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Characterizing the Blaschke connection*

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Abstract: It is proved that the Blaschke connection attached to a planar three-web is the unique natural connection attached to planar three-webs.

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1. Introduction and statement of the result

In [3], S.S. Chern reformulated the notion of a three-web on a surface (which from now on we will call simply a web) introduced by W. Blaschke [2] (also see [6, 7]) in terms of G-structures as follows. Let $F^\star M$ be the bundle of coframes of a surface M. Let $\mathbb{R}^\star$ be the subgroup of the full linear group $GL(2,\mathbb{R})$ consisting in all non-zero multiples of the identity matrix,

$$\mathbb{R}^\star = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} : \lambda \in \mathbb{R} - \{0\} \right\}.$$

Giving a planar web is equivalent to giving an $\mathbb{R}^\star$-structure over M, $Q \to M$; i.e., to giving a reduction of the bundle of coframes of M to a subbundle with structure group $\mathbb{R}^\star$. A section of Q defines a moving coframe (ω_1, ω_2) and the curves of three families of one-dimensional foliations are defined by the equations $\omega_1 = 0$, $\omega_2 = 0$, $\omega_1 + \omega_2 = 0$. Equivalently, we can define a web as a reduction of the bundle of linear frames $\pi : FM \to M$ to a subbundle $P \to M$ with structure group $\mathbb{R}^\star$. Blaschke showed that a linear connection can be attached canonically to each web which is characterized—like the Levi-Civita connection of a metric—by the property of being adapted to the web and being symmetric.

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According to a well-known philosophy that has motivated many works since the fifties, the important operators in differential geometry (such as exterior differential, Levi-Civita connection, etc.) should be exclusively characterized by means of their natural properties ([13, 14, 16, 1, 4, 21, 18, 5]; for a general setting on the topic also see [11]). The aim of this work is to prove that the only natural connection attached to a web is the Blaschke connection. To state the result we first need some preliminaries and definitions.

Let $p^r\colon J^r(N) \to M$ be the bundle of r-jets of local sections of a surjective submersion $p\colon N \to M$. We denote by $J^\infty(N)$ the inverse limit of the system $(J^r(N);\, p^r_{r'}\colon J^r(N) \to J^{r'}(N), r \geqslant r')$. A continuous map $f\colon J^\infty(N) \to P$ with values into a finite-dimensional manifold P, is said to be differentiable if for every $j^\infty_x s \in J^\infty(N)$, there exists a non-negative integer r, an open neighbourhood U_r of $j^r_x s \in J^r(N)$ and a C^∞ map $f_r\colon U_r \to P$, such that $f|_{(p^\infty_r)^{-1}(U_r)} = f_r \circ p^\infty_r$. For further details on the topology and differentiable structure of $J^\infty(N)$ we refer to [20]. Also recall (e.g., see [9]) that G-structures over M are in bijection with the sections of the quotient bundle $\bar\pi\colon FM/G \to M$, whenever G is a closed subgroup of the full linear group. In fact, the bijection $s \leftrightarrow P_s$ between sections and G-structures is stated by

$$P_s = \{u \in FM\colon [u] = (\pi(u))\},$$

where $[u] = u \cdot G$ stands for the coset of $u \in FM$ in FM/G and $\pi\colon FM \to M$ is the canonical projection. For each diffeomorphism $\varphi\colon M \to M'$, let $\tilde\varphi\colon FM \to FM'$ be the $GL(n,\mathbb{R})$-principal bundle isomorphism, with $n = \dim M = \dim M'$, given by (cf. [10, Chapter VI, p. 226]): $\tilde\varphi(X_1, \dots, X_n) = (\varphi_* X_1, \dots, \varphi_* X_n)$, for every frame $(X_1, \dots, X_n) \in F_x M$. By passing to the quotient, $\tilde\varphi$ induces a diffeomorphism $\bar\varphi\colon FM/G \to FM'/G$, such that $\bar\pi' \circ \bar\varphi = \varphi \circ \bar\pi$, and for every G-structure $s\colon M \to FM/G$ over M we can define a G-structure over M'—the G-structure obtained by transporting s to M' via φ—by setting $\varphi \cdot s = \bar\varphi \circ s \circ \varphi^{-1}$. Finally, let us denote by $\mathcal{C}(M) \to M$ the bundle of linear connections over M. This is an affine bundle modelled over the vector bundle $T^\star(M) \otimes T^\star(M) \otimes T(M)$ whose global sections are identified to the linear connections on M. Bearing this in mind, we can give a precise definition of the notion of a natural connection as follows.

