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Classification of differential symmetry breaking operators for differential forms ☆

Classification des opérateurs de brisure de symétrie pour les formes différentielles

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A R T I C L E I N F O

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A B S T R A C T

We give a complete classification of conformally covariant differential operators between the spaces of differential i-forms on the sphere S^n and j-forms on the totally geodesic hypersphere S^{n-1} by analyzing the restriction of principal series representations of the Lie group $O(n+1,1)$. Further, we provide explicit formulæ for these matrix-valued operators in the flat coordinates and find factorization identities for them.

R É S U M É

Nous présentons une classification complète des opérateurs différentiels conformément covariants agissant entre les espaces des i-formes différentielles sur la sphère S^n et ceux des j-formes sur la hypershère totalement géodésique S^{n-1} en analysant les restrictions des représentations des séries principales du groupe de Lie $O(n+1,1)$. Pour de tels opórateurs à valeurs matricielles, nous donnons des formules explicites dans les coordonnées plates et trouvons des identités de factorisation.

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1. Introduction

Suppose a Lie group G acts conformally on a Riemannian manifold (X, g). This means that there exists a positive-valued function $\Omega \in C^\infty(G \times X)$ (*conformal factor*) such that

$$L_h^* g_{h\cdot x} = \Omega(h,x)^2 g_x \quad \text{for all } h \in G \text{ and } x \in X,$$

where $L_h : X \to X$, $x \mapsto h \cdot x$ denotes the action of G on X. Since Ω satisfies a cocycle condition, we can form a family of representations $\varpi_u^{(i)}$ for $u \in \mathbb{C}$ and $0 \le i \le \dim X$ on the space $\mathcal{E}^i(X)$ of differential i-forms on X by

$$\varpi_u^{(i)}(h)\alpha := \Omega(h^{-1}, \cdot)^u L_{h^{-1}}^* \alpha \quad (h \in G). \tag{1}$$

The representation $\varpi_u^{(i)}$ of the conformal group G on $\mathcal{E}^i(X)$ will be simply denoted by $\mathcal{E}^i(X)_u$.

If Y is a submanifold of X, then we can also define a family of representations $\varpi_v^{(j)}$ on $\mathcal{E}^j(Y)$ ($v \in \mathbb{C}, 0 \le j \le \dim Y$) of the subgroup

$$G' := \{h \in G : h \cdot Y = Y\},$$

which acts conformally on the Riemannian submanifold $(Y, g|_Y)$.

We study differential operators $\mathcal{D} : \mathcal{E}^i(X) \longrightarrow \mathcal{E}^j(Y)$ that intertwine the two representations $\varpi_u^{(i)}|_{G'}$ and $\varpi_v^{(j)}$ of G'. Here $\varpi_u^{(i)}|_{G'}$ stands for the restriction of the G-representation $\varpi_u^{(i)}$ to the subgroup G'. We say that such $\mathcal{D}$ is a *differential symmetry breaking operator*, and denote by $\mathrm{Diff}_{G'}(\mathcal{E}^i(X)_u, \mathcal{E}^j(Y)_v)$ the space of all differential symmetry breaking operators. We address the following problems:

Problem 1. Determine the dimension of the space $\mathrm{Diff}_{G'}\left(\mathcal{E}^i(X)_u, \mathcal{E}^j(Y)_v\right)$. In particular, find a necessary and sufficient condition on a quadruple (i, j, u, v) such that there exist nontrivial differential symmetry breaking operators.

Problem 2. Construct explicitly a basis of $\mathrm{Diff}_{G'}\left(\mathcal{E}^i(X)_u, \mathcal{E}^j(Y)_v\right)$.

In the case where $X = Y$, $G = G'$, and $i = j = 0$, a classical prototype of such operators is a second-order differential operator called the Yamabe operator

$$\Delta + \frac{n-2}{4(n-1)}\kappa \in \mathrm{Diff}_G(\mathcal{E}^0(X)_{\frac{n}{2}-1}, \mathcal{E}^0(X)_{\frac{n}{2}+1}),$$

where Δ is the Laplace–Beltrami operator, n is the dimension of X, and κ is the scalar curvature of X. Conformally covariant differential operators of higher order are also known: the Paneitz operator (fourth order) [11], which appears in four dimensional supergravity [2], or more generally, the so-called GJMS operators [3] are such examples. Analogous conformally covariant operators on forms ($i = j$ case) were studied by Branson [1]. On the other hand, the insight of representation theory of conformal groups is useful in studying Maxwell's equations, see [10], for instance.

