first derivatives. The superiority of the new method is shown in numerical examples.

## 2. Main results

In this paper, we consider the following iteration scheme

$$
\begin{aligned}
z_n &= x_n - J_f(x_n)\frac{f(x_n)}{f'(x_n)}, \\
x_{n+1} &= z_n - \frac{f(z_n)}{f'(z_n)},
\end{aligned} \tag{3}
$$

where $y_n = x_n - \frac{2}{3}f(x_n)/f'(x_n)$. This iteration scheme consists of a Jarratt iterate to get $z_n$ from $x_n$, followed by a Newton iterate to calculate $x_{n+1}$ from the new point $z_n$. However, this method may not require the first derivative at the point $z_n$. We can use various approximations of $f'(z_n)$ in (3) as

$$
f'(z_n) \approx \psi(z_n). \tag{4}
$$

where $\psi(z_n)$ may be computed without any new evaluation of the function or its first derivative. So we want to find such schemes as

$$
x_{n+1} = z_n - \frac{f(z_n)}{\psi(z_n)}, \tag{5}
$$

where $y_n = x_n - \frac{2}{3}f(x_n)/f'(x_n)$. Now we consider the constructions of $\psi(z_n)$ and then obtain many efficient methods, which are displayed in the following.

(1) $\psi(x)$ may be considered as the linear interpolation function using two points $(x_n, f'(x_n))$ and $(y_n, f'(y_n))$, namely

$$
\psi(x) = \frac{x - x_n}{y_n - x_n}f'(x_n) + \frac{x - y_n}{x_n - y_n}f'(y_n). \tag{6}
$$

Then one approximation of $f'(z_n)$ can be obtained

$$
\psi(z_n) = \frac{3}{2}J_f(x_n)f'(y_n) + \left(1 - \frac{3}{2}J_f(x_n)\right)f'(x_n), \tag{7}
$$

so the variant of Jarratt's method, which has sixth-order convergence, is obtained

$$
x_{n+1} = z_n - \frac{f(z_n)}{\frac{3}{2}J_f(x_n)f'(y_n) + \left(1 - \frac{3}{2}J_f(x_n)\right)f'(x_n)}. \tag{8}
$$

This is the method presented in [3].

(2) Ref. [4] presents the approximation

$$
\psi(x) = ax^2 + bx + c \tag{9}
$$

which agrees with $f'$ at two points $(x_n, f'(x_n))$ and $(y_n, f'(y_n))$. Since the points $(x_n, f'(x_n))$ and $(y_n, f'(y_n))$ are on the graph of $h$, then it is easy to see that the constants $b$, $c$ are determined by

$$
b = \frac{x_n - y_n}{f'(x_n) - f'(y_n)} - a(x_n + y_n), \tag{10}
$$

$$
c = f'(x_n) + ax_ny_n - x_n\frac{x_n - y_n}{f'(x_n) - f'(y_n)}, \tag{11}
$$

so the approximation of $f'(z_n)$ is

$$
\psi(z_n) = a(z_n - x_n)(z_n - y_n) + \frac{3}{2}J_f(x_n)f'(y_n) + \left(1 - \frac{3}{2}J_f(x_n)\right)f'(x_n), \tag{12}
$$

and the scheme is given by

$$x_{n+1} = z_n - \frac{f(z_n)}{a(z_n - x_n)(z_n - y_n) + \frac{3}{2}J_f(x_n)f'(y_n) + \left(1 - \frac{3}{2}J_f(x_n)\right)f'(x_n)}. \tag{13}$$

where $a \in \mathbb{R}$. Note that the method of Kou et al. defined by (9) is just a special case of the presented family (15) when $a = 0$.

(3) Here, we can use a new approximation of $f'(z_n)$ in (3) as

$$\psi(z_n) = \frac{1}{\frac{\theta}{f'(y_n)} + \frac{1-\theta}{f'(x_n)}}. \tag{14}$$

The parameter $\theta$ is determined from the following convergence theorem.

