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Local indicability and relative presentations of groups

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Abstract

A relative presentation is a triple $\mathscr{P} = (A, X : R)$ where A is a group, X is a set, and R is a set of words in the free product $A * F(X)$ where $F(X)$ is the free group with basis X. Under certain hypotheses on the relative presentation $\mathscr{P}$, we show that (1) the group presented by $\mathscr{P}$ is locally indicable; (2) the pre-aspherical model for $\mathscr{P}$ is aspherical; (3) the Freiheitssatz holds for $\mathscr{P}$. The result has applications in the computation of cohomology of groups and the field of equations over groups.

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1. Introduction

In this paper, we will extend three foundational results of one-relator group theory, Magnus' Freiheitssatz, Lyndon's Identity Theorem [8], and a theorem by Brodskĭi [3] which states that each torsion free one-relator group is locally indicable. Each of these results has previously been generalized by Howie [5–7] to the setting of one-relator products $(A * B)/r$ of locally indicable groups A and B which completed the one-relator case. A group is locally indicable if every non-trivial, finitely generated subgroup admits a surjection onto the integers.

Anshel [1] proved an extension of the Freiheitssatz for a class of two-relator presentations in 1990. Bogley [2] extended Anshel's Freiheitssatz to a class of multi-relator presentations and proved an analogue of the Identity Theorem for these presentations.

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We will continue in the generalization of the multi-relator case begun by Anshel and Bogley by considering relative presentations.

A relative presentation is a triple $\mathscr{P} = (A, X : R)$, where A is a group, X is a set, and R is a set of words in the free product $A * F(X)$, where $F(X)$ is the free group with basis X. The group presented by the relative presentation $\mathscr{P}$ is the quotient group $G(\mathscr{P}) = (A * F(X))/N$ where N is the normal closure of R. Our results concern the case when the coefficient group A is locally indicable. Following Anshel [1] and Bogley [2], we formulate hypotheses on a relative presentation $\mathscr{P}$ under which (i) the Freiheitssatz holds for $\mathscr{P}$, (ii) there exists an analogue to the Identity Theorem by construction of a $K(G(\mathscr{P}), 1)$, and (iii) the group $G(\mathscr{P})$ is locally indicable.

The paper is organized in the following manner. The second section is an overview of the results from one-relator group theory. The third section includes the multi-relator results, the statement of the new results introduced by this paper, and a method of constructing examples. The final section contains the proofs of all the new theorems.

2. One-relator results

Given the relative presentation $\mathscr{P} = (A, X : R)$, we say that the *Freiheitssatz holds for* $\mathscr{P}$ if the inclusion induced homomorphism $j : A \to G(\mathscr{P})$ is injective. This property was originally introduced by Magnus in [9] where he showed that the Freiheitssatz held for one-relator groups.

The following theorem of Howie generalizes Magnus' Freiheitssatz to the setting of one-relator products of locally-indicable groups. A group G is said to be *locally indicable* if every non-trivial, finitely generated subgroup of G admits a surjection onto the integers.

Theorem 2.1 (Howie [5, Theorem 4.3]). *Suppose* $G = (A * B)/N$, *where* A *and* B *are locally indicable groups and* N *is the normal closure in the free product* $A * B$ *of a cyclically reduced word* r *of length at least* 2. *Then the canonical maps* $A \to G$ *and* $B \to G$ *are injective.*

Now we will define the Identity Property which was first introduced by Lyndon in [8]. The Identity Property was originally defined for ordinary group presentations, but the definition can be generalized to the setting of relative presentations. For a relative presentation $\mathscr{P} = (A, X : R)$, let N be the normal closure of the set of relators R in the free product $A * F(X)$. Then define N^{ab} to be the abelianization of the group N. If $G = G(\mathscr{P})$, then define a G-action on N^{ab} that is induced by conjugation in $A * F(X)$. Under this action, the abelian group is a $\mathbb{Z}G$-module. Moreover, it is generated as a $\mathbb{Z}G$-module by the set of elements $\{r[N,N] : r \in R\}$ which are determined by the relators of the presentation $\mathscr{P}$. We refer to the $\mathbb{Z}G$-module N^{ab} as the *relation module* for the relative presentation $\mathscr{P}$.

For every relator $r \in R$, write $r = q_r^{e(r)}$ where $e(r)$ is maximal. We say that q_r is the root of the relator r in the free product $A * F(X)$. We will always have the relations, $(q_r - 1) * r\,[N,N] = 0$ when our relators are proper powers. These relations are often referred to as the *trivial relations*. The relative presentation $\mathscr{P}=(A,X:R)$ is said to have the *Identity Property* if under the generating set $\{r[N,N]:r\in R\}$ the set of trivial relations $\{(q_r-1)[N,N]:r\in R\}$ are defining relations for N^{ab}. The original definition of a relation module can be recovered by taking the group A to be the trivial group. In [8], Lyndon showed that the Identity property held for one-relator presentations and a special class of multi-relator presentations that are referred to as staggered presentations.

Recall that a topological space K is aspherical if every spherical map $S^n\to K$ is null-homotopic for $n\geqslant 2$. There is a strong connection between the Identity Property and asphericity. Given a relative presentation $\mathscr{P}=(A,X:R)$, build a topological space as follows. Start with the one point union of a $K(A,1)$–complex K_A and the one point union of a collection of circles that are in 1–1 correspondence with the elements of the set X, denoted $K_A\vee(\bigvee_{x\in X}S^1_x)$. Now for every $r\in R$, there exists a based loop

$$\alpha(r):S^1\to\left(K_A\vee\left(\bigvee_{x\in X}S^1_x\right)\right)^{(1)}$$

that represents q_r where $r=q_r^{e(r)}$ and q_r is the root of the relator r. Let the CW-complex D_e be the $K(C_e,1)$-complex where C_e is the cyclic group of order e. Attach the CW-complex $D=\bigvee_{r\in R}D_{e(r)}$ to $(K_A\vee(\bigvee_{x\in X}S^1_x))^{(1)}$ by $\alpha=\bigvee_{r\in R}\alpha(r)$ and let K equal this complex.