Let us consider the more general case where $Y \ne X$ and $G' \ne G$. An obvious example of symmetry breaking operators is the restriction operator Rest_Y which belongs to $\mathrm{Diff}_{G'}\left(\mathcal{E}^i(X)_u, \mathcal{E}^i(Y)_u\right)$ for all $u \in \mathbb{C}$. Another elementary example is $\mathrm{Rest}_Y \circ \iota_{N_Y(X)} \in \mathrm{Diff}_{G'}\left(\mathcal{E}^i(X)_u, \mathcal{E}^{i-1}(Y)_v\right)$ if $v = u + 1$ where $\iota_{N_Y(X)}$ denotes the interior multiplication by the normal vector field to Y when Y is of codimension one in X.

In the model space where $(X, Y) = (S^n, S^{n-1})$, the pair (G, G') of conformal groups amounts to $(O(n+1,1), O(n,1))$ modulo center, and Problems 1 and 2 have been recently solved for $i = j = 0$ by Juhl [4], see also [5,7] and [9] for different approaches by the F-method and the residue calculus, respectively.

Problems 1 and 2 for general i and j for the model space can be reduced to analogous problems for (nonspherical) principal series representations by the isomorphism (3) below. In this note we shall give complete solutions to Problems 1 and 2 in those terms (see Theorems 3 and 4).

Notation: $\mathbb{N} = \{0, 1, 2, \cdots\}$, $\mathbb{N}_+ = \{1, 2, \cdots\}$.

2. Principal series representations of $G = O(n+1, 1)$

We set up notations. Let $P = MAN$ be a Langlands decomposition of a minimal parabolic subgroup of $G = O(n+1, 1)$. For $0 \le i \le n$, $\delta \in \mathbb{Z}/2\mathbb{Z}$, and $\lambda \in \mathbb{C}$, we extend the outer tensor product representation $\bigwedge^i(\mathbb{C}^n) \otimes (-1)^\delta \otimes \mathbb{C}_\lambda$ of $MA \simeq (O(n) \times O(1)) \times \mathbb{R}$ to P by letting N act trivially, and form a G-equivariant vector bundle $\mathcal{V}^i_{\lambda,\delta} := G \times_P \left(\bigwedge^i(\mathbb{C}^n) \otimes (-1)^\delta \otimes \mathbb{C}_\lambda\right)$ over the real flag variety $X = G/P \simeq S^n$. Then we define an unnormalized principal series representations

$$I(i,\lambda)_\delta := \mathrm{Ind}_P^G\left(\bigwedge^i(\mathbb{C}^n) \otimes (-1)^\delta \otimes \mathbb{C}_\lambda\right) \tag{2}$$

of G on the Fréchet space $C^\infty(X, \mathcal{V}^i_{\lambda,\delta})$ of smooth sections.

In our parameterization, $I(i, n-2i)_\delta$ and $I(i,i)_\delta$ have the same infinitesimal character with the trivial one-dimensional representation of G. Then, for all $u \in \mathbb{C}$, we have a natural G-isomorphism

$$\varpi_u^{(i)} \simeq I(i, u+i)_{i \bmod 2}. \tag{3}$$

Similarly, for $0 \le j \le n-1$, $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$ and $\nu \in \mathbb{C}$, we define an unnormalized principal series representation $J(j,\nu)_\varepsilon := \operatorname{Ind}_{P'}^{G'}\left(\bigwedge^j(\mathbb{C}^{n-1}) \otimes (-1)^\varepsilon \otimes \mathbb{C}_\nu\right)$ of the subgroup $G' = O(n,1)$ on $C^\infty(Y, \mathcal{W}^j_{\nu,\varepsilon})$, where $\mathcal{W}^j_{\nu,\varepsilon} := G' \times_{P'} \left(\bigwedge^j(\mathbb{C}^{n-1}) \otimes (-1)^\varepsilon \otimes \mathbb{C}_\nu\right)$ is a G'-equivariant vector bundle over $Y = G'/P' \simeq S^{n-1}$.