**Theorem 1.** *Assume that the function $f : D \subset \mathbb{R} \to \mathbb{R}$ for an open interval $D$ has a simple root $x^* \in D$. If $f(x)$ is sufficiently smooth in the neighborhood of the root $x^*$, then the method defined by (3), in which $f'(z_n)$ is approximated by (14), is of order six if $\theta = 3/2$.*

**Proof.** Using Taylor expansion and taking into account $f(x^*) = 0$, we have

$$f(x_n) = f'(x^*)\left[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + O(e_n^5)\right] \tag{15}$$

where $e_n = x_n - x^*$ and $c_k = (1/k!)f^{(k)}(x^*)/f'(x^*)$, $k \geq 2$. Furthermore, we have

$$f'(x_n) = f'(x^*)\left[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + O(e_n^4)\right]. \tag{16}$$

Dividing (15) by (16) gives us

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^2 + 2\left(c_2^2 - c_3\right)e_n^3 + \left(7c_2c_3 - 4c_2^3 - 3c_4\right)e_n^4 + O(e_n^5), \tag{17}$$

and hence, we have

$$y_n - x^* = \frac{1}{3}e_n + \frac{2}{3}\left[c_2e_n^2 - 2\left(c_2^2 - c_3\right)e_n^3 - \left(7c_2c_3 - 4c_2^3 - 3c_4\right)e_n^4\right] + O(e_n^5). \tag{18}$$

Expanding $f'(y_n)$ about $x^*$ and from (18), we have

$$f'(y_n) = f'(x^*)\left[1 + \frac{2}{3}c_2e_n + \frac{1}{3}\left(4c_2^2 + c_3\right)e_n^2 - \left(\frac{8}{3}c_2^3 - 4c_2c_3 - \frac{4}{27}c_4\right)e_n^3 + O(e_n^4)\right]. \tag{19}$$

From (16) and (19), we have

$$-\frac{3}{4}(f'(y_n) - f'(x_n)) = f'(x^*)\left[c_2e_n - \left(c_2^2 - 2c_3\right)e_n^2 + \left(2c_2^3 - 3c_2c_3 + \frac{26}{9}c_4\right)e_n^3 + O(e_n^4)\right], \tag{20}$$

$$\frac{3}{2}f'(y_n) - \frac{1}{2}f'(x_n) = f'(x^*)\left[1 + \left(2c_2^2 - c_3\right)e_n^2 + O(e_n^3)\right]. \tag{21}$$

Dividing (20) by (21) gives us

$$-\frac{3}{2}\frac{f'(y_n) - f'(x_n)}{3f'(y_n) - f'(x_n)} = c_2e_n - \left(c_2^2 - 2c_3\right)e_n^2 - 2\left(c_2c_3 - \frac{13}{9}c_4\right)e_n^3 + O(e_n^4). \tag{22}$$

From (17) and (22), we have

$$-\frac{3}{2}\frac{f'(y_n) - f'(x_n)}{3f'(y_n) - f'(x_n)}\frac{f(x_n)}{f'(x_n)} = c_2e_n^2 - 2\left(c_2^2 - c_3\right)e_n^3 + \left(3c_2^3 - 6c_2c_3 + \frac{26}{9}c_4\right)e_n^4 + O(e_n^5). \tag{23}$$

Thus from (17) and (23), we have

$$z_n - x^* = e_n - \left(1 - \frac{3}{2}\frac{f'(y_n) - f'(x_n)}{3f'(y_n) - f'(x_n)}\right)\frac{f(x_n)}{f'(x_n)} = \left(c_2^3 - c_2c_3 + \frac{1}{9}c_4\right)e_n^4 + O(e_n^5). \tag{24}$$

Again expanding $f(z_n)$ about $x^*$, we have

$$f(z_n) = f'(x^*)\left[(z_n - x^*) + O\left((z_n - x^*)^2\right)\right]. \tag{25}$$

**Table 1**
$|f(x_n)|$ for Example 1.

| $n$ | Newton | Jarratt | KM | CM1 | CM2 | Present method |
|---|---|---|---|---|---|---|
| 1 | 3.57 | 3.07 | 8.76 | 82.65 | 5.87 | 1.40 |
| 2 | 0.23 | 4.26e−3 | 7.60e−2 | 19.35 | 1.68e−2 | 3.39e−5 |
| 3 | 4.27e−3 | 4.22e−14 | 2.17e−13 | 1.41 | 0 | 0 |
| 4 | 1.46e−6 | 0 | 0 | 1.82e−5 | | |
| 5 | 1.71e−13 | | | 0 | | |
| 6 | 0 | | | | | |

**Table 2**
$|f(x_n)|$ for Example 2.

| $n$ | Newton | Jarratt | KM | CM1 | CM2 | Present method |
|---|---|---|---|---|---|---|
| 1 | 0.16 | 1.69e−4 | 2.93e−6 | 1.65e−5 | 2.81e−5 | 2.07e−6 |
| 2 | 5.00e−4 | 0 | 0 | 0 | 0 | 0 |
| 3 | 4.68e−9 | | | | | |
| 4 | 4.44e−16 | | | | | |
| 5 | 0 | | | | | |

