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RESEARCH PROBLEMS

Problems 121–127. Posed by G. Rayindra.

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A graph G is *perfect* if for each of its induced subgraphs H, the chromatic number of H is equal to the maximum cardinality of a clique of H. A graph is *strongly perfect* [1] if each of its induced subgraphs H contains an independent set of vertices which meets all the maximal cliques of H. The strongly perfect graphs form an interesting class of perfect graphs, as the complement of a strongly perfect graph need not be strongly perfect, unlike a perfect graph. For example, the cycle C_{2n}, $n \geqslant 3$, is strongly perfect, but its complement is not strongly perfect. A graph G is *costrongly perfect* if G and its complement $\bar{G}$ are strongly perfect. A graph G is *p-critical* if G is not perfect and every proper induced subgraph of G is perfect. Similarly, G is *sp-critical* (*csp-critical*) if G is not strongly (costrongly) perfect and every proper induced subgraph of G is strongly (costrongly) perfect. The *normal product* of graphs G and H is the graph $G \cdot H$ whose vertex-set is $V(G) \times V(H)$ and two vertices (u, v) and $(u'v')$ are adjacent in $G \cdot H$ if and only if one of the following holds:

(i) $u = u'$ and vv' is an edge in H,
(ii) uu' is an edge in G and $v = v'$, or
(iii) uu' is an edge in G and vv' is an edge in H.

Berge's famous strong perfect graph conjecture states that a graph G is perfect if and only if neither G nor $\bar{G}$ contains an induced cycle of odd length five or more. If the conjecture is true, it is an excellent characterization of perfect graphs in terms of forbidden subgraphs. Detailed information regarding perfect graphs and the strong perfect graph conjecture is available in [2, 3].

The poser [7] has proved that a graph is strongly perfect if each of its cycles of odd length at least five has at least two chords, and has characterized strongly perfect line graphs [6]. Berge and Duchet [1] have characterized those perfect graphs which are strongly perfect by assigning an integer to each maximal clique in a perfect graph in a particular manner. However, there is no elegant characterization of strongly perfect graphs in terms of forbidden subgraphs. Thus, the problem of attempting to characterize strongly perfect graphs in terms of forbidden subgraphs, originally posed by Berge, is worth considering.

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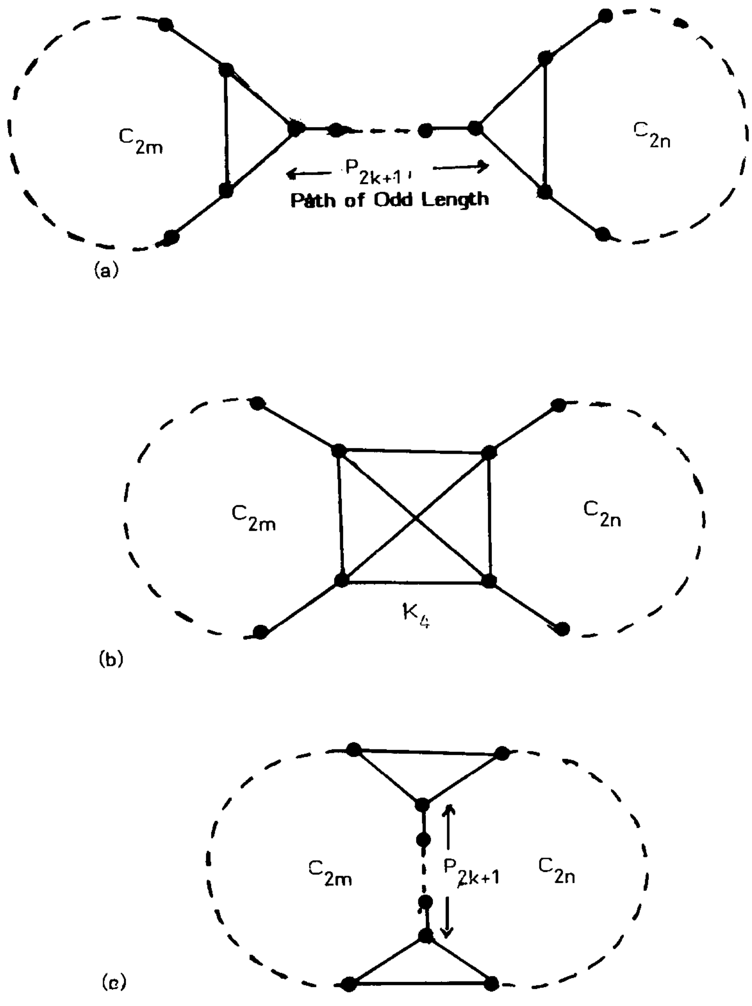