3. Existence condition for differential symmetry breaking operators

A continuous G'-intertwining operator $T : I(i,\lambda)_\delta \longrightarrow J(j,\nu)_\varepsilon$ is said to be a *symmetry breaking operator* (SBO). We say that T is a differential operator if T satisfies $\operatorname{Supp}(Tf) \subset \operatorname{Supp} f$ for all $f \in C^\infty(X, \mathcal{V}^i_{\lambda,\delta})$, and $\operatorname{Diff}_{G'}(I(i,\lambda)_\delta, J(j,\nu)_\varepsilon)$ denotes the space of differential SBOs. We give a complete solution to Problem 1 for $(X,Y) = (S^n, S^{n-1})$ in terms of principal series representations:

Theorem 3. *Let $n \ge 3$. Suppose $0 \le i \le n$, $0 \le j \le n-1$, $\lambda, \nu \in \mathbb{C}$, and $\delta, \varepsilon \in \mathbb{Z}/2\mathbb{Z}$. Then the following three conditions on 6-tuple $(i, j, \lambda, \nu, \delta, \varepsilon)$ are equivalent:*

(i) $\operatorname{Diff}_{O(n,1)}(I(i,\lambda)_\delta, J(j,\nu)_\varepsilon) \ne \{0\}$.
(ii) $\dim \operatorname{Diff}_{O(n,1)}(I(i,\lambda)_\delta, J(j,\nu)_\varepsilon) = 1$.
(iii) *The 6-tuple belongs to one of the following six cases:*

Case 1. $j = i$, $0 \le i \le n-1$, $\nu - \lambda \in \mathbb{N}$, $\varepsilon - \delta \equiv \nu - \lambda \bmod 2$.
Case 2. $j = i-1$, $1 \le i \le n$, $\nu - \lambda \in \mathbb{N}$, $\varepsilon - \delta \equiv \nu - \lambda \bmod 2$.
Case 3. $j = i+1$, $1 \le i \le n-2$, $(\lambda, \nu) = (i, i+1)$, $\varepsilon \equiv \delta + 1 \bmod 2$.
Case 3′. $(i,j) = (0,1)$, $-\lambda \in \mathbb{N}$, $\nu = 1$, $\varepsilon \equiv \delta + \lambda + 1 \bmod 2$.
Case 4. $j = i-2$, $2 \le i \le n-1$, $(\lambda,\nu) = (n-i, n-i+1)$, $\varepsilon \equiv \delta + 1 \bmod 2$.
Case 4′. $(i,j) = (n, n-2)$, $-\lambda \in \mathbb{N}$, $\nu = 1$, $\varepsilon \equiv \delta + \lambda + 1 \bmod 2$.

We set $\Xi := \{(i,j,\lambda,\nu)$: the 6-tuple $(i,j,\lambda,\nu,\delta,\varepsilon)$ satisfies one of the equivalent conditions of Theorem 3 for some $\delta, \varepsilon \in \mathbb{Z}/2\mathbb{Z}\}$.

4. Construction of differential symmetry breaking operators

In this section, we describe an explicit generator of the space of differential SBOs if one of the equivalent conditions in Theorem 3 is satisfied. For this, we use the *flat picture* of the principal series representations $I(i,\lambda)_\delta$ of G, which realizes the representation space $C^\infty(X, \mathcal{V}^i_{\lambda,\delta})$ as a subspace of $C^\infty(\mathbb{R}^n, \bigwedge^i(\mathbb{C}^n))$ by trivializing the bundle $\mathcal{V}^i_{\lambda,\delta} \longrightarrow X$ on the open Bruhat cell

$$\mathbb{R}^n \hookrightarrow X, \quad (x_1, \cdots, x_n) \mapsto \exp\left(\sum_{j=1}^n x_j N_j^-\right) P.$$

Here $\{N_1^-, \cdots, N_n^-\}$ is an orthonormal basis of the nilradical $\mathfrak{n}_-(\mathbb{R})$ of the opposite parabolic subalgebra with respect to an M-invariant inner product. Without loss of generality, we may and do assume that the open Bruhat cell $\mathbb{R}^{n-1} \hookrightarrow Y \simeq G'/P'$ is given by putting $x_n = 0$. Then the flat picture of the principal series representation $J(j,\nu)_\varepsilon$ of G' is defined by realizing $C^\infty(Y, \mathcal{W}^j_{\nu,\varepsilon})$ as a subspace of $C^\infty(\mathbb{R}^{n-1}, \bigwedge^j(\mathbb{C}^{n-1}))$. For the construction of explicit generators of matrix-valued SBOs, we begin with a scalar-valued differential operator. For $\alpha \in \mathbb{C}$ and $\ell \in \mathbb{N}$, we define a polynomial of two variables (s,t) by

$$\left(I_\ell \widetilde{C}^\alpha_\ell\right)(s,t) := s^{\frac{\ell}{2}} \widetilde{C}^\alpha_\ell\left(\frac{t}{\sqrt{s}}\right),$$

where $\widetilde{C}^\alpha_\ell(z)$ is the renormalized Gegenbauer polynomial given by

$$\widetilde{C}^\alpha_\ell(z) := \frac{1}{\Gamma\left(\alpha + \left[\frac{\ell+1}{2}\right]\right)} \sum_{k=0}^{\left[\frac{\ell}{2}\right]} (-1)^k \frac{\Gamma(\ell - k + \alpha)}{k!(\ell - 2k)!} (2z)^{\ell - 2k}.$$