Note that for each relator $r\in R$, the CW-complex K has a 2-cell $c^2_r\subseteq D_{e(r)}$ that is attached along path $\beta(r):S^1\to K^{(1)}$ which traverses the path $\alpha(r)\,e(r)$ times. By the Seifert–Van Kampen theorem, $\pi_1(K)\cong G=(A*F(X))/U$ where $U=\langle\langle R\rangle\rangle$ is the normal closure of the set of relators R in the group $A*F(X)$. The complex K is called the *pre-aspherical model* and was first introduced by Dyer and Vasquez in [4]. If A is the trivial group, then the 2-skeleton of K is equivalent to the standard 2-complex associated to the ordinary group presentation $(X:R)$. It is easy to show that a presentation has the Identity Property if and only if the pre-aspherical model is aspherical.

Go back to the setting of $G=(A*B)/N$ where N is the normal closure of the single relator r in the free product $A*B$. This group is represented by the generalized presentation $\mathscr{P}=(A,B:r)$ where A and B are groups. Write $r=q^e$ where q is the root of the relator r. We can modify the pre-aspherical model to generalized presentations by starting our construction with the one point union of a $K(A,1)$ and $K(B,1)$.

Theorem 2.2 (Howie [7, Theorem 1]). *Let $\mathscr{P}=(A,B:r)$ be a one-relator generalized presentation in which the relator r has free product length at least 2 and A and B are locally indicable groups. Then the pre-aspherical model of this generalized presentation is aspherical.*

This theorem is equivalent to saying that the generalized presentation $\mathscr{P}$ has the identity property when A and B are locally indicable groups and r is a word of free product length at least 2.

The last result we will offer a generalization of is the following theorem proved by Brodskĭi.

Theorem 2.3 (Brodskĭi [3]). *Torsion-free* 1*-relator groups are locally indicable.*

During the course of the paper, the following result of Howie will be used repeatedly.

Theorem 2.4 (Howie [6]). *Let A and B be locally indicable groups, and let G be the quotient of the free product $A * B$ by the normal closure of a cyclically reduced word r of length at least* 2. *Then the following are equivalent*:

(i) *G is locally indicable*;
(ii) *G is torsion-free*;
(iii) *r is not a proper power in $A * B$.*

A reduced word $w = x_1 x_2 \dots x_n$ is said to be cyclically reduced if $x_n \neq x_1^{-1}$. A word w is not a proper power if $w = q^e$ implies that $e = 1$.

3. Semi-staggered presentations

3.1. Definition of semi-staggered presentations

In 1990, Anshel [1] published a Freiheitssatz statement for a class of two-relator groups. She extended Magnus' approach to the one-relator case by developing what she termed an *independence hypothesis* for a two-relator presentation of the form $\mathscr{P} = (X, y, z : R, S)$ and proving that the inclusion of the free group with basis X into the group $G(\mathscr{P})$ is injective. Her methods, like Magnus', were combinatorial. This theorem was a first step in attempting to generalize 1-relator group theory to multi-relator groups. In 1991, by interpreting Anshel's conditions in a topological setting, Bogley proved that a larger class of multi-relator presentations which include Anshel's two-relator presentations have the Identity Property. He also extended her Freiheitssatz to this class of multi-relator groups.

Let $\mathscr{P} = (A, X : R)$ be a relative presentation where A is a group, X is a set, and R is a set of cyclically reduced words representing elements in $A * F(X)$ where $F(X)$ is the free group with basis X. Let $G(\mathscr{P}) = (A * F(X))/U$ where $U = \langle\langle R \rangle\rangle$ is the normal closure of the set R in the group $A * F(X)$. Also, let $H = (A * F(X))/N$ where $N = \langle\langle A \cup R \rangle\rangle$. Thus, H is obtained from G by "killing" the (normal closure) of A.

After cyclic permutation, we can assume that each $r \in R$ has the cyclically reduced from

$$r = x_1 a_1 x_2 a_2 \dots x_n a_n,$$

where x_i is a word in $F(X)$, $a_i \in A$, and $i \geqslant 1$. Now define P_r to be the subset of cosets of N

$$P_r = \{x_1 N, x_1 x_2 N, \dots, x_1 \dots x_n N = 1N\}.$$

The set P_r is the set of initial segments of the relator r modulo A. Let $\Pi = \bigcup_{r\in R} P_r \subseteq H$. If Ω is a subset of H, define $\Omega^\bullet = \Omega - \{1N\}$. Now we are ready for the definition of a semi-staggered presentation.

Definition 1. A relative presentation $\mathscr{P} = (A, X : R)$ *is semi-staggered if* the following three conditions are satisfied:
(S1) $P_r^\bullet \neq \emptyset$ *for every* $r \in R$;
(S2) *There exists linear orderings on R and* $\Pi^\bullet$ *such that for* $r, s \in R$, *if* $r < s$ *then* $\min P_r^\bullet < \min P_s^\bullet$ *and* $\max P_r^\bullet < \max P_s^\bullet$;
(S3) $\Pi^\bullet$ *is a basis for a free subgroup of* H.

Anshel and Bogley's results are stated in the following theorem.

