On the hamiltonian index and the radius of a graph

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Abstract

Catlin et al. [1, Corollary 9A] characterised the graphs $G$ with the property that $\text{ham}(G) > \text{rad}(G) + 1$ where $\text{ham}(G)$ and $\text{rad}(G)$ stand for the hamiltonian index and the radius of $G$, respectively. Here a slightly stronger result is presented. In effect, the graphs for which $\text{ham}(G) > \text{rad}(G)$ holds are characterised in a similar way.

1. Introduction

Although the notion of line graph is well-known, it is worth starting with its definition. If $G$ is any graph with nonempty set $E(G)$ of edges, then the line graph $L(G)$ of $G$ is a graph such that $V(L(G)) = E(G)$ and two vertices are adjacent in $L(G)$, if and only if they are adjacent as edges in $G$. Iterated line graphs are defined by recursion as $L^0(G) = G$, $L^n(G) = L(L^{n-1}(G))$ for $n \geq 1$. Due to our concern about hamiltonicity in line graphs, we restrict ourselves to connected graphs. Clearly, $L(G)$ is connected if $G$ is connected. In the infinite sequence $G, L(G), L^2(G), \ldots$ of iterated line graphs there always exists a hamiltonian one, except when $G$ is a path (a result of Chartrand [2]). Since the line graph of hamiltonian graph is again hamiltonian, it makes sense to define the hamiltonian index of $G$ as follows [2]:

$\text{ham}(G) = \min \{ n : L^n(G) \text{ is hamiltonian} \}$.

The paths and cycles will be excluded from the rest of this article for obvious reasons.

A sequence of vertices $u_0, u_1, \ldots, u_k$ in $G$ with $u_0 = u_k$ is called a circuit if $u_{i-1}u_i \in E(G)$ ($i = 1, \ldots, k$) are pairwise distinct edges. If $k = 0$, the circuit consists of a single vertex $u_0$ without edges and it is said to be trivial. A circuit $D$ is dominating if every edge of $G$ is incident to a vertex of $D$. For instance, hamiltonian cycles and
eulerian circuits are special cases of dominating circuits, but the reverse is generally not true. According to Harary and Nash-Williams [3], $L(G)$ is hamiltonian, if and only if $G$ contains a dominating circuit.

Let $Q$ be a path in $G$. The length of $Q$ is defined as the number of edges belonging to $Q$, i.e. $|E(Q)|$. For $u, v \in V(G)$, the distance from $u$ to $v$ is the length of the shortest path from $u$ to $v$ and is denoted by $\text{dist}(u, v)$. If $u \in V(G)$, then the eccentricity of $u$ is the number $\text{ecc}(u) = \max_{v \in V(G)} \text{dist}(u, v)$, and the radius of $G$ is the minimal eccentricity among all vertices. The aim of this article is to show that the graphs with the property $\text{ham}(G) > \text{rad}(G)$ can be efficiently characterised.

2. Branch graphs

We are now going to present a short exposé of the theory of branch graphs, as developed in [4]. Define

$$E_T(G) = \{e \in E(G): e \text{ belongs to some triangle}\},$$
$$V_D(G) = \{u \in V(G): \deg(u) \neq 2\}.$$ 

Let $T_G$ be the subgraph spanned by $E_T(G)$ and $D_G$ the subgraph such that $V(D_G) = V_D(G)$, $E(D_G) = \emptyset$. A connected component of the union $T_G \cup D_G$ is called the 3-component of $G$. Obviously, if $H$ is a 3-component, then $H$ is either a maximal subgraph with the property that every edge of $H$ belongs to a triangle, or $H$ is trivial with its single vertex having degree different from 2 and not lying on any triangle. Suppose that $Q$ is a path in $G$ such that

- no edge of $Q$ belongs to any triangle,
- every internal vertex of $Q$ has degree 2,
- both endvertices of $Q$ belong to $V_D(G)$.

Then $Q$ is said to be a branch (cf. [1]). The set $E(G) \setminus E(T_G \cup D_G)$ is exactly the set of all edges which belong to branches. Moreover, the endpoints of branches always belong to $T_G \cup D_G$. Thus, we can define the branch graph $B(G)$ of $G$ with $V(B(G)) = \{H_1, \ldots, H_s\}$, the set of all 3-components, as its vertex set, and $E(B(G)) = \{Q_1, \ldots, Q_t\}$, the set of all branches, as its edge set. Denote $\eta(Q_i) = |E(Q_i)|$, $i = 1, \ldots, t$.