Then $\widetilde{C}_\ell^\alpha(z)$ is a nonzero polynomial for all $\alpha \in \mathbb{C}$ and $\ell \in \mathbb{N}$, and a (normalized) *Juhl's conformally covariant operator* $\widetilde{\mathbb{C}}_{\lambda,\nu} : C^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^{n-1})$ is defined by

$$\widetilde{\mathbb{C}}_{\lambda,\nu} := \mathrm{Rest}_{x_n=0} \circ \left(I_\ell \widetilde{C}_\ell^{\lambda - \frac{n-1}{2}} \right) \left(-\Delta_{\mathbb{R}^{n-1}}, \frac{\partial}{\partial x_n} \right),$$

for $\lambda, \nu \in \mathbb{C}$ with $\ell := \nu - \lambda \in \mathbb{N}$. For instance,

$$\widetilde{\mathbb{C}}_{\lambda,\nu} = \mathrm{Rest}_{x_n=0} \circ \begin{cases} \mathrm{id} & \text{if } \nu = \lambda, \\ 2\dfrac{\partial}{\partial x_n} & \text{if } \nu = \lambda + 1, \\ \Delta_{\mathbb{R}^{n-1}} + (2\lambda - n + 3)\dfrac{\partial^2}{\partial x_n^2} & \text{if } \nu = \lambda + 2. \end{cases}$$

For $(i, j, \lambda, \nu) \in \Xi$, we introduce a new family of matrix-valued differential operators

$$\widetilde{\mathbb{C}}_{\lambda,\nu}^{i,j} : C^\infty(\mathbb{R}^n, \bigwedge^i(\mathbb{C}^n)) \longrightarrow C^\infty(\mathbb{R}^{n-1}, \bigwedge^j(\mathbb{C}^{n-1})),$$

by using the identifications $\mathcal{E}^i(\mathbb{R}^n) \simeq C^\infty(\mathbb{R}^n) \otimes \bigwedge^i(\mathbb{C}^n)$ and $\mathcal{E}^j(\mathbb{R}^{n-1}) \simeq C^\infty(\mathbb{R}^{n-1}) \otimes \bigwedge^j(\mathbb{C}^{n-1})$, as follows. Let $d_{\mathbb{R}^n}^*$ be the codifferential, which is the formal adjoint of the differential $d_{\mathbb{R}^n}$, and $\iota_{\frac{\partial}{\partial x_n}}$ the inner multiplication by the vector field $\frac{\partial}{\partial x_n}$. Both operators map $\mathcal{E}^i(\mathbb{R}^n)$ to $\mathcal{E}^{i-1}(\mathbb{R}^n)$. For $\alpha \in \mathbb{C}$ and $\ell \in \mathbb{N}$, let $\gamma(\alpha, \ell) := 1$ (ℓ is odd); $= \alpha + \frac{\ell}{2}$ (ℓ is even). Then we set

$$\mathbb{C}_{\lambda,\nu}^{i,i} := \widetilde{\mathbb{C}}_{\lambda+1,\nu-1} d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* - \gamma(\lambda - \frac{n}{2}, \nu - \lambda) \widetilde{\mathbb{C}}_{\lambda,\nu-1} d_{\mathbb{R}^n} \iota_{\frac{\partial}{\partial x_n}} + \frac{1}{2}(\nu - i)\widetilde{\mathbb{C}}_{\lambda,\nu} \qquad \text{for } 0 \le i \le n-1.$$

$$\mathbb{C}_{\lambda,\nu}^{i,i-1} := -\widetilde{\mathbb{C}}_{\lambda+1,\nu-1} d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}} - \gamma(\lambda - \frac{n-1}{2}, \nu - \lambda) \widetilde{\mathbb{C}}_{\lambda+1,\nu} d_{\mathbb{R}^n}^* + \frac{1}{2}(\lambda + i - n)\widetilde{\mathbb{C}}_{\lambda,\nu} \iota_{\frac{\partial}{\partial x_n}} \qquad \text{for } 1 \le i \le n.$$