Theorem 3.1 (Anshel [1]; Bogley [2]). *If* $\mathscr{P}$ *is semi-staggered in A as defined above and A is a free group, then*
(1) *the Freiheitssatz holds for* $\mathscr{P}$, *and*
(2) *the presentation* $\mathscr{P}$ *has the Identity Property.*

3.2. New results

In this paper, we will adapt the arguments of Anshel and Bogley to prove the following theorems.

Theorem 3.2. *If the relative presentation* $\mathscr{P} = (A, X : R)$ *is semi-staggered and A is a locally indicable group, then the pre-aspherical model of* $\mathscr{P}$ *is aspherical, i.e.* $\mathscr{P}$ *has the Identity Property.*

Theorem 3.3 (Freiheitssatz). *If the relative presentation* $\mathscr{P} = (A, X : R)$ *is semi-staggered and A is a locally indicable group, then the inclusion of A into* $G(\mathscr{P})$ *is an injection.*

Theorem 3.4. *If the relative presentation* $\mathscr{P} = (A, X : R)$ *is semi-staggered, A is a locally indicable group, and no relator is a proper power, then the subgroup* N/U *of the group* $G(\mathscr{P})$ *is locally indicable.*

Corollary 3.5. *If in addition to the assumptions made in Theorem* 3.4 *the group* $H = (A * F(X))/\langle\langle A \cup R\rangle\rangle$ *is locally indicable, then the group* $G(\mathscr{P})$ *is locally indicable.*

3.3. Constructing examples

To construct examples of semi-staggered presentations $\mathscr{P} = (A, X : R)$, we will start with the group $H \cong G(\mathscr{P})/U$ where U is the normal closure of the set $A \cup R$. For H, one must choose a group which has a free subgroup with a basis Π. The first step is to define a linear ordering on the basis Π. Now let A be any locally indicable group and your basis for the free subgroup of H be the set $\Pi = \{b_1, b_2, b_3, b_4, \ldots\}$ with indicated linear ordering. Construct your relators in the following manner:

$$r_1 = b_1 a_{1,1} b_2 a_{1,2} b_3 a_{1,3} \ldots b_k a_{1,k},$$
$$r_2 = b_l a_{2,1} b_{l+1} a_{2,2} \ldots b_m a_{2,m},$$
$$\vdots$$

where each relator has free product length at least $2, 2 \leqslant l \leqslant k < m$, and each $a_{i,j}$ is an element of the group A. One can continue this process of "staggering" the basis elements to build a set of relators. Note, if you start with an infinite basis for the free subgroup of H, you can build an infinite number of relators. The presentation $\mathscr{P} = (A, \Pi : r_i)$ is semi-staggered. At this point, none of the relators are proper powers. One can construct a new semi-staggered presentation by replacing any non-empty subset of the relators $S \leqslant \{r_i\}$ by the set

$$\{s^{e_s}\colon\ e_s \geqslant 2,\ s \in S\}$$

which adds proper powers to the presentation. In the case where none of the relators are proper powers, if you choose H to be a locally indicable group, then by Corollary 3.5, the group $G(\mathscr{P})$ is locally indicable.

4. Proofs of theorems

4.1. A preliminary lemma

To prove Theorem 3.4, we will need the following lemma that shows that a direct limit of locally indicable groups is locally indicable.

Lemma 4.1. *Let $\{K_\alpha\colon \alpha \in A\}$ be set CW-complexes and let K be the CW-complex such that the complex $K = \bigcup_{\alpha \in A} K_\alpha$, every compact subcomplex of K is contained in K_α for some $\alpha \in \mathscr{A}$, and for every $\alpha, \beta \in \mathscr{A}$ there exists a γ such that $K_\alpha \cup K_\beta \subseteq K_\gamma$. If $\pi_1 K_\alpha$ is locally indicable for each K_α, then $\pi_1 K$ is locally indicable.*

Proof. Let K and $\{K_\alpha\colon \alpha \in \mathscr{A}\}$ be as above and let H be a finitely generated subgroup of $\pi_1(K)$. Assume that H does not admit a surjection onto the integers. We will show that H is trivial in $\pi_1(K)$. Since H is finitely generated, there exist $x_1, \ldots, x_n$ in $\pi_1(K)$ such that $H = \langle x_1, \ldots, x_n \rangle$. The complex K is a union of subcomplexes K_α and each compact subcomplex of K is contained in some K_α. Therefore, we can find an element $N \in \mathscr{A}$ and elements x_i' in $\pi_1(K_N)$ such that the homomorphism induced by the inclusion of K_N into K sends each x_i' to the element x_i in $\pi_1(K)$.

Since H does not admit a surjection onto the integers, the abelianization of H, denoted H^{ab}, is finite. Then, for every i, there exists an integer e_i such that

$$x_i^{e_i} \in [H, H] \leqslant \pi_1(K).$$

Say that $x_i^{e_i} = w_i$ where w_i is a product of commutators of H. Then $x_i^{-e_i} w_i = 1$ in $\pi_1 K$, i.e. $x_i^{-e_i} w_i$ is a trivial loop in K so, without loss of generality, we can assume that there exists a disk map $d_i : B^2 \to K$ such that the boundary of d_i is equal to $x_i^{-e_i} w_i$.

