Applied Mathematics Letters 24 (2011) 1063–1068

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Construction of stratified L-fuzzy topological structures

S.E. Abbas *, M.A. Hebeshi

Department of Mathematics, Faculty of Science, Sohag 82524, Egypt
Department of Mathematics, Faculty of Science, Jazan University, Saudi Arabia

A R T I C L E I N F O

Article history:
Received 24 October 2010
Received in revised form 15 January 2011
Accepted 25 January 2011

Keywords:
L-fuzzy quasi-proximity
L-fuzzy topogenous order spaces
Stratification

A B S T R A C T

In this paper, we study the relationships between L-fuzzy quasi-proximity (resp. L-fuzzy topogenous order spaces) and L-grill (resp. L-filter) and the stratification of them.

© 2011 Elsevier Ltd. All rights reserved.

1. Introduction and preliminaries

Csaszar [1] gave a new method for the foundation of general topology based on the theory of syntopogenous structure to develop a unified approach to the three main structures of set-theoretic topology: topologies, uniformities and proximities. This enabled him to evolve a theory including the foundations of the three classical theories of topological spaces, uniform spaces and proximity spaces. In the case of the fuzzy structures there are at least two notions of fuzzy syntopogenous structures, the first notion worked out in [2–4] presents a unified approach to the theories of Chang fuzzy topological spaces [5], Hutton fuzzy uniform spaces [6], Katsaras fuzzy proximity spaces [7–9] and Artico fuzzy proximity [10]. The second notion worked out in Katsaras [11,12] agree very well with Lowen fuzzy topological spaces [13], Lowen–Höhle fuzzy uniform spaces [14] and Artico–Moresco fuzzy proximity spaces [10].

In this paper, we study the relationships between L-fuzzy quasi-proximity (resp. L-fuzzy topogenous order spaces) and L-grill (resp. L-filter) and the stratification of them.

Throughout this paper, let X be a nonempty set. Let a complete lattice $L = (L, \leq, \vee, \wedge, ')$ be a complete distributive complete lattice with an order-reversing involution on it, and with a smallest element $\bot$ and largest element $\top (\bot \neq \top)$. For $\alpha \in L$, $\underline{\alpha}(x) = \alpha$ for all $x \in X$.

2. Stratified L-fuzzy quasi-proximity spaces

Definition 2.1 (*[15]*). A map $\delta : L^X \times L^X \to L$ is said to be an L-fuzzy quasi-proximity on X if it satisfies the following conditions:

(LP1) $\delta(1_\emptyset, 1_X) = \bot$.
(LP2) If $\delta(f, g) \neq \top$, then $f \leq g'$.
(LP3) $\delta(f_1 \vee f_2, g) = \delta(f_1, g) \vee \delta(f_2, g)$ and $\delta(g, f_1 \vee f_2) = \delta(g, f_1) \vee \delta(g, f_2)$.
(LP4) $\delta(f, g) \geq \bigwedge_{h \in L^X} \{\delta(f, h) \vee \delta(h', g)\}$.

An L-fuzzy quasi-proximity δ is said to be stratified iff δ satisfies the following condition:

(LPS) $\delta(\underline{\alpha}, \underline{\alpha}') = \bot$, for all $\alpha \in L$.

Let δ_1 and δ_2 be L-fuzzy quasi-proximities on X. We say δ_1 is finer than δ_2 (δ_2 is coarser than δ_1) if $\delta_1(f, g) \leq \delta_2(f, g)$ for all $f, g \in L^X$.

* Corresponding author at: Department of Mathematics, Faculty of Science, Sohag 82524, Egypt.
E-mail address: sabbas73@yahoo.com (S.E. Abbas).

0893-9659/$ – see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2011.01.023

Theorem 2.2 ([16]). *Let (X, δ) be an L-fuzzy quasi-proximity space. Define $\delta^{st} : L^X \times L^X \to L$ by*

$$\delta^{st}(f, g) = \bigwedge_{\{(f_i, g_i, \underline{\alpha_i}) | i \in N\} \in \mathcal{W}(f, g)} \left\{ \bigvee_{(f_l, g_l, \underline{\alpha_l}) \in \{(f_i, g_i, \underline{\alpha_i}) | i \in N\}} \delta(f_l, g_l) \right\},$$

where $\mathcal{W}(f, g) = \{\{(f_i, g_i, \underline{\alpha_i}) \mid i \in N, N \text{ is finite index set}\} \mid f \le \bigvee_{i \in N}(f_i \wedge \underline{\alpha_i}) \text{ and } g \le \bigwedge_{i \in N}(g_i \vee \underline{\alpha_i}')\}$. Then δ^{st} is the coarsest stratified L-fuzzy quasi-proximity on X which is finer than δ.