Fig. 1

Problem 121. Is the following conjecture true?

Conjecture: If a graph does not contain $K_{1,3}$ as an induced subgraph, then it is strongly perfect if and only if it does not contain a cycle of odd length at least five, the complement of a cycle of length at least five, or any of the graphs in *Fig. 1.*

The following problem was revised after the adition of the last forbidden subgraph (*Fig. 2(b)*) provided to the poser by Chvátal.

Problem 122. Is the following conjecture true?

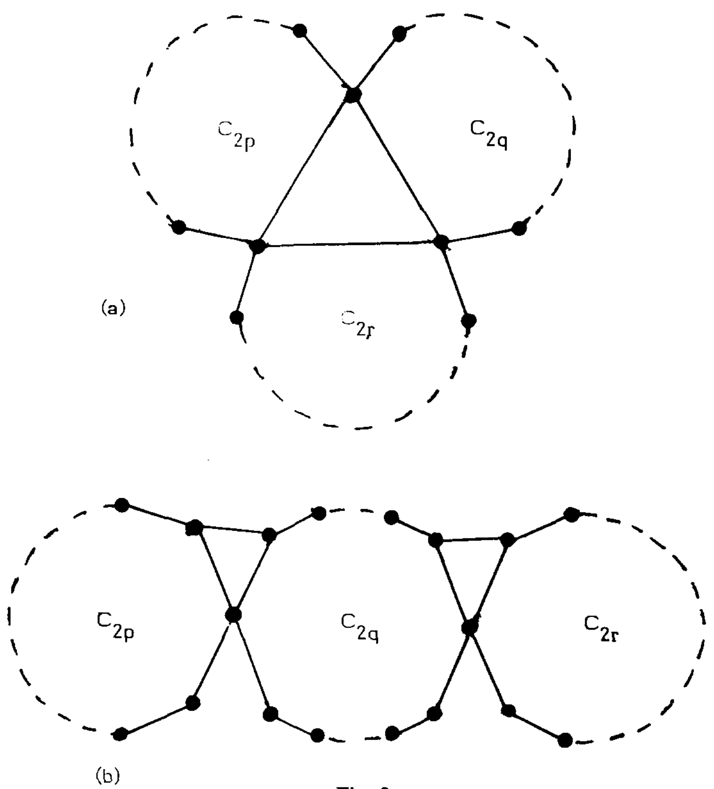

Fig. 2

Conjecture: A graph is strongly perfect if and only if it does not contain any of the forbidden subgraphs mentioned in Problem B or the graph of *Fig. 2* as an induced subgraph.

Problem 123. Is the following conjecture true?

Conjecture: If G is p-critical, then it is sp-critical.

If the conjecture of Problem 123 is true, the characterization of strongly perfect graphs in terms of forbidden subgraphs leads immediately to a solution of the strong perfect graph conjecture.

Some classes of costrongly perfect graphs have been identified in [8, 9, 10], but there is no complete characterization of costrongly perfect graphs. This leads to the next problem.

Problem 124. Characterize co-strongly perfect graphs.

Problem 125. Is the following conjecture true?