Let $T \in \mathscr{A}$ so that $K_N \subseteq K_T$ and K_T supports each disk map d_i. Since $K_N \subseteq K_T$, the image of x_i' under the inclusion induced homomorphism $i_\# : \pi_1(K_N) \to \pi_1(K_T)$ is an element x_i'' of $\pi_1(K_T)$. Let H_T be the subgroup of $\pi_1(K_T)$ that is generated by the x_i''. Note that this is a finitely generated subgroup of the locally indicable group $\pi_1(K_T)$. Since K_T supports each disk map d_i, each element $(x_1'')^{e_i} \in [H_T, H_T]$. It follows that the abelianization of H_T is finite, therefore, there exists no surjective homomorphism from H_T onto the integers. Since $\pi_1(K_T)$ is locally indicable, we conclude that H_T is the trivial group. The inclusion of K_T into K induces a surjective homomorphism from H_T onto the subgroup H of $\pi_1(K)$, therefore, H must be trivial. □

4.2. Topological models for the proofs of the theorems

The proofs involve examining the structure of a particular covering space of the pre-aspherical model K. Let $p : \bar{K} \to K$ be the regular covering of K such that $p_\sharp(\pi_1(\bar{K})) = N/U \trianglelefteq \pi_1 K = G(\mathscr{P})$.

The 0-cells of $\bar{K}$ are in one-to-one correspondence with $H = (A * F)/N$, so we can choose a labeling of the 0-cells by elements of the group H. At each vertex of $\bar{K}$, there will be a lift of each 1-cell of K. For every $x \in X$, the lift of S_x^1 at the vertex $h = wN$ will be a 1-cell of $\bar{K}$ with initial vertex wN and terminal vertex wxN. Let $T = p^{-1}(\bigvee_{x \in X} S_x^1)$.

At each vertex there will also be a lift of $K(A,1) \subseteq K$. We will call the lift of the subcomplex $K(A,1)$ of K at the vertex $h \in H$ the "rose" at vertex h, denoted $V(h)$. Then $p^{-1}(K(A,1))$ is the disjoint union of the set $\{V(h) : h \in H\}$. In fact $p^{-1}(K(A,1)) = K(A,1) \times H$ where H represents the discrete set of 0-cells of $\bar{K}$. The following lemma examines the lift, for each $r \in R$, of the subcomplex $D_{e(r)}$ of K.

Note, for the remainder of the paper, we assume that the relative presentation $\mathscr{P} = (A, X : R)$ is semi-staggered.

Lemma 4.2. Let $\mathscr{P} = (A, X : R)$ *be semi-staggered. For every* $r \in R$ *and for every* $h \in H$, *the loop* $\alpha(r)$ *lifts at* 0-*cell* h in $\bar{K}$ *to a loop* $\alpha(r,h)$ *in* $\bar{K}^{(1)}$. *Moreover, the image of* $\alpha(r,h)$ *is contained in* $T \cup (\bigcup_{k \in hP_r} V(K))$ *and* $\alpha(r,h)$ *strictly involves at least one* 1-*cell from each rose* $V(K)$ *for every* $k \in hP_r$. *The loop* $\alpha(r,h)$ *does not represent a proper power in* $\pi_1 \bar{K}^{(1)}$.

Proof. The path $\beta(r)$ lifts to a path $\beta(r,h)$ in $\bar{K}$ which begins at $h = wN$ and traverses a path in T that covers the non-empty path x_1 and ends at vertex wx_1N. Then it travels a lift of a_1 in the rose $V(wx_1N)$. For $m = 1, \ldots, n-1$, the path $\beta(r,h)$ travels from $wx_1 \ldots x_m N$ to $wx_1 \ldots x_m x_{m+1} N$ and lifts to an essential loop in the rose $V(wx_1 \ldots x_{m+1} N)$ covering a_{m+1}.

This implies that $im(\beta(r,h)) \subseteq T \cup (\bigcup_{k\in hP_r} V(K))$ and strictly involves at least one 1-cell from each rose $V(k)$ for each $k \in hP_r$. Since $im(\alpha(r,h)) = im(\beta(r,h))$, we also know that $im(\alpha(r,h))$, involves at least one 1-cell from each rose $V(K)$ for each $k \in hP_r$. Now we will show that $\alpha(r,h)$ lifts to a loop at $h = wN \in H$.

Since $r = q_r^{e(r)}$, it suffices to show that $q_r \in N$. Note that $q_rN \in P_r$ and $(q_rN)^{e(r)} = q_r^{e(r)}N = rN = 1N$. However, by assumption, the subgroup generated by Π is free, hence torsion free so $q_rN = 1N$. Therefore, $q_r \in N$ and $\alpha(r)$ lifts to a loop at h. We will show that $\alpha(r,h)$ is not a proper power in $\pi_1\bar{K}^{(1)}$ by way of contradiction. If $\alpha(r,h)$ was a proper power it would transverse a loop $\gamma(r,h): S^1 \to \bar{K}$ at least 2 times. Since the covering map p is continuous, the image of $\gamma(r,h)$ under p would be a loop γ in K. Then the image of $\alpha(r,h)$ under p would transverse the loop γ at least 2 times. But the image of $\alpha(r,h)$ is α which is not a proper power, therefore, we have a contradiction and conclude that $\alpha(r,h)$ is not a proper power. □

Recall that for any subset Ω of H, we defined $\Omega^\bullet = \Omega - \{1N\}$. Also we defined $\Pi = \bigcup_{r\in R} P_r \subseteq H$. Now for $r \in R$, let

$$\bar{P}_r = \{p \in \Pi^\bullet : \min P_r^\bullet \leqslant p \leqslant \max P_r^\bullet\} \cup \{1N\} \subseteq \Pi.$$

For $h \in H$, let

$$\bar{K}(r,h) = D_{e(r)} \cup T \cup \left(\bigcup_{k\in h\bar{P}r} V(k)\right),$$

where $S^1_{e(r)} \subseteq D_{e(r)}$ is identified with its image in

$$T \cup \left(\bigcup_{k\in hP_r} V(k)\right) \subseteq T \cup \left(\bigcup_{k\in h\bar{P}_r} V(k)\right)$$

by $\alpha(r,h)$. This identification is well defined by Lemma 4.2. If $e(r) = 1$, then D_e is a single 2-cell, denoted $c^2(r,h)$ attached by $\alpha(r,h)$.