Definition 2.3. A map $\mathcal{G} : L^X \to L$ is said to be an L-grill on X if it satisfies the following conditions:

(LG1) $\mathcal{G}(1_\emptyset) = \bot$ and $\mathcal{G}(1_X) = \top$,
(LG2) $\mathcal{G}(f \vee g) \le \mathcal{G}(f) \vee \mathcal{G}(g)$, for each $f, g \in L^X$,
(LG3) If $f \le g$, then $\mathcal{G}(f) \le \mathcal{G}(g)$.
An L-grill $\mathcal{G}$ is said to be stratified iff $\mathcal{G}$ satisfies the following condition:
(LGS) $\mathcal{G}(f \vee \underline{\alpha}) \le \mathcal{G}(f) \vee \alpha$, for each $f \in L^X$ and $\alpha \in L$.
Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be L-grills on X. We say $\mathcal{G}_1$ is finer than $\mathcal{G}_2$ ($\mathcal{G}_2$ is coarser than $\mathcal{G}_1$) if $\mathcal{G}_1(f) \le \mathcal{G}_2(f)$ for all $f \in L^X$.

Theorem 2.4. *Let $\mathcal{G}$ be an L-grill on X. Define $\mathcal{G}^{st} : L^X \to L$ by*

$$\mathcal{G}^{st}(f) = \bigwedge_{\{(f_i, \underline{\alpha_i}) | i \in N\} \in \mathcal{W}(f)} \left\{ \bigvee_{(f_l, \underline{\alpha_l}) \in \{(f_i, \underline{\alpha_i}) | i \in N\}} \mathcal{G}(f_l) \vee \alpha_l \right\},$$

where $\mathcal{W}(f) = \{\{(f_i, \underline{\alpha_i}) \mid i \in N, N \text{ is finite index set }\} \mid f \le \bigwedge_{i \in N}(f_i \vee \underline{\alpha_i})\}$. Then $\mathcal{G}^{st}$ is the coarsest stratified L-grill on X which is finer than $\mathcal{G}$.

Proof. First, we will prove that $\mathcal{G}^{st}$ is stratified L-grill on X.

(LG1) For all $\{(f_k, \underline{\alpha_k}) \mid k \in N\} \in \mathcal{W}(1_X)$, we have $f_k = 1_X$ or $\underline{\alpha_k} = 1_X$. Thus, $\mathcal{G}^{st}(1_X) = \top$. Also, $\mathcal{G}^{st}(1_\emptyset) = \bot$.
(LG2) Suppose that there exist $f, g \in L^X$ such that

$$\mathcal{G}^{st}(f \vee g) \not\le \mathcal{G}^{st}(f) \vee \mathcal{G}^{st}(g).$$

By the definition of $\mathcal{G}^{st}$, there exist $\{(f_k, \underline{\alpha_k}) \mid k \in N\} \in \mathcal{W}(f)$ and $\{(g_m, \underline{\beta_m}) \mid m \in M\} \in \mathcal{W}(g)$ such that

$$\mathcal{G}^{st}(f \vee g) \not\le \left(\bigvee_{(f_i, \underline{\alpha_i}) \in \{(f_k, \underline{\alpha_k}) | k \in N\}} \mathcal{G}(f_i) \vee \alpha_i \right) \vee \left(\bigvee_{(g_j, \underline{\beta_j}) \in \{(g_m, \underline{\beta_m}) | m \in M\}} \mathcal{G}(g_j) \vee \beta_j \right).$$

Put $l \in N \cup M$ such that

$$h_l \vee \underline{\sigma_l} = \begin{cases} f_l \vee \underline{\alpha_l}, & \text{if } l \in N - (N \cap M) \\ g_l \vee \underline{\beta_l}, & \text{if } l \in M - (N \cap M) \\ (f_l \vee g_l) \vee (\underline{\alpha_l} \vee \underline{\beta_l}), & \text{if } l \in (N \cap M). \end{cases}$$