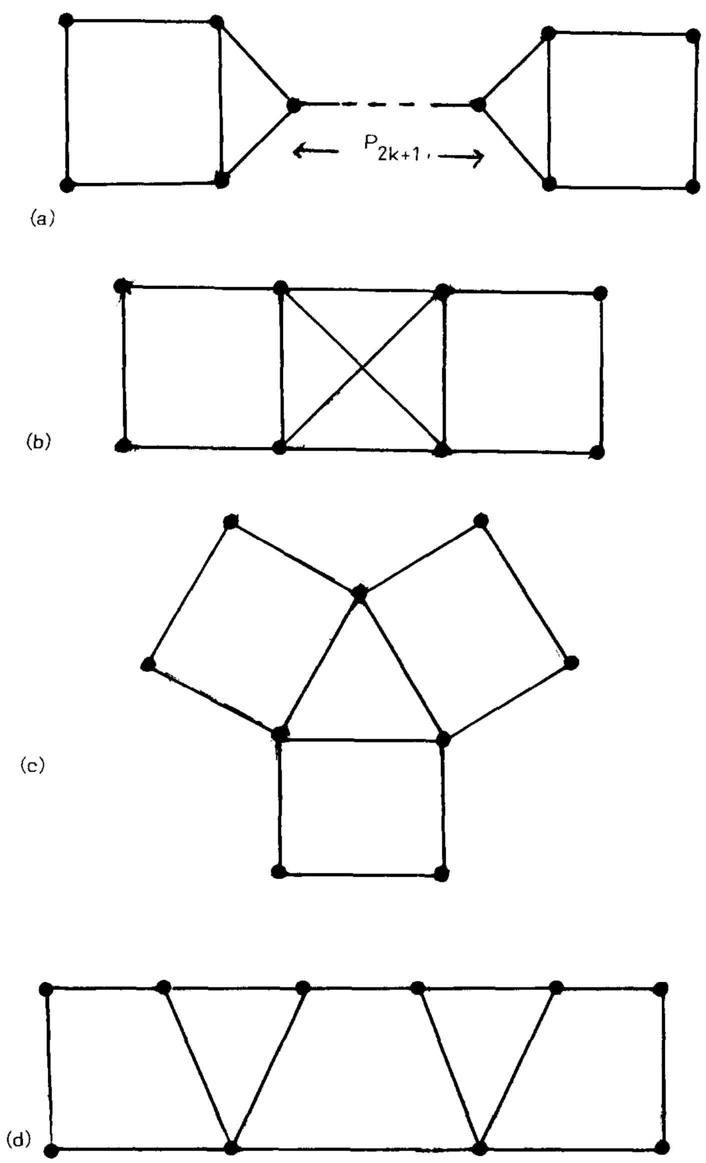

Fig. 3

Conjecture: A strongly perfect graph is costrongly perfect if and only if it does not contain a cycle of even length at least six, or the complement of any of the graphs of *Fig. 3* as an induced subgraph.

An *sp*-critical graph is not necessarily *csp*-critical. For example, a graph of the type shown in *Fig. 1(a)* is *sp*-critical but not *csp*-critical. Contrast this with the conjecture of Problem 123.

Several perfect product graphs have been characterized [4], but only partial characterizations of perfect normal product graphs are available [5]. This leads to the following problem.

Problem 126. Characterize perfect normal products $G_1 \cdot G_2$ in terms of the structures of G_1 and G_2.

Let $\alpha(G)$ denote the independence number of G. The graph G is said to be α-partitionable if either $\alpha(G) = 1$ or the vertex-set of G has a partition $V_1 \cup V_2$ such that $\alpha(G_1) + \alpha(G_2) = \alpha(G)$, where G_1 and G_2 are the subgraphs induced by G on V_1 and V_2, respectively. Let

$$E_{cr}(G) = \{e \in E(G);\ \alpha(G - e) > \alpha(G)\}.$$

Problem 127. If $E_{cr}(G) = \emptyset$, then G is α-partitionable.

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