From (16) and (19), we have

$$\frac{\theta}{f'(y_n)} + \frac{1-\theta}{f'(x_n)} = \frac{1}{f'(x^*)}\left\{1 + \left(\frac{4}{3}\theta - 2\right)c_2 e_n + \left[(4c_2^2 - 3c_3) - \left(\frac{44}{9}c_2^2 - \frac{8}{3}c_3\right)\theta\right]e_n^2 + O(e_n^3)\right\} \tag{26}$$

Since from (3) and (14), we have

$$e_{n+1} = z_n - x^* - f(z_n)\left[\frac{\theta}{f'(y_n)} + \frac{1-\theta}{f'(x_n)}\right],$$

from (24)–(26), we have

$$\begin{aligned} e_{n+1} &= \left\{\left(2 - \frac{4}{3}\theta\right)c_2 e_n - \left[(4c_2^2 - 3c_3) - \left(\frac{44}{9}c_2^2 - \frac{8}{3}c_3\right)\theta\right]e_n^2\right\}(z_n - x^*) + O(e_n^7) \\ &= \left\{\left(2 - \frac{4}{3}\theta\right)c_2 - \left[(4c_2^2 - 3c_3) - \left(\frac{44}{9}c_2^2 - \frac{8}{3}c_3\right)\theta\right]e_n\right\}\left(c_2^3 - c_2c_3 + \frac{1}{9}c_4\right)e_n^5 + O(e_n^7). \end{aligned} \tag{27}$$

This means that the method defined by (3), in which $f'(z_n)$ is approximated by (14), is at least of fifth-order for any $\theta \in \mathbb{R}$. Furthermore, when we take $\theta = 3/2$, the order of convergence is six and from (27), we have the error equation

$$e_{n+1} = \left(\frac{10}{3}c_2^2 - c_3\right)\left(c_2^3 - c_2c_3 + \frac{1}{9}c_4\right)e_n^6 + O(e_n^7). \tag{28}$$

This ends the proof. □

If we use the approximation (14) in (3) and take $\theta = 3/2$, we can obtain a new sixth-order method

$$x_{n+1} = z_n - \left[\frac{3}{2f'(y_n)} - \frac{1}{2f'(x_n)}\right]f(z_n), \tag{29}$$

where $y_n = x_n - \frac{2}{3}f(x_n)/f'(x_n)$ and $z_n = x_n - J_f(x_n)f'(x_n)^{-1}f(x_n)$.

Thus the present scheme (29) improves the local order of convergence of its classical predecessor, Jarratt method, by an additional evaluation of the function at another point iterated by Jarratt method.

## 3. Numerical examples

Now, we employ the present method defined by (29) to solve some non-linear equations and compare it with Newton method and Jarratt method, Kou et al.'s method defined by (8) (KM), and the methods (13) with $a = 1$ (CM1) and $a = -1$ (CM2) introduced in [4]. All computations are carried out with double arithmetic precision. All problems are solved taking a given initial value $x_0$. Displayed in Tables 1–3 are the absolute values of $f(x_n)$ computed by various methods.

The numerical results show that the present method improves the local order of convergence of Jarratt method and therefore it requires less iterations.

**Table 3**
$|f(x_n)|$ for Example 3.

| $n$ | Newton | Jarratt | KM | CM1 | CM2 | Present method |
|---|---|---|---|---|---|---|
| 1 | 0.31 | 3.47e−2 | 1.76e−2 | 2.68e−2 | 4.62e−2 | 3.21e−3 |
| 2 | 1.39e−2 | 7.40e−9 | 7.99e−15 | 3.09e−12 | 8.95e−11 | 0 |
| 3 | 2.09e−5 | 0 | 0 | 0 | 0 | |
| 4 | 4.63e−11 | | | | | |
| 5 | 2.22e−16 | | | | | |
| 6 | 0 | | | | | |

**Example 1.**

$$f(x) = x^2 - (2-x)^3 = 0, \qquad x_0 = 3, \qquad x^* = 1.$$

The results of this problem are displayed in Table 1.

**Example 2.**

$$f(x) = 3x - e^{-x} - 2 = 0, \qquad x_0 = 2.0, \qquad x^* = 0.8143143142996808\cdots$$

The results of this problem are displayed in Table 2.

**Example 3.**

$$f(x) = e^{-x}\sin(x) + \ln(x^2+1) - 2 = 0, \qquad x_0 = 0.5, \qquad x^* = 2.4477482864524247\cdots$$

The results of this problem are displayed in Table 3.

## 4. Conclusions

We have obtained a new variant of Jarratt method. The improvement of the local order goes from four for the Jarratt method to six for the new method. The high-order convergence is also corroborated by numerical tests. Finally, we note that the present methods can be directly extended to the systems of equations.

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