On the other hand,

$$f \vee g \le \left(\bigwedge_{k \in N} (f_k \vee \underline{\alpha_k}) \right) \vee \left(\bigwedge_{m \in M} (g_m \vee \underline{\beta_m}) \right) = \bigwedge_{l \in N \cup M} (h_l \vee \underline{\sigma_l}),$$

$\{(h_l, \underline{\sigma_l}) \mid l \in N \cup M\} \in \mathcal{W}(f \vee g)$. Then we have

$$\begin{aligned} \mathcal{G}^{st}(f \vee g) &\le \bigvee_{(h_n, \underline{\sigma_n}) \in \{(h_l, \underline{\sigma_l}) | l \in N \cup M\}} \mathcal{G}(h_n) \vee \sigma_n \\ &= \left(\bigvee_{(f_i, \underline{\alpha_i}) \in \{(f_k, \underline{\alpha_k}) | k \in N\}} \mathcal{G}(f_i) \vee \alpha_i \right) \vee \left(\bigvee_{(g_j, \underline{\beta_j}) \in \{(g_m, \underline{\beta_m}) | m \in M\}} \mathcal{G}(g_j) \vee \beta_j \right). \end{aligned}$$

It is a contradiction. Hence, $\mathcal{G}^{st}(f \vee g) \le \mathcal{G}^{st}(f) \vee \mathcal{G}^{st}(g)$.
(LG3) Obvious.
(LGS) Suppose there exist $f \in L^X$ and $\alpha \in L$ such that

$$\mathcal{G}^{st}(f \vee \underline{\alpha}) \not\le \mathcal{G}^{st}(f) \vee \alpha.$$

By the definition of $\mathcal{G}^{st}$, there exists $\{(f_k, \underline{\alpha_k}) \mid k \in N\} \in \mathcal{W}(f)$ such that

$$\mathcal{G}^{st}(f \vee \underline{\alpha}) \not\leq \left(\bigvee_{(f_i,\underline{\alpha_i}) \in \{(f_k,\underline{\alpha_k}) \mid k \in N\}} \mathcal{G}(f_i) \vee \alpha_i \right) \vee \alpha.$$

On the other hand, $f \vee \underline{\alpha} \leq \bigwedge_{k \in N}(f_k \vee \underline{\sigma_k})$, where $\sigma_k = \alpha_i \vee \alpha$, then $\{(f_k, \underline{\sigma_k}) \mid k \in N\} \in \mathcal{W}(f \vee \underline{\alpha})$. Then we have

$$\begin{aligned}\mathcal{G}^{st}(f \vee \underline{\alpha}) &\leq \left(\bigvee_{(f_i,\underline{\sigma_i}) \in \{(f_k,\underline{\sigma_k}) \mid k \in N\}} \mathcal{G}(f_i) \vee \sigma_i \right) \\ &= \left(\bigvee_{(f_i,\underline{\alpha_i}) \in \{(f_k,\underline{\alpha_k}) \mid k \in N\}} \mathcal{G}(f_i) \vee \alpha_i \right) \vee \alpha.\end{aligned}$$

It is a contradiction. Hence $\mathcal{G}^{st}(f \vee \underline{\alpha}) \leq \mathcal{G}^{st}(f) \vee \alpha$. Thus, $\mathcal{G}^{st}$ is stratified.

Second, for each $f \in L^X$, there exists a family $\{\underline{\alpha}\}$ with $f \leq f \vee \underline{\alpha}$ such that $\mathcal{G}^{st}(f) \leq \mathcal{G}(f)$. Hence $\mathcal{G}^{st}$ is finer than $\mathcal{G}$.