In particular, $\bar{K}(r,h)$ has a single 2-cell outside $T \cup (\bigcup_{k\in h\bar{P}_r} V(K))$ with attaching map $\beta(r,h)$. Moreover, the subcomplex $\bar{K}(r,h)$ of $\bar{K}$ contains the lifts at h of all k-cells in $D_{e(r)} \subseteq K$ for $k \geqslant 2$.

Lemma 4.3. *For every $k \in hP_r^\bullet$, the inclusion induced homomorphisms*

$$\pi_1\left(T \cup \left(\bigcup_{k\neq l\in hP_r^\bullet} V(l)\right)\right) \to \pi_1(\bar{K}(r,h))$$

and

$$\pi_1(V(k)) \to \pi_1(\bar{K}(r,h))$$

are injective.

Proof. By Lemma 4.2, the attaching map $\beta(r,h)$ for the 2-cell $c^2(r,h)$ strictly involves the rose $V(K)$. By condition (S1) of the definition of a semi-staggered presentation, the attaching map $\beta(r,h)$ also strictly involves the rose $V(l)$ for some $k \neq l \in hP_r^{\bullet}$. The Seifert–Van Kampen Theorem [10] implies that

$$\pi_1(\bar{K}(r,h)) \cong \left[\pi_1(V(k)) * \pi_1\left(T \cup \left(\bigcup_{k\neq j\in hP_r^*} V(j)\right)\right)\right] \Big/ \langle\langle r\rangle\rangle.$$

Theorem 2.1. implies that the inclusion of each factor into $\pi_1(\bar{K}(r,h))$ is injective. □

For $h \in H$, let $\bar{K}(h) = \bigcup_{r\in R} \bar{K}(r,h)$. Note that if $g \in H = Aut(p)$, then $g\bar{K}(r,h) = \bar{K}(r,gh)$ and so $g\bar{K}(h) = \bar{K}(gh)$ and $\bar{K} = \bigcup_{h\in H} \bar{K}(h)$.

4.3. The proofs of Theorems 3.2–3.4

The method of proof used is to construct the covering space $\bar{K}$ as a union of smaller pieces. The following lemmas will show that the conclusions hold for each of these pieces. Then compact supports, covering space properties, and Lemma 4.1 will provide the final step to prove Theorems 3.2–3.4.

Lemma 4.4. *Let $h \in H$. Then*
(1) *$\bar{K}(h)$ is aspherical*;
(2) *if $r \in R$, then the inclusion of $\bar{K}(r,h)$ into $\bar{K}(h)$ induces a monomorphism of fundamental groups*; *and*
(3) *if $e(r) = 1$ for every $r \in R$, then $\pi_1(\bar{K}(h))$ is locally indicable.*

Proof. To show that $\bar{K}(h)$ is aspherical, we consider maps of the n-sphere S^n into $\bar{K}(h)$. The image of each of these maps is a compact set in $\bar{K}(h)$. Compact supports says that each compact set in $\bar{K}(h)$ is contained in a finite subcomplex of $\bar{K}(h)$. Moreover, every finite subcomplex is contained in $X = T \cup (\bigcup_{k\in h\Pi} V(K)) \cup (\bigcup_{i=1}^{n} c^2(r_i,h))$ for some subset $\{r_1,\dots,r_n\}$ where each $r_i \in R$. If the topological space X is aspherical for every finite subset $\{r_1,\dots,r_n\}$ then we can conclude that $\bar{K}(h)$ is aspherical. Therefore, it suffices to show $(1)'$ X is aspherical. Similarly, it suffices to show that $(2)'$ the inclusion induced homomorphism from $\pi_1(\bar{K}(r,h))$ into $\pi_1(X)$ is injective. For part (3), since the collection of complexes X for each finite subset of relators satisfies the conditions of Lemma 4.1, it suffices to show that $(3)'$ if $e(r) = 1$ for every $r \in R$, then $\pi_1(X)$ is locally indicable. We will now prove $(1)'$, $(2)'$ and $(3)'$, by induction on the number n of relators.

For $n = 1$, the Lemma 4.2 provides that $\alpha(r_1,h)$ does not represent a proper power in $\pi_1(\bar{K}(r_1,h)^{(1)})$. It follows that $X = \bar{K}(r_1,h)$ is aspherical by Theorem 2.2. The result $(2)'$ is trivial in the case $n = 1$.

If $e(r)=1$, then $\bar{K}(r,h) = T\cup(\bigcup_{k\in hP_r} V(k))\cup c^2(r,h)$. By (S1) and Lemma 4.2, there exists $k_0 \in hP_r$ such that $\alpha(r,h)$ strictly involves at least one 1-cell of $V(k_0)$. Consider the following decomposition of $\bar{K}(r,h)$:

$$\bar{K}(r,h) = \left(T \cup \left(\bigcup_{k_0\neq k\in hP_r} V(K)\right)\right) \cup V(k_0) \cup c^2(r,h).$$

By the Seifert–Van Kampen Theorem [10],

$$\pi_1(\bar{K}(r,h)) = \pi_1\left(T \cup \left(\bigcup_{k_0\neq k\in hP_r} V(k)\right)\right) * \pi_1 V(k_0)/\langle\langle r\rangle\rangle.$$

Therefore, by Theorem 2.4, $\pi_1\bar{K}(r,h)$ is locally indicable. This completes the $n=1$ case.