Definition 1.1. A *natural connection* attached to the G-structures over M is a presheaf morphism $s \mapsto \Gamma(s)$, from the sheaf of sections of $FM/G \to M$ into the presheaf of sections of $\mathcal{C}(M) \to M$, such that:

(1) If $s\colon U \to FM$ is a section on an open subset $U \subseteq M$, then $\Gamma(s)$ is a linear connection on U adapted to the G-structure defined by s; i.e., $\Gamma(s)$ is reducible to the subbundle P_s.

(2) Let $U' \subseteq M'$ be an open subset of an other manifold M'. If $\varphi\colon U \to U'$ is a diffeomorphism, then $\Gamma(\varphi \cdot s) = \varphi \cdot \Gamma(s)$, where the right hand side stands for the direct image of $\Gamma(s)$ by φ (cf. [10, II.Proposition 6.1]).

(3) Γ factors smoothly through $J^\infty(FM/G)$.

The third item above means that the value of the section $\Gamma(s)$ of $\mathcal{C}(U) \to U$ at a point $x \in U$ only depends on $j^\infty_x s$, and that the induced map $\Gamma^\infty\colon J^\infty(FM/G) \to \mathcal{C}(M)$, $\Gamma^\infty(j^\infty_x s) = \Gamma(s)(x)$, is differentiable in the sense previously introduced. Also note that $\Gamma = \Gamma^\infty \circ j^\infty$. This item can be substituted by apparently less restrictive conditions; for example, by only imposing that

$\Gamma(s)(x)$ depends on the germ of s at x, not on the ∞-jet of the section. Nevertheless, standard techniques working in a very general setting (see [11, Chapter V]) readily shows the equivalence of both conditions.

The result can thus be stated as follows:

Theorem 1.2. *Blaschke's connection is the only natural connection which can be attached to two-dimensional three-webs.*

2. Invariants of a linear frame

In order to prove the above result, the basic tool that we used is the determination of the differential invariants of a linear frame. This is known classically as the "local equivalence problem for complete parallelisms" and it plays an important role in the theory of G-structures of finite type (see [15, Chapters 8, 9], [19, VII.4], [22, VIII]). We shall use the approach to this topic given in [8] as its jet formulation is specially adapted to our purposes.

For every vector field $X \in \mathfrak{X}(M)$ let us denote by $\tilde{X} \in \mathfrak{X}(FM)$ the natural lift of X to the bundle of linear frames (cf. [10, VI. Proposition 2.1]). Moreover, given a surjective submersion $p: N \to M$, for every vector field $Y \in \mathfrak{X}(N)$ the notation Y^∞ stands for the natural lift of Y to $J^\infty(N)$ by infinitesimal contact transformations (cf. [12, §12], [20, (2.6)]). A *differential invariant* on the bundle of linear frames is a differentiable function $F: J^\infty(FM) \to \mathbb{R}$ such that $\tilde{X}^\infty(F) = 0$, for every vector field $X \in \mathfrak{X}(M)$. The above equation is the infinitesimal version of the condition stating that F is invariant under the action of the group of diffeomorphisms of M on $J^\infty(FM)$; namely, $F(j^\infty_x(\varphi \cdot \sigma)) = F(j^\infty_x \sigma)$ for every diffeomorphism φ of M and every $j^\infty_x \sigma \in J^\infty(FM)$.

The main example of a differential invariant in $J^\infty(FM)$ is built as follows. Let us fix a linear frame $\sigma = (X_1, \ldots, X_n)$, $n = \dim M$, defined on an open neighbourhood of $x \in M$. We set

$$-[X_i, X_j] = f^k_{ij} X_k, \quad 1 \leqslant i \leqslant j \leqslant n. \tag{1}$$

The sign on the left hand side ensures that the functions f^k_{ij} are exactly the components of the torsion tensor field of the flat connection associated to the given frame when expressed in the basis $(X_1, \ldots, X_n)$. For each sequence of integers $1 \leqslant l_h \leqslant n$, $1 \leqslant h \leqslant m$, we further define a function

$$f^k_{ij;l_1,\ldots,l_m} = X_{l_m}\left(\cdots\left(X_{l_1}(f^k_{ij})\right)\cdots\right) \tag{2}$$

and the scalars $f^k_{ij;l_1,\ldots,l_m}(x)$ only depend on $j^{m+1}_x \sigma$, indeed. (Of course, for $m = 0$ the above assertion means that $f^k_{ij}(x)$ only depends on $j^1_x \sigma$.) Therefore we can define functions $t^k_{ij;l_1,\ldots,l_m} \in C^\infty(J^\infty(FM))$ by setting

$$t^k_{ij;l_1,\ldots,l_m}(j^\infty_x \sigma) = f^k_{ij;l_1,\ldots,l_m}(x). \tag{3}$$

For each positive integer r, let $\mathfrak{F}^r$ be the set of functions $t^k_{ij;l_1,\ldots,l_m}$ whose indices satisfy the following two conditions:

1. $0 \leqslant m \leqslant r-1$ (for $m = 0$, $t^k_{ij;l_1,\dots,l_m} = t^k_{ij}$).
2. $l_1 \geqslant \cdots \geqslant l_m \geqslant i < j$.