Finally, consider $\mathcal{G}^*$ is a stratified L-grill on X which is finer than $\mathcal{G}$. And we will show that $\mathcal{G}^{st} \geq \mathcal{G}^*$. Suppose there exists $f \in L^X$ such that $\mathcal{G}^*(f) \not\leq \mathcal{G}^{st}(f)$. By the definition of $\mathcal{G}^{st}$, there exists $\{(f_k, \underline{\alpha_k}) \mid k \in N\} \in \mathcal{W}(f)$ such that

$$\mathcal{G}^*(f) \not\leq \bigvee_{(f_i,\underline{\alpha_i}) \in \{(f_k,\underline{\alpha_k}) \mid k \in N\}} (\mathcal{G}(f_i) \vee \alpha_i).$$

On the other hand, $\mathcal{G}^*$ is stratified, then we have

$$\begin{aligned}\mathcal{G}^*(f) &\leq \mathcal{G}^*\left(\bigwedge_{k \in N} (f_k \vee \underline{\alpha_k}) \right) \\ &\leq \bigvee_{k \in N} \mathcal{G}^*(f_k \vee \underline{\alpha_k}) \\ &\leq \bigvee_{k \in N} (\mathcal{G}^*(f_k) \vee \alpha_k) \\ &\leq \bigvee_{(f_i,\underline{\alpha_i}) \in \{(f_k,\underline{\alpha_k}) \mid k \in N\}} (\mathcal{G}(f_i) \vee \alpha_i).\end{aligned}$$

It is a contradiction. Thus $\mathcal{G}^{st}$ is the coarsest stratified L-grill on X which is finer than $\mathcal{G}$. □

Now, let δ be an L-fuzzy quasi-proximity, we can identify the relation δ_f on L^X with the map $\delta_f : L^X \to L$ such that

$$\delta_f(g) = \begin{cases} \delta(f, g), & \text{if } g \neq 1_X \\ \top, & \text{if } g = 1_X. \end{cases}$$

It is clearly that δ_f is L-grill.

Theorem 2.5. *Let $\Omega(X)$ and $\Psi(X)$ be families of all L-fuzzy quasi-proximities and L-grills, respectively. Define $\mathcal{H} : \Omega(X) \times \Psi(X) \to \Psi(X)$ as follows:*

$$\mathcal{H}(\delta, \mathcal{G})(f) = \bigwedge_{g \in L^X} \{\delta(f, g) \vee \mathcal{G}(f)\},$$

where $\delta \in \Omega(X)$ and $\mathcal{G} \in \Psi(X)$. Then, we have the following properties:

(1) $\mathcal{H}(\delta, \mathcal{G}) \in \Psi(X)$.
(2) $\mathcal{G} \leq \mathcal{H}(\delta, \mathcal{G})$, *for any* $\mathcal{G} \in \Psi(X)$.
(3) $\mathcal{H}(\delta, \delta_f) = \delta_f$.
(4) $\mathcal{H}(\delta^{st}, \mathcal{G}^{st}) = (\mathcal{H}(\delta, \mathcal{G}))^{st}$.

Proof. (1) (LG1) Since $\mathcal{G}(1_\emptyset) = \bot$ and $\mathcal{G}(1_X) = \top$,

$$\mathcal{H}(\delta, \mathcal{G})(1_\emptyset) = \bigwedge_{g \in L^X} \{\delta(g, 1_\emptyset) \vee \mathcal{G}(1_\emptyset)\} = \bot,$$

$$\mathcal{H}(\delta, \mathcal{G})(1_X) = \bigwedge_{g \in L^X} \{\delta(g, 1_X) \vee \mathcal{G}(1_X)\} = \top.$$

(LG2) Let $f, g \in L^X$. Then we have

$$\begin{aligned}\mathcal{H}(\delta, \mathcal{G})(f \vee g) &= \bigwedge_{h\in L^X}\{\delta(f \vee g, h) \vee \mathcal{G}(f \vee g)\}\\ &\leq \bigwedge_{h\in L^X}\{\{\delta(f, h) \vee \delta(g, h)\} \vee \{\mathcal{G}(f) \vee \mathcal{G}(g)\}\}\\ &= \bigwedge_{h\in L^X}\{\delta(f, h) \vee \mathcal{G}(f)\} \vee \bigwedge_{h\in L^X}\{\delta(g, h) \vee \mathcal{G}(g)\}\\ &= \mathcal{H}(\delta, \mathcal{G})(f) \vee \mathcal{H}(\delta, \mathcal{G})(g).\end{aligned}$$

(LG3) If $f \leq g$, then

$$\mathcal{H}(\delta, \mathcal{G})(f) = \bigwedge_{h\in L^X}\{\delta(f, h) \vee \mathcal{G}(f)\} \leq \bigwedge_{h\in L^X}\{\delta(g, h) \vee \mathcal{G}(g)\} = \mathcal{H}(\delta, \mathcal{G})(g).$$

(2) It is clear from the definition.