We will usually say that $\eta(Q_i)$ is the length of the edge $Q_i$ (in $B(G)$). It should be noted here that branch graphs may have loops and multiple edges.

If $e \in E(G)$ such that $G - e$ is disconnected (while $G$ is, of course, connected), then $e$ is called a bridge. Let $Q$ be a branch in $G$ and $s = \eta(Q)$. When every edge of $Q$ is a bridge of $G$, we say that $Q$ is an $s$-bridge. If one of the endvertices of $Q$ is of degree 1, then $Q$ is an $s$-bridge of the first kind, otherwise it is said to be of the second kind.
Another type of circuit can be defined which plays the same role in branch graphs as dominating circuits do in usual graphs, as far as hamiltonicity is concerned [4].

**Definition 1.** The circuit $M$ in branch graph $B(G)$ is said to be *main* if it satisfies the following conditions:

- $M$ traverses every $Q \in E(B(G))$ such that $\eta(Q) \geq 3$,
- $M$ traverses every $H \in V(B(G))$ such that $\deg_{B(G)}(H) \geq 2$,
- if $H \in V(B(G))$, $\deg_{B(G)}(H) = 1$, and $Q \in E(B(G))$ is incident to $H$, then $H$ is a trivial 3-component of $G$ and $\eta(Q) = 1$.

The connection between dominating and main circuits is established by the following theorem proved in [4].

**Theorem 2.** Consider the following two propositions:

(a) $G$ contains a dominating circuit,
(b) $B(G)$ contains a main circuit.

It is always true that (b) $\implies$ (a). If $G$ is the line graph of some other graph, then (a) $\implies$ (b) as well.

Let us briefly remind the reader of the notion of graph contractions. Assume $F$ is an arbitrary graph and $e \in E(F)$. The *elementary contraction of $F$ by $e$* (denoted by $F/e$) is the graph obtained from $F$ by deleting $e$ and identifying its end vertices. If $H$ is any subgraph of $F$, then $F/H$ is the *contraction of $F$ by $H$* obtained after performing a sequence of elementary contractions using every $e \in E(H)$. Since we will use contractions in branch graphs where all edges (and their lengths) are important, we adopt the practice of preserving loops and multiple edges if they emerge as a result of contractions, instead of deleting them.

We also need a connection between line and branch graphs. More precisely, what is the interplay between a graph, its line graph, and the branch graphs of the two? Let $H$ be an arbitrary 3-component of $G$. It follows that $L(H)$, which is of course a subgraph of $L(G)$, appears as a subgraph in a 3-component $H'$ of $L(G)$. If $H$ is trivial, then $H'$ denotes the 3-component containing the complete subgraph of $L(G)$ induced by edges incident to $H$. We say that $H'$ is *generated by $H$*. Likewise, if $Q$ is a branch in $G$, then $Q' = L(Q)$ is a branch *generated by $Q$* with length $\eta(Q') = \eta(Q) - 1$. However, this fails for $\eta(Q) = 1$ because in this case the 3-components $H_1'$ and $H_2'$, generated by "endvertices" $H_1$ and $H_2$ of $Q$, merge together into one. Let $F^*$ be the subgraph spanned by all branches of length 1 in $B(G)$ and define $\Lambda(B(G)) = B(G)/F^*$. It follows from the preceding argument that $B(L(G)) = \Lambda(B(G))$ and that the lengths of all branches are diminished by 1 in $\Lambda(B(G))$.

Take $H_1, H_2 \in V(B(G))$ and an arbitrary path $\mathcal{P}$ from $H_1$ to $H_2$. Define

$$d(\mathcal{P}) = \max_{Q \in E(\mathcal{P})} \eta(Q),$$

$$d(H_1, H_2) = \min\{d(\mathcal{P}). \text{ over all paths } \mathcal{P} \text{ from } H_1 \text{ to } H_2\}.$$
The function $d$ can be regarded as distance on the set $V(B(G))$. Denote by $A^n(H_1), A^n(H_2)$ the 3-components in $A^n(B(G))$ generated by $H_1$ and $H_2$, respectively. The following lemma is essentially Lemma 5 in [4] and its proof is straightforward.

**Lemma 3.** For any $n \geq 1$, $B(L^n(G)) = A^n(B(G))$. Let $H'_1 = A^n(H_1), H'_2 = A^n(H_2)$ where $n \geq 1$ is arbitrary fixed number. If $d(H_1, H_2) > n$, then $d(H'_1, H'_2) = d(H_1, H_2) - n$, otherwise $H'_1 = H'_2$.