Now suppose $n>1$. Without loss of generality, we may assume that

$$r_1 < r_2 < \cdots < r_n$$

in the ordering on R. Set $Y=\bigcup_{m=1}^{n-1}\bar{K}(r_m,h)$ so that $X = Y\cup\bar{K}(r_n,h)$. The complexes Y and $\bar{K}(r_n,h)$ are aspherical by our inductive hypothesis. Also, if $e(r_i)=1$ for every $i\in\{1,\ldots,n\}, \pi_1 Y$ and $\pi_1\bar{K}(r,h)$ are locally indicable by our inductive hypothesis.

Claim. *Let $W = h\bar{P}_{r_{n-1}}\cap h\bar{P}_{r_n}$. Then*

$$Y\cap\bar{K}(r_n,h) = T\cup\left(\bigcup_{k\in W} V(k)\right).$$

Reason. From the definitions, it is clear that

$$Y\cap\bar{K}(r_n,h) = T\cup\left(\bigcup_{k\in S} V(k)\right),$$

where $S=(\bigcup_{m=1}^{n-1} h\bar{P}_{r_m})\cap h\bar{P}_{r_n}$. It would suffice to show for each $1\leqslant m\leqslant n-1$, that $h\bar{P}_{r_m}\cap h\bar{P}_{r_n}\subseteq h\bar{P}_{r_{n-1}}$. If $1N\neq p\in h\bar{P}_{r_m}\cap h\bar{P}_{r_n}$ then by (S2)

$$h\min P^\bullet_{r_{n-1}} < h\min P^\bullet_{r_n} \leqslant p \leqslant h\max P^\bullet_{r_m} < h\max P^\bullet_{r_{n-1}}$$

and so $p\in h\bar{P}_{r_{n-1}}$ and the claim follows.

By the claim, the intersection $Y\cap\bar{K}(r_n),h)$ is contained in $\bar{K}(r_{n-1},h)\subseteq Y$. Also, recall that

$$Y\cap\bar{K}(r_n,h) = T\cup\left(\bigcup_{k\in W} V(K)\right),$$

where $W = h\bar{P}_{r_{n-1}} \cap h\bar{P}_{r_n}$. Therefore, the inclusion of the intersection $Y \cap \bar{K}(r_n, h)$ into Y is the composition

$$Y \cap \bar{K}(r_n,h) \xrightarrow{i} T \cup \left(\bigcup_{k \in h\bar{P}_{r_{n-1}}} V(k) \right) \xrightarrow{j} \bar{K}(r_{n-1,h}) \xrightarrow{k} Y$$

which gives the induced composition on fundamental groups

$$\pi_1(Y \cap \bar{K}(r_n,h)) \xrightarrow{i_\#} \pi_1 \left(T \cup \left(\bigcup_{k \in h\bar{P}_{r_{n-1}}} V(K) \right) \right) \xrightarrow{j_\#} \pi_1(\bar{K}(r_{n-1},h)) \xrightarrow{k_\#} \pi_1(Y).$$

By applying the theory of free products, Theorem 2.1 and our inductive hypothesis, the inclusion of the intersection $Y \cap \bar{K}(r_n,h)$ into Y induces a monomorphism on fundamental groups. By a similar argument, one can show that the inclusion of the intersection $Y \cap \bar{K}(r_n,h)$ into $\bar{K}(r_n,h)$ also induces a monomorphism on fundamental groups. Since

$$Y \cap \bar{K}(r_n,h) = T \cup \left(\bigcup_{k \in W} V(K) \right)$$

is aspherical we see that X is aspherical by Whitehead Amalgamation [11]. The Seifert–Van Kampen Theorem [10] tells us that $\pi_1(X)$ is the free product with amalgamation

$$\pi_1(Y) *_{\pi_1(Y \cup \bar{K}(r_n,h))} \pi_1(\bar{K}(r_n,h)).$$

Therefore, the induced homomorphism from $\pi_1(\bar{K}(r_n,h))$ into $\pi_1(X)$ is injective by the theory of free products with amalgamation which proves $(2)'$.

By conditions (S2) of a semi-staggered presentation, the map $\beta(r,h)$ associated to the 2-cell corresponding to the relator r_n that is lifted at the 0-cell h to $\bar{K}$ strictly involves a 1-cell of the rose $V(h \max P^{\bullet}_{r_n})$. Moreover, by the previous claim, the rose $V(h \max P^{\bullet}_{r_n})$ is not contained in the complex Y. Let the set $M = h\bar{P}_{r_n} - h\bar{P}_{r_{n-1}}$ and the set $M' = M - h \max P^{\bullet}_{r_n}$. For $(3)'$, consider the following decomposition of X:

$$X = \left(Y \cup \left(\bigcup_{k \in M'} V(k) \right) \right) \cup V(h \max P^{\bullet}_{r_n}) \cup hc^2_{r_n}.$$

Then, by the Seifert–Van Kampen Theorem [10],

$$\pi_1 X = \pi_1 \left(Y \cup \left(\bigcup_{k \in M'} V(k) \right) \right) * (\pi_1(V(h \max P^{\bullet}_{r_n})))/\langle\langle r_n \rangle\rangle.$$

By Theorem 2.4, $\pi_1 X$ is locally indicable. □.

Now let Φ denote the subgroup of H that is generated by Π. By (S3), Φ is a free group with basis $\Pi^{\bullet}$.