(3) From (2), $\mathcal{H}(\delta, \delta_f) \geq \delta_f$, we need only show that $\mathcal{H}(\delta, \delta_f) \leq \delta_f$. Let $1_\emptyset \neq g \in L^X$. Then we have

$$\begin{aligned}\mathcal{H}(\delta, \delta_f)(g) &= \bigwedge_{h\in L^X}\{\delta(h, g) \vee \delta_f(g)\}\\ &= \bigwedge_{h\in L^X}\{\delta(h, g) \vee \delta(f, g)\}\\ &\leq \delta(f, g) \vee \delta(f, g) = \delta(f, g) = \delta_f(g).\end{aligned}$$

(4) Let $f, g \in L^X$. From Theorems 2.2 and 2.4, we have for all finite families $\{f_k \mid f \leq \bigwedge_{k\in N}(f_k \vee \underline{\alpha_k}')\}$ and $\{g_k \mid g \leq \bigvee_{k\in N}(g_k \wedge \underline{\alpha_k})\}$, we have

$$\begin{aligned}\mathcal{H}(\delta^{st}, \mathcal{G}^{st})(f) &= \bigwedge_{g\in L^X}\{\delta^{st}(g, f) \vee \mathcal{G}^{st}(f)\}\\ &= \bigwedge_{g\in L^X}\left(\left(\bigwedge\left\{\bigvee_{k\in N}\delta(g_k, f_k)\right\}\right) \vee \left(\bigwedge\left\{\bigvee_{k\in N}\mathcal{G}(f_k) \vee \alpha_k\right\}\right)\right)\\ &= \left(\bigwedge\left(\left\{\bigvee_{k\in N}\left(\bigwedge_{g_k\in L^X}\delta(g_k, f_k)\right)\right\}\right) \vee \left(\bigwedge\left\{\bigvee_{k\in N}\mathcal{G}(f_k) \vee \alpha_k\right\}\right)\right)\\ &= \bigwedge\left(\bigvee_{k\in N}\left(\bigwedge_{g_k\in L^X}\delta(g_k, f_k) \vee \mathcal{G}(f_k)\right) \vee \alpha_k\right)\\ &= \bigwedge\left(\bigvee_{k\in N}\mathcal{H}(\delta, \mathcal{G})(f_k) \vee \alpha_k\right)\\ &= (\mathcal{H}(\delta, \mathcal{G}))^{st}(f). \quad \square\end{aligned}$$

3. Stratified *L*-fuzzy topogenous order spaces

Definition 3.1 ([17]). A map $\mathcal{N} : L^X \times L^X \to L$ is said to be an L-fuzzy topogenous order on X if it satisfies the following conditions:

(LN1) $\mathcal{N}(1_X, 1_X) = \mathcal{N}(1_\emptyset, 1_\emptyset) = \top$,
(LN2) If $\mathcal{N}(f, g) \neq \bot$, then $f \leq g$,
(LN3) If $f \leq f_1$ and $g_1 \leq g$, then $\mathcal{N}(f_1, g_1) \leq \mathcal{N}(f, g)$,
(LN4) (i) $\mathcal{N}(f_1 \vee f_2, g_1 \vee g_2) \geq \mathcal{N}(f_1, g_1) \wedge \mathcal{N}(f_2, g_2)$,
(ii) $\mathcal{N}(f_1 \wedge f_2, g_1 \wedge g_2) \geq \mathcal{N}(f_1, g_1) \wedge \mathcal{N}(f_2, g_2)$.

The pair $(X, \mathcal{N})$ is called L-fuzzy topogenous order space.

An L-fuzzy topogenous order $\mathcal{N}$ is said to be stratified iff $\mathcal{N}$ satisfies the following condition:

(LNS) $\mathcal{N}(\underline{\alpha}, \underline{\alpha}) = \top$, for all $\alpha \in L$.

Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be L-fuzzy topogenous orders on X. We say $\mathcal{N}_1$ is finer than $\mathcal{N}_2$ ($\mathcal{N}_2$ is coarser than $\mathcal{N}_1$) if $\mathcal{N}_2(f, g) \leq \mathcal{N}_1(f, g)$ for all $f, g \in L^X$.