If $F$ is an arbitrary graph, let $\bar{V}(F)$ be the set of its vertices of odd degree. Recall the well-known fact that this set is of even cardinality. Define

$$\xi(F) = \max \{d(H, H') : H, H' \in \bar{V}(B(F))\},$$

if $\bar{V}(B(F))$ is nonempty, and $\xi(F) = 0$ otherwise (cf. [1, p. 357]). The key tool used in the proof of the main result is stated below.

**Lemma 4.** For any graph $G$,

$$\text{ham}(G) \leq \xi(G) + 1.$$

This is actually Theorem 7 of [4].

3. The radius

We are now ready to prove the following:

**Theorem 5.** Assume that $G$ is a graph such that $\text{ham}(G) > \text{rad}(G)$. Then exactly one of the two statements holds:

(i) $G$ contains a $\xi(G)$-bridge of the first kind without having any $\xi(G)$-bridge of the second kind, and $\text{ham}(G) = \xi(G)$;

(ii) $G$ contains a $\xi(G)$-bridge of the second kind and $\text{ham}(G) = \xi(G) + 1$.

**Proof.** If $\text{rad}(G) = 1$, then there is a vertex $u_0 \in V(G)$ with $\text{ecc}(u_0) = 1$ which means that $u_0$ is adjacent to all other vertices. We shall construct a dominating circuit in $G$ as follows. Take arbitrary $w, z \in V(G)$ such that $w, z \neq u_0$. If $wz \notin E(G)$ for every such $w$ and $z$, then $G$ is isomorphic to $K_{1,m}$ for some $m$ and so it contains a trivial dominating circuit. If this is not the case, then $wz \in E(G)$ for some $w, z$. Start a circuit at $u_0$ and traverse the edges $u_0w, wz, zu_0$. If there is another edge $w'z'$ such that $w' \neq u_0$ and neither $w'$ nor $z'$ have been traversed yet, then traverse $u_0w', w'z', z'u_0$. Repeat this step many times until every edge of $G$ has at least one endvertex lying on the circuit. This means that $G$ contains the dominating circuit, hence $L(G)$ is hamiltonian by [3]. But then $\text{ham}(G) \leq 1 = \text{rad}(G)$, a contradiction.
We can therefore assume that \( \text{rad}(G) \geq 2 \). Since \( \text{rad}(G) < \text{ham}(G) \), it follows from Lemma 4 that \( 2 \leq \text{rad}(G) \leq \xi(G) \). Now take two 3-components \( H, H' \in \mathcal{V}(B(G)) \) such that \( \xi(G) = d(H, H') \). This means that every path from \( H \) to \( H' \) in \( B(G) \) contains at least one edge \( Q \) with length \( \eta(Q) = \xi(G) \), and at least one of these paths does not contain edges whose length would exceed \( \xi(G) \). Let \( \mathcal{S} = \{ Q \in E(B(G)): \eta(Q) \geq \xi(G) \text{ and } Q \text{ belongs to some path from } H \text{ to } H' \} \). It follows that \( B(G) - \mathcal{S} \) is disconnected. Let \( \mathcal{S} \) be the minimal subset such that \( B(G) - \mathcal{S} \) is disconnected. This graph consists of exactly two components, say \( B_1 \) and \( B_2 \). Let \( \mathcal{S} = \{ Q_1, \ldots, Q_t \} \) and recall that \( Q_i \) are branches in \( G \) with lengths \( \xi(G) \) or more (at least one having length exactly \( \xi(G) \)). If \( u \in V(G) \) does not belong to any of the branches in \( \mathcal{S} \), then it is clear that \( \text{ecc}(u) > \xi(G) > \text{rad}(G) \), hence the vertex \( u_0 \) of minimal eccentricity is contained in a branch \( Q_j \). Consider now three possibilities.