Lemma 4.5. *Let $h_0, h_1, \ldots, h_n$ be distinct elements of H Where n is a positive integer. Then*
(1) $\bigcup_{m=0}^{n} \bar{K}(h_m)$ *is aspherical*;
(2) *for $i=0,\ldots,n$, the inclusion of $\bar{K}(h_i)$ into $\bigcup_{m=0}^{n} \bar{K}(h_m)$ induces a monomorphism of fundamental groups; and*
(3) *if $e(r)=1$ for every $r \in R$, then $\pi_1(\bigcup_{m=0}^{n} \bar{K}(h_m))$ is locally indicable.*

Proof. First, assume that $n=1$. Partition H into the cosets of Φ. Note that if the cosets

$$h_0\Phi \neq h_1\Phi \quad \text{for } h_0, h_1 \in H,$$

then $\bar{K}(h_0) \cap \bar{K}(h_1) = T$. The inclusion of T into $T \cup V(k)$ induces a monomorphism on fundamental groups by the theory of free products since

$$\pi_1(T \cup V(k)) \cong \pi_1(T) * \pi_1(V(K)).$$

Then by Lemma 4.3, the inclusion of $T \cup V(k)$ for some $k \in h_0 P_r^{\bullet}$ into $K(r, h_0)$ for any relator r and any 0-cell h_0 of $\bar{K}$ induces a monomorphism on fundamental groups. By Lemma 4.4, the inclusion of $K(r, h_0)$ into $K(h_0)$ induces a monomorphism on fundamental groups. Therefore, the inclusion of T into $K(h_0)$ induces a monomorphism on fundamental groups. Once we know that this induced homomorphism is injective, we can show that $\bar{K}(h_0) \cup \bar{K}(h_1)$ is aspherical by Whitehead Amalgamation [11] which proves (1) for this case. Furthermore, we know that the inclusions of $\pi_1(\bar{K}(h_0))$ and $\pi_1(\bar{K}(h_1))$ into $\pi_1(\bar{K}(h_0) \cup \bar{K}(h_1))$ are injective by the theory of free products with amalgamation since $\pi_1(\bar{K}(h_0) \cup \bar{K}(h_1)) \cong \pi_1(\bar{K}(h_0)) *_{\pi_1(T)} (\bar{K}(h_1))$ by the Seifert–Van Kampen Theorem [10], therefore (2) is satisfied for this case.

The free product with amalgamation structure of $\pi_1(\bar{K}(h_0) \cup \bar{K}(h_1))$ is unfortunately not enough to show that this group is locally indicable. To see this, we must consider the following collection of subcomplexes of $\bar{K}(h_0) \cup \bar{K}(h_1)$

$$\Sigma = T \cup \left(\bigcup_{k \in h_0 \Pi} V(K) \right) \cup \left(\bigcup_{k \in h_1 \Pi} v(K) \right) \cup \left(\bigcup_{i=1}^{n} c^2(r_i, h_0) \right) \cup \left(\bigcup_{j=1}^{m} c^2(s_j, h_1) \right)$$

such that $r_1, \ldots, r_n$ and $s_1, \ldots, s_m$ are finite subsets of R and then apply Lemma 4.1.

The lemma is now proved for the case where $n=1$ and the intersection $\bar{K}(h_0) \cap \bar{K}(h_1) = T$. If $n > 1$, assume $h_0, h_1, \ldots, h_n$ splits into two sublists $h_0, \ldots, h_j$ and $h_{j+1}, \ldots, h_n$ such that $\bigcup_{i \leqslant j} h_i\Phi$ and $\bigcup_{i>j} h_i\Phi$ are disjoint. Now we can follow the previous argument replacing $\bar{K}(h_0)$ and $\bar{K}(h_1)$ with $\bigcup_{i \leqslant j} \bar{K}(h_j)$ and $\bigcup_{i>j} \bar{K}(h_i)$, respectively.

Now recall that $\bar{K}(h)$ is homeomorphic to its translate $g\bar{K}(h) = \bar{K}(gh)$ so it suffices to prove the lemma in the case where each $h_0, \ldots, h_n$ are distinct elements of the trivial coset $1\Phi = \Phi$. Under this assumption, the result is proven by induction on n.

For the case $n = 0$, all three results are consequences of Lemma 4.4. Now assume that $n > 0$. Without loss of generality, we may assume that $|h_0| \geqslant |h_i|$ *for* $i = 1, \ldots, n$ where $|h|$ indicates the length of the element h in the free group Φ. Set $X = \bigcup_{i=0}^{n} \bar{K}(h_i)$ and $Y = \bigcup_{i=1}^{n} \bar{K}(h_i)$ so that $X = Y \cup \bar{K}(h_0)$.

Let $U = (\bigcup_{i=1}^{n} h_i\Pi) \cap h_0\Pi$. Lemma 1 in [2] implies that this intersection U is contained in a singleton. This implies that there exists an element $k_0 \in h_0\Pi$ such that

$$T \subseteq Y \cap \bar{K}(h_0) \subseteq T \cup V(k_0).$$

If $Y \cap \bar{K}(h_0) = T$, then the result follows by the same arguments given above. Assume $Y \cap \bar{K}(h_0) \neq T$ and let $r \in R$ such that $k_0 \in h_0 P_r$. The attaching map $\beta(r, h_0)$ for the 2-cell $c^2(r, h_0)$ of $\bar{K}(r, h_0)$ strictly involves some 1-cell of a rose other than $V(k_0)$ by (S1) and Lemma 4.2. Then Lemma 4.3 implies that the inclusion of $Y \cap \bar{K}(h_0) = T \cup V(k_0)$ into $\bar{K}(r, h_0)$ induces a monomorphism of fundamental groups. Lemma 4.4 implies that the inclusion of $\bar{K}(r, h_0)$ into $\bar{K}(h_0)$ induces a monomorphism on fundamental groups, therefore, we conclude that the inclusion of $Y \cap \bar{K}(h_0)$ into $\bar{K}(h_0)$ induces a monomorphism of fundamental groups. Also, there exists an $m \in \{1, \ldots, n\}$ such that $k_0 \in h_m\Pi$ and a similar argument shows that the inclusion of $Y \cap \bar{K}(h_0)$ into $\bar{K}(h_m)$ induces a monomorphism of fundamental groups. By part 2 of the inductive hypothesis, the inclusion of $\bar{K}(h_m)$ into Y induces a monomorphism of fundamental groups, therefore, the inclusion-induced homomorphism from $\pi_1(Y \cap \bar{K}(h_0))$ into $\pi_1(Y)$ is injective.