Theorem 3.2 ([16]). *Let $(X, \mathcal{N})$ be an L-fuzzy topogenous order space. Define $\mathcal{N}^{\mathrm{st}} : L^X \times L^X \to L$ by*

$$\mathcal{N}^{\mathrm{st}}(f, g) = \bigvee_{\{(f_i, g_i, \underline{\alpha_i}) | i \in N\} \in \mathcal{M}(f, g)} \left\{ \bigwedge_{(f_l, g_l, \underline{\alpha_l}) \in \{(f_i, g_i, \underline{\alpha_i}) | i \in N\}} \mathcal{N}(f_l, g_l) \right\},$$

where $\mathcal{M}(f, g) = \{\{(f_i, g_i, \underline{\alpha_i}) \mid i \in N, N \text{ is finite index set}\} \mid f \leq \bigvee_{i \in N}(f_i \wedge \underline{\alpha_i}) \text{ and } g \geq \bigvee_{i \in N}(g_i \wedge \underline{\alpha_i})\}$. Then $\mathcal{N}^{\mathrm{st}}$ is the coarsest stratified L-fuzzy topogenous order on X which is finer than $\mathcal{N}$.

Definition 3.3 ([18,19]). A map $\mathcal{F} : L^X \to L$ is said to be an L-filter on X if it satisfies the following conditions:

(LF1) $\mathcal{F}(1_\emptyset) = \bot$ and $\mathcal{F}(1_X) = \top$,
(LF2) $\mathcal{F}(f \wedge g) \geq \mathcal{F}(f) \wedge \mathcal{F}(g)$, for each $f, g \in L^X$,
(LF3) If $f \leq g$, then $\mathcal{F}(f) \leq \mathcal{F}(g)$.
An L-filter $\mathcal{F}$ is said to be stratified iff $\mathcal{F}$ satisfies the following condition:
(LFS) $\mathcal{F}(f \wedge \underline{\alpha}) \geq \mathcal{F}(f) \wedge \alpha$, for each $f \in L^X$ and $\alpha \in L$.
Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be L-filters on X. We say $\mathcal{F}_1$ is finer than $\mathcal{F}_2$ ($\mathcal{F}_2$ is coarser than $\mathcal{F}_1$) if $\mathcal{F}_2(f) \leq \mathcal{F}_1(f)$ for all $f \in L^X$.

Theorem 3.4 ([20]). *Let $\mathcal{F}$ be an L-filter on X. Define $\mathcal{F}^{\mathrm{st}} : L^X \to L$ by*

$$\mathcal{F}^{\mathrm{st}}(f) = \bigvee_{\{(f_i, \underline{\alpha_i}) | i \in N\} \in \mathcal{W}(f)} \left\{ \bigwedge_{(f_l, \underline{\alpha_l}) \in \{(f_i, \underline{\alpha_i}) | i \in N\}} \mathcal{F}(f_l) \wedge \alpha_l \right\},$$

where $\mathcal{W}(f) = \{\{(f_i, \underline{\alpha_i}) \mid i \in N, N \text{ is finite index set}\} \mid f \geq \bigvee_{i \in N}(f_i \wedge \underline{\alpha_i})\}$. Then $\mathcal{F}^{\mathrm{st}}$ is the coarsest stratified L-filter on X which is finer than $\mathcal{F}$. □

Now, let $\mathcal{N}$ be an L-fuzzy topogenous, we can identify the relation $\mathcal{N}_f$ on L^X with the map $\mathcal{N}_f : L^X \to L^{L^X \times L^X}$ such that

$$\mathcal{N}_f(g) = \begin{cases} \mathcal{N}(f, g), & \text{if } g \neq 1_\emptyset \\ \bot, & \text{if } g = 1_\emptyset. \end{cases}$$

It is clear that $\mathcal{N}_f$ is L-filter.