As a part of the results in [8], for $n = \dim M = 2$ we have $\#\mathfrak{F}^r = r(r+1)$. Let $I^r\colon J^\infty(FM) \to \mathbb{R}^{r(r+1)}$ be the map whose components are exactly the functions in $\mathfrak{F}^r$. It also follows from [8] that I^r is a surjective submersion and that the ring of differential invariants on the bundle of linear frames of a surface M is

$$\bigcup_{r=1}^{\infty}(I^r)^\star\bigl(C^\infty(\mathbb{R}^{r(r+1)})\bigr).$$

In other words, the functions in $\mathfrak{F}^r$, $r \in \mathbb{N}$ are functionally independent and generate "differentiably" the ring of differential invariants of linear frames on a surface (in the sense that every differential invariant can be written as a differentiable function of some functions in $\mathfrak{F}^r$).

3. Proof of the result

The first reduction in the problem of finding natural connections attached to webs consists in proving that, in fact, such a natural connection only depends on the 1-jet of the $\mathbb{R}^\star$-structure.

Lemma 3.1. *Let $s \mapsto \Gamma(s)$ be a natural connection attached to $\mathbb{R}^\star$-structures over M. There exists a differentiable map $\Gamma^1\colon J^1(FM/G) \to \mathcal{C}(M)$ such that $\Gamma^\infty(j^\infty_x s) = \Gamma^1(j^1_x s)$ for every local section s of $\bar{\pi}\colon FM/G \to M$.*

Proof. Let us first define functions $F_i\colon J^\infty(FM) \to \mathbb{R}$, $i = 1, 2$, as follows. By composing the canonical map $FM \to FM/\mathbb{R}^\star$, $u \mapsto [u]$, with a given linear frame $\sigma\colon U \subseteq M \to FM$, we obtain a section $s\colon U \to FM/\mathbb{R}^\star$ which will be called the web canonically attached to σ. The connection $\Gamma(s)$ has a connection form $\gamma_s\colon T(FU) \to \mathfrak{gl}(2,\mathbb{R})$ which, when restricted to $T(P_s)$, takes values in $\mathbb{R}$ since it is adapted to P_s by virtue of Definition 1.1(1) (note that we are identifying the Lie algebra of $\mathbb{R}^\star$ to $\mathbb{R}$). If $\sigma = (X_1, X_2)$, we define

$$F_i(j^\infty_x\sigma) = (\sigma^\star\gamma_s)((X_i)_x), \quad i = 1, 2.$$

From the fact that Γ factorizes through Γ^∞ it follows that F_i is well defined. We claim that F_i is a differential invariant. In fact, as Γ^∞ is smooth by virtue of the hypothesis, we conclude that F_i also is smooth. Moreover, as γ_s depends functorially on s we have $\gamma_{\varphi\cdot s} = (\tilde{\varphi}^{-1})^\star\gamma_s$ for every diffeomorphism $\varphi\colon M \to M$ (cf. Definition 1.1(2)) and hence

$$\begin{aligned}
F_i\bigl(j^\infty_x(\varphi\cdot\sigma)\bigr) &= F_i\bigl(j^\infty_{\varphi(x)}(\tilde{\varphi}\circ\sigma\circ\varphi^{-1})\bigr)\\
&= \gamma_{\varphi\cdot\sigma}\bigl((\tilde{\varphi}\circ\sigma\circ\varphi^{-1})_\star\varphi_\star(X_i)_x\bigr)\\
&= (\tilde{\varphi}^{-1})^\star\gamma_s\bigl(\tilde{\varphi}_\star\sigma_\star(X_i)_x\bigr)\\
&= \gamma_s\bigl(\sigma_\star(X_i)_x\bigr)\\
&= F_i(j^\infty_x\sigma).
\end{aligned}$$

Accordingly, the function F_i can be written as a differentiable function of the torsions $t^j_{kl;l_1\dots l_m} \in \mathfrak{F}^r$; i.e., there exist functions $G_i \in C^\infty(\mathbb{R}^{r(r+1)})$ such that

$$F_i = G_i \circ I^r = G_i(t^j_{kl}, t^j_{kl;l_1}, \dots, t^j_{kl;l_1\dots l_{r-1}}).$$

For every differentiable function $f\colon M \to \mathbb{R}^\star$ and every linear frame $\sigma = (X_1, X_2)$ we set $f\sigma = (fX_1, fX_2)$. Note that the web canonically attached to the frames σ and $f\sigma$ is the same. We have

$$(f\sigma)^\star\gamma_s = \sigma^\star\gamma_s + \frac{\mathrm{d}f}{f}. \tag{4}$$