The complexes Y and $\bar{K}(h_0)$ are aspherical by part 1 of the inductive hypothesis. By applying Whitehead Amalgamation [11], $X = Y \cup \bar{K}(h_0)$ is aspherical, therefore proving part 1 of the lemma. To show part 2, note that the Seifer–Van Kampen Theorem [10] implies that the group $\pi_1 X$ is a free product of $\pi_1 Y$ and $\pi_1 \bar{K}(h_0)$ with free subgroup amalgamated which implies that $\pi_1 \bar{K}(h_0)$ embeds in $\pi_1 X$. By the inductive hypothesis and the theory of free products, if $m \in \{1, \ldots, n\}$ the inclusion of $\bar{K}(h_m)$ into Y and then into X induces a monomorphism of fundamental groups.

To show part (3), by Lemma 4.1, it suffices to show that $\pi_1 X$ is locally indicable where $X = Y \cup (\bigcup_{k \in h_0\Pi} V(k)) \cup c^2(r_1, h_0) \cup \cdots \cup c^2(r_m, h_0)$ with $\{r_1, \ldots, r_m\}$ being any finite subset of the set R of relators. We will show this by induction on m. For $m = 1$, let k_* be a vertex of $\bar{K}$ such that the attaching map $\beta(r, h_0)$ strictly involves the rose $V(k_*)$. Consider the decomposition $X = (Y \cup (\bigcup_{k_* \neq k \in h_0\Pi} V(k))) \cup V(k_*) \cup h_0 c_r^2$. By the Seifert–Van Kampen Theorem [10],

$$\pi_1 X = \pi_1 \left(Y \cup \left(\bigcup_{k_* \neq k \in h_0\Pi} V(k) \right) * \pi_1(V(k_*)) \right) \bigg/ \langle\langle r \rangle\rangle.$$

By Theorem 2.4, $\pi_1 X$ is locally indicable.

Now consider the general case. Let $m > 1$. Without loss of generality, we can assume that

$$r_1 < r_2 < \cdots < r_m$$

under the liner ordering given by (S2). By (S1) and (S2) there exists a $k_0 \in h_0\Pi$ such that the attaching map for $c^2(r_m, h_0)$ strictly involves a 1-cell of $V(k_0)$ and no other 2-cell outside the rose involves it. Then consider the following decomposition of X:

$$X = \left(Y \cup \left(\bigcup_{k \neq k_0 \in h_0 \Pi} V(k) \right) \cup \left(\bigcup_{i=1}^{m-1} h_0 c_{r_i}^2 \right) \right) \cup V(k_0) \cup h_0 c_{r_m}^2.$$

Then by the Seifert–Van Kampen Theorem [10],

$$\pi_1 X = \left(\pi_1 \left(Y \cup s \left(\bigcup_{k \neq k_0 \in h_0 \Pi} V(k) \right) \cup \left(\bigcup_{i=1}^{m-1} h_0 c_{r_i}^2 \right) \right) * \pi_1 V(k_0) \right) \Big/ \langle\langle r \rangle\rangle.$$

Applying Theorem 2.4, we conclude that $\pi_1 X$ is locally indicable. □

Theorem 3.2. *If the relative presentation $\mathscr{P} = (A, X : R)$ is semi-staggered and A is a locally indicable group, then the pre-aspherical model of $\mathscr{P}$ is aspherical, i.e. has the Identity Property.*

Proof. By compact supports and Lemma 4.5, $\bar{K}$ is aspherical. It then follows that K is aspherical. □

Theorem 3.3. *If the relative presentation $\mathscr{P} = (A, X : R)$ is semi-staggered and A is a locally indicable group, then the inclusion of A into $G(\mathscr{P})$ is an injection.*

Proof. The result follows once it is shown that the inclusion of $K(A,1)$ into K induces a monomorphism of fundamental groups. We can translate the problem to $\bar{K}$ by lifting through the covering p at the 0-cell $1N$. We must now show that the inclusion of $V(1N)$ into $\bar{K}$ induces a monomorphism of fundamental groups. Let r be any element of R. We know from (S1) and Lemma 4.2 that the attaching map for the 2-cell of $\bar{K}(r, 1N)$ strictly involves a 1-cell of a rose other than $V(1N)$. By applying Theorem 4.3 in [5], we find that the inclusion of $V(1N)$ into $\bar{K}(r, 1N)$ induces a monomorphism of fundamental groups. Now apply Lemmas 4.4 and 4.5 to show that the inclusion of $V(1N)$ into $\bar{K}(h)$ induces a monomorphism of fundamental groups. The result now follows from compact supports. □

Theorem 3.4. *If the relative presentation $\mathscr{P} = (A, X : R)$ is semi-staggered, A is a locally indicable group, and no relator is a proper power, then the subgroup $N/U \leqslant G(\mathscr{P})$ is locally indicable.*

Proof. Let the collection Ω of subcomplexes of $\bar{K}$ be defined to be

$$\Omega = \left\{ \bigcup_{h \in M} \bar{K}(h) : M \text{ is a finite subset of} H \right\}.$$

Every finite subcomplex of $\bar{K}$ is contained in an element of Ω for some finite subset $\{h_1, \ldots, h_n\} \subseteq H$. Also, the union of any two elements of Ω is also a union of complexes $\bar{K}(h_i)$ for a finite number of elements h_i, therefore, an element of the collection Ω. The result follows by applying Lemma 4.1 to the collection Ω. □

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