Theorem 3.5. *Let $\Omega(X)$ and $\Psi(X)$ be families of all L-fuzzy topogenous and L-filters, respectively. Define $\mathcal{H} : \Omega(X) \times \Psi(X) \to \Psi(X)$ as follows:*

$$\mathcal{H}(\mathcal{N}, \mathcal{F})(f) = \bigvee_{g \in L^X} \{\mathcal{N}(g, f) \wedge \mathcal{F}(f)\},$$

where $\mathcal{N} \in \Omega(X)$ and $\mathcal{F} \in \Psi(X)$. Then, we have the following properties:

(1) $\mathcal{H}(\mathcal{N}, \mathcal{F}) \in \Psi(X)$.
(2) $\mathcal{F} \geq \mathcal{H}(\mathcal{N}, \mathcal{F})$, *for any* $\mathcal{F} \in \Psi(X)$.
(3) $\mathcal{H}(\mathcal{N}, \mathcal{N}_f) = \mathcal{N}_f$.
(4) $\mathcal{H}(\mathcal{N}^{\mathrm{st}}, \mathcal{F}^{\mathrm{st}}) = (\mathcal{H}(\mathcal{N}, \mathcal{F}))^{\mathrm{st}}$.

Proof. (1) (LF1) Since $\mathcal{F}(1_\emptyset) = \bot$ and $\mathcal{F}(1_X) = \top$,

$$\mathcal{H}(\mathcal{N}, \mathcal{F})(1_\emptyset) = \bigvee_{g \in L^X} \{\mathcal{N}(g, 1_\emptyset) \wedge \mathcal{F}(1_\emptyset)\} = \bot,$$

$$\mathcal{H}(\mathcal{N}, \mathcal{F})(1_X) = \bigvee_{g \in L^X} \{\mathcal{N}(g, 1_X) \wedge \mathcal{F}(1_X)\} = \top.$$

(LF2) Let $f, g \in L^X$. Then we have

$$\begin{aligned} \mathcal{H}(\mathcal{N}, \mathcal{F})(f \wedge g) &= \bigvee_{h \in L^X} \{\mathcal{N}(h, f \wedge g) \wedge \mathcal{F}(f \wedge g)\} \\ &\geq \bigvee_{h \in L^X} \{\{\mathcal{N}(h, f) \wedge \mathcal{N}(h, g)\} \wedge \{\mathcal{F}(f) \wedge \mathcal{F}(g)\}\} \\ &= \bigvee_{h \in L^X} \{\mathcal{N}(h, f) \wedge \mathcal{F}(f)\} \wedge \bigvee_{h \in L^X} \{\mathcal{N}(h, g) \wedge \mathcal{F}(g)\} \\ &= \mathcal{H}(\mathcal{N}, \mathcal{F})(f) \wedge \mathcal{H}(\mathcal{N}, \mathcal{F})(g). \end{aligned}$$

(LF3) If $f \le g$, then

$$\mathcal{H}(\mathcal{N}, \mathcal{F})(f) = \bigvee_{h \in L^X} \{\mathcal{N}(h, f) \wedge \mathcal{F}(f)\} \le \bigvee_{h \in L^X} \{\mathcal{N}(h, g) \wedge \mathcal{F}(g)\} = \mathcal{H}(\mathcal{N}, \mathcal{F})(g).$$

(2) It is clear from the definition.

(3) From (2), $\mathcal{H}(\mathcal{N}, \mathcal{N}_f) \le \mathcal{N}_f$, we need only to show that $\mathcal{H}(\mathcal{N}, \mathcal{N}_f) \ge \mathcal{N}_f$. Let $1_\emptyset \ne g \in L^X$. Then we have

$$\begin{aligned} \mathcal{H}(\mathcal{N}, \mathcal{N}_f)(g) &= \bigvee_{h \in L^X} \{\delta(h, g) \wedge \mathcal{N}_f(g)\} \\ &= \bigvee_{h \in L^X} \{\mathcal{N}(h, g) \wedge \mathcal{N}(f, g)\} \\ &\ge \mathcal{N}(f, g) \wedge \mathcal{N}(f, g) = \mathcal{N}(f, g) = \mathcal{N}_f(g). \end{aligned}$$

(4) Let $f, g \in L^X$. From Theorems 3.2 and 3.4, we have for all finite families $\{f_j \mid f \ge \bigvee_{j \in N}(f_j \wedge \underline{\alpha_j})\}$ and $\{g_k \mid g \le \bigvee_{k \in N}(g_k \wedge \underline{\alpha_k})\}$,