In fact, as σ takes values in P_s we can define a function $y\colon P_s \to \mathbb{R}^\star$ by setting $u = \sigma(\pi(u))\cdot y(u)$ for every $u \in P_s$. The restriction of γ_s to P_s can be written as $y^{-1}\,\mathrm{d}y + \eta_s$, where η_s is a π-horizontal 1-form on P_s and the formula (4) follows by taking pull-backs along σ and $f\sigma$, as $y \circ \sigma = 1$. In particular, for an scalar $\lambda \in \mathbb{R}^\star$ we obtain $(\lambda\sigma)^\star\gamma_s = \sigma^\star\gamma_s$ and hence,

$$F_i\big(j^\infty_x(\lambda\sigma)\big) = (\lambda\sigma)^\star\gamma_s((\lambda X_i)_x) = \lambda F_i(j^\infty_x\sigma).$$

Moreover, from the definition of the functions $t^j_{kl;l_1\dots l_m}$ (cf. formula (3)) we obtain

$$t^j_{kl;l_1\dots l_m}\big(j^\infty_x(\lambda\sigma)\big) = \lambda^{m+1}\, t^j_{kl;l_1\dots l_m}(j^\infty_x\sigma)$$

and taking into account that I^r is a surjective map, we can therefore conclude that for any $(b^j_{kl}, b^j_{kl;l_1}, \dots, b^j_{kl;l_1\dots l_{r-1}}) \in \mathbb{R}^{r(r+1)}$ we have

$$G_i(\lambda b^j_{kl}, \lambda^2 b^j_{kl;l_1}, \dots, \lambda^{r+1} b^j_{kl;l_1\dots l_{r-1}}) = \lambda G_i(b^j_{kl}, b^j_{kl;l_1}, \dots, b^j_{kl;l_1\dots l_{r-1}}).$$

Derivation of the above equation with respect to λ and evaluation at $\lambda = 0$, yields

$$G_i(b^j_{kl}, b^j_{kl;l_1}, \dots, b^j_{kl;l_1\dots l_{r-1}}) = \sum_{j,k,l} b^j_{kl}\,\frac{\partial G_i}{\partial b^j_{kl}}(0).$$

We thus see that G_i is a homogeneous function only depending on b^j_{kl}. Therefore the connection form—which can be quickly recovered from the functions G_i—only depends on the 1-jet of the web. $\square$

Remark 3.2. Note that the above lemma also holds true for any $\mathbb{R}^\star$-structure over an n-dimensional manifold.

Proof of Theorem 1.1. Let $f\colon M \to \mathbb{R}^\star$ be a differentiable function. From formula (4) we conclude that

$$F_i\big(j^1_x(f\sigma)\big) = f(x)F_i(j^1_x\sigma) + X_i(f)(x). \tag{5}$$

Furthermore, in the proof of the lemma we saw that the functions F_i are homogeneous in t^j_{kl}, and hence there exist constants c^{ij}_{kl}, $k < l$, such that:

$$F_i = \sum_j \sum_{k<l} c^{ij}_{kl} t^j_{kl}. \tag{6}$$

We set $c^{ij}_{kk} = 0$; $c^{ij}_{kl} = -c^{ij}_{lk}$, for $k > l$. Replacing (6) in (5) and taking into account that

$$t^j_{kl}\big(j^1_x(f\sigma)\big) = f(x)\, t^j_{kl}(j^1_x\sigma) + (X_l f)(x)\,\delta_{jk} - (X_k f)(x)\,\delta_{jl},$$

we obtain the system

$$\sum_j c^{ij}_{jk} = \delta_{ik},$$

whose only solution is $c^{11}_{12} = c^{22}_{12} = 0$, $-c^{12}_{12} = c^{21}_{12} = 1$, and accordingly, $F_1 = -t^2_{12},\ F_2 = t^1_{12}$. Denoting by (ω_1, ω_2) the dual coframe of the linear frame $\sigma = (X_1, X_2)$, we obtain

$$\sigma^\star\gamma_s = (F_1 \circ j^1 s)\,\omega_1 + (F_2 \circ j^1 s)\,\omega_2 = -(t^2_{12} \circ j^1 s)\,\omega_1 + (t^1_{12} \circ j^1 s)\,\omega_2,$$

and from the very definition of the functions t^i_{jk} (cf. formulas (1), (3)) we have

$$\sigma^\star\gamma_s = \omega_2\,([X_1, X_2])\,\omega_1 - \omega_1\,([X_1, X_2])\,\omega_2.$$

From this equation we easily conclude that the torsion of the connection must vanish and hence the connection coincides with Blaschke's connection of the web. □

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