Case 1: \( \text{rad}(G) = \xi(G) \) and \( u_0 \) is an internal vertex of \( Q_j \). Denote by \( H_1, H_2 \) the 3-components which are incident to \( Q_j \) in \( B(G) \) and belong to \( B_1 \) and \( B_2 \), respectively. Since \( \text{rad}(G) = \text{ecc}(u_0) \), it follows that the distance from \( H_1 \) to any other vertex in \( B_1 \) is strictly less than \( \text{rad}(G) \). The same holds for \( H_2 \) and \( B_2 \), too. By Lemma 3 the branch graph \( F = A_{1}^{\text{rad}(G)-1}(B(G)) \) contains exactly two vertices \( H_1', H_2' \) (generated by \( H_1 \) and \( H_2 \), respectively) and edges \( Q_i' \), generated by \( Q_i \), between \( H_1' \) and \( H_2' \). If the set \( \mathcal{S} \) contains more than one element, then \( F \) possesses a main circuit by Definition 1, since at least one of \( Q_i \) has length equal to 1. It follows from \([3]\) and Theorem 2 that \( L_{\text{rad}(G)}(G) \) is hamiltonian, but this is contradictory to the assumption that \( \text{ham}(G) > \text{rad}(G) \). Hence \( t = 1 \) and there is only one branch \( Q_1 \) in the set \( \mathcal{S} \) which is clearly a \( \xi(G) \)-bridge. If it is of the first kind, then one of two vertices of \( F \) represents a trivial 3-component and \( F \) contains the main circuit (\( Q_1' \) has length equal to one), which leads us to contradiction as above. Therefore, \( Q_1 \) is of the second kind and \( F \) does not have a main circuit, which means that \( L_{\text{rad}(G)}(G) \) is not hamiltonian. Hence it follows from Lemma 4 that \( \text{ham}(G) = \xi(G) + 1 \) and (ii) holds.

Case 2: \( \text{rad}(G) = \xi(G) \) and \( u_0 \) is an endvertex of \( Q_j \). Suppose, without loss of generality, that \( u_0 \) belongs to a 3-component \( H_2 \) of \( B_2 \). As in Case 1, all edges in \( B_1 \) have lengths strictly less than \( \text{rad}(G) \). Likewise, the edges in \( B_2 \) not incident to \( H_2 \) are strictly shorter than \( \text{rad}(G) \). It follows from Lemma 3 that \( F = A_{2}^{\text{rad}(G)-1}(B(G)) \) is isomorphic to \( K_2 \) or \( K_{1,m} \) for some \( m \geq 2 \) with multiple edges and loops, and for every pair of adjacent vertices at least one edge joining them has length 1. Moreover, if some edge in \( F \) is a bridge, then one of its end vertices is a trivial 3-component due to the fact that \( \text{ecc}(u_0) = \text{rad}(G) = \xi(G) \). This means that \( F \) satisfies the condition in Definition 1, so it contains a main circuit. Thus, \( L_{\text{rad}(G)}(G) \) is hamiltonian, a contradiction.

Case 3. \( \text{rad}(G) < \xi(G) \). Suppose that there are two branches \( Q_i \) and \( Q_k \) which lie on a common cycle. Obviously, \( \text{ecc}(u) \geq \xi(G) \geq \text{rad}(G) + 1 \) for every vertex \( u \in V(Q_i) \cup V(Q_k) \). From this it can be quickly derived that \( \text{ecc}(u) > \text{rad}(G) + 1 \) for all vertices \( u \) belonging to the branch \( Q_i \in \mathcal{S} \), for every \( i = 1, \ldots, t \). If \( u \) does not belong to any branch \( Q_i \in \mathcal{S} \), then it is even easier to see that \( \text{ecc}(u) > \text{rad}(G) + 1 \), too. Thus, \( \text{ecc}(u) > \text{rad}(G) \) for every \( u \in V(G) \) which is impossible. It follows that no two branches
from the set \( \mathcal{Q} \) belong to a common cycle, hence all of them are bridges. Since, there is a vertex \( u_0 \in V(Q_1) \) with \( \text{ecc}(u) = \text{rad}(G) < \xi(G) \), the set \( \mathcal{Q} \) cannot contain more than one branch \( Q_1 \), and so \( u_0 \) is an internal vertex of \( Q_1 \). Following the same line of argument as in Case 1, it is readily shown that the branch graph \( F = A^{(G)-1}(B(G)) \) is isomorphic to \( K_2 \) and that the length of its single edge equals 1. If \( Q_1 \) is a \( \xi(G) \)-bridge of the first kind, then \( F \) contains a main circuit. Hence \( L^{(G)-1}(G) \) contains a dominating circuit, yet it is not hamiltonian. Thus, \( \text{ham}(G) = \xi(G) \) and (i) holds. If \( Q_1 \) is of the second kind, then \( F \) does not contain a main circuit, hence \( \text{ham}(G) = \xi(G) + 1 \) by Lemma 4 and (ii) holds. \( \square \)

4. Conclusions

In the present paper, we investigated the relationship between the hamiltonian index and the radius of a graph. Using the concept of branch graphs, as developed in [4], we determined all graphs for which \( \text{ham}(G) > \text{rad}(G) \) and also calculated the exact value of \( \text{ham}(G) \) (Theorem 5). Thus, the inequality \( \text{ham}(G) \leq \text{rad}(G) \) holds for all other graphs, which is an improvement over the result of Catlin et al. [1, Corollary 9A].

References