$$\begin{aligned} \mathcal{H}(\mathcal{N}^{st}, \mathcal{F}^{st})(f) &= \bigvee_{g \in L^X} \{\mathcal{N}^{st}(g, f) \wedge \mathcal{F}^{st}(f)\} \\ &= \bigvee_{g \in L^X} \left(\left(\bigvee \left\{ \bigwedge_{j,k} \mathcal{N}(g_k, f_j) \right\} \right) \wedge \left(\bigvee \left\{ \bigwedge_j \mathcal{F}(f_j) \wedge \alpha_j \right\} \right) \right) \\ &= \left(\bigvee \left(\left\{ \bigwedge_{j,k} \left(\bigvee_{g_k \in L^X} \mathcal{N}(g_k, f_j) \right) \right\} \right) \wedge \left(\bigvee \left\{ \bigwedge_j \mathcal{F}(f_j) \wedge \alpha_j \right\} \right) \right) \\ &= \bigvee \left(\bigwedge_{j,k} \left(\bigvee_{g_k \in L^X} \mathcal{N}(g_k, f_j) \wedge \mathcal{F}(f_j) \right) \wedge \alpha_j \right) \\ &= \bigvee \left(\bigwedge_j \mathcal{H}(\mathcal{N}, \mathcal{F})(f_j) \wedge \alpha_j \right) \\ &= (\mathcal{H}(\mathcal{N}, \mathcal{F}))^{st}(f). \quad \square \end{aligned}$$

Acknowledgements

The authors would like to thank to anonymous referee for his comments, which helped us to improve the final version of this paper.

References

[1] Csaszar, Foundations of General Topology, Pergamon press, 1963.
[2] A.K. Katsaras, On fuzzy syntopogenous structures, J. Math. Anal. Appl. 99 (1983) 219–236.
[3] A.K. Katsaras, On fuzzy syntopogenous spaces, Rev. Roumaine Math. Pure Appl. 30 (1985) 419–431.
[4] A.K. Katsaras, Totally bounded fuzzy syntopogenous structures, Fuzzy Sets and Systems 28 (1988) 91–105.
[5] C.L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl. 24 (1968) 182–190.
[6] B. Hutton, Uniformities on fuzzy topological spaces, J. Math. Anal. Appl. 58 (1977) 559–571.
[7] A.K. Katsaras, Fuzzy proximity spaces, J. Math. Anal. Appl. 68 (1979) 100–110.
[8] A.K. Katsaras, On fuzzy proximity spaces, J. Math. Anal. Appl. 75 (1980) 571–583.
[9] A.K. Katsaras, Fuzzy proximities and fuzzy completely regular spaces, J. Anal. St. Univ. Iasi 26 (1985) 31–41.
[10] G. Artico, R. Moresco, Fuzzy proximities and totally bounded fuzzy uniformities, J. Math. Anal. Appl. 99 (1984) 320–337.
[11] A.K. Katsaras, Fuzzy syntopogenous structures compatible with Lowen fuzzy uniformities and Artico–Moresco fuzzy proximities, Fuzzy Sets and Systems 36 (1990) 375–393.
[12] A.K. Katsaras, Operation on fuzzy syntopogenous structures, Fuzzy Sets and Systems 47 (1991) 199–218.
[13] R. Lowen, Fuzzy topological spaces and fuzzy compactness, J. Math. Anal. Appl. 56 (1976) 621–633.
[14] R. Lowen, Fuzzy uniform spaces, J. Math. Anal. Appl. 82 (1981) 370–385.
[15] M.H. Ghanim, O.A. Tantawy, F.M. Selim, Gradations of uniformity and gradation of proximity, Fuzzy Sets and Systems 79 (1996) 373–382.
[16] A.M. Zahran, S.E. Abbas, M.A. Hebeshi, On stratification of *L*-fuzzy topological structures, Punjab Univ. J. Math. 39 (2007) 55–75.
[17] A.A. Ramadan, M. El-Darder, Y.C. Kim, Fuzzy semi-topogenous order, J. Fuzzy Math 11 (4) (2003) 769–790.
[18] U. Höhle, Many Valued Topology and its Applications, Kluwer, Boston, Dordrecht, London, 2001.
[19] U. Höhle, A.P. Šostak, Axiomatics foundations of fixed-basis fuzzy topology, in: U. Höhle, S.E. Rodabaugh (Eds.), Mathematics of Fuzzy Sets: Logic, Topology and Measure Theory, Kluwer, Dordrecht, 1999.
[20] A.M. Zahran, S.E. Abbas, M.A. Hebeshi, Stratified *L*-filter structures, J. Egyptian Math. Soc. 17 (2) (2009) 135–147.