We note that there exist isolated parameters (λ, ν) for which $\mathbb{C}_{\lambda,\nu}^{i,i} = 0$ or $\mathbb{C}_{\lambda,\nu}^{i,i-1} = 0$. For instance, $\mathbb{C}_{\lambda,\nu}^{0,0} = \frac{1}{2}\nu\widetilde{\mathbb{C}}_{\lambda,\nu}$, and thus $\mathbb{C}_{\lambda,\nu}^{0,0} = 0$ if $\nu = 0$. To be precise, we have the following:

$$\mathbb{C}_{\lambda,\nu}^{i,i} = 0 \text{ if and only if } \lambda = \nu = i \text{ or } \nu = i = 0; \qquad \mathbb{C}_{\lambda,\nu}^{i,i-1} = 0 \text{ if and only if } \lambda = \nu = n - i \text{ or } \nu = n - i = 0.$$

We renormalize these operators by

$$\widetilde{\mathbb{C}}_{\lambda,\nu}^{i,i} := \begin{cases} \mathrm{Rest}_{x_n=0} & \text{if } \lambda = \nu, \\ \widetilde{\mathbb{C}}_{\lambda,\nu} & \text{if } i = 0, \\ \mathbb{C}_{\lambda,\nu}^{i,i} & \text{otherwise,} \end{cases} \quad \text{and} \quad \widetilde{\mathbb{C}}_{\lambda,\nu}^{i,i-1} := \begin{cases} \mathrm{Rest}_{x_n=0} \circ \iota_{\frac{\partial}{\partial x_n}} & \text{if } \lambda = \nu, \\ \widetilde{\mathbb{C}}_{\lambda,\nu} \circ \iota_{\frac{\partial}{\partial x_n}} & \text{if } i = n, \\ \mathbb{C}_{\lambda,\nu}^{i,i-1} & \text{otherwise.} \end{cases}$$

Then $\widetilde{\mathbb{C}}_{\lambda,\nu}^{i,i}$ $(0 \le i \le n-1)$ and $\widetilde{\mathbb{C}}_{\lambda,\nu}^{i,i-1}$ $(1 \le i \le n)$ are nonzero differential operators of order $\nu - \lambda$ for any $\lambda, \nu \in \mathbb{C}$ with $\nu - \lambda \in \mathbb{N}$.

The differential operators $\widetilde{\mathbb{C}}_{\lambda,\nu}^{i,i+1}$ and $\widetilde{\mathbb{C}}_{\lambda,\nu}^{i,i-2}$ are defined only for special parameters (λ, ν) as follows.

$$\widetilde{\mathbb{C}}_{\lambda,i+1}^{i,i+1} := \begin{cases} \mathrm{Rest}_{x_n=0} \circ d_{\mathbb{R}^n} & \text{for } 1 \le i \le n-2, \lambda = i, \\ d_{\mathbb{R}^{n-1}} \circ \widetilde{\mathbb{C}}_{\lambda,0} & \text{for } i = 0, \lambda \in -\mathbb{N}, \end{cases} \qquad \widetilde{\mathbb{C}}_{\lambda,n-i+1}^{i,i-2} := \begin{cases} \mathrm{Rest}_{x_n=0} \circ \iota_{\frac{\partial}{\partial x_n}} d_{\mathbb{R}^n}^* & \text{for } 2 \le i \le n, \lambda = n - i, \\ -d_{\mathbb{R}^{n-1}}^* \circ \mathbb{C}_{\lambda,0}^{n,n-1} & \text{for } i = n, \lambda \in -\mathbb{N}. \end{cases}$$

Then we give a complete solution to Problem 2 for the model space $(X, Y) = (S^n, S^{n-1})$ in terms of the flat picture of principal series representations as follows:

Theorem 4. *Suppose a 6-tuple $(i, j, \lambda, \nu, \delta, \varepsilon)$ satisfies one of the equivalent conditions in Theorem 3. Then the operators $\widetilde{\mathbb{C}}_{\lambda,\nu}^{i,j} : C^\infty(\mathbb{R}^n) \otimes \bigwedge^i(\mathbb{C}^n) \longrightarrow C^\infty(\mathbb{R}^{n-1}) \otimes \bigwedge^j(\mathbb{C}^{n-1})$ extend to differential SBOs $I(i, \lambda)_\delta \longrightarrow J(j, \nu)_\varepsilon$, to be denoted by the same letters. Conversely, any differential SBO from $I(i, \lambda)_\delta$ to $J(j, \nu)_\varepsilon$ is proportional to the following differential operators: $\widetilde{\mathbb{C}}_{\lambda,\nu}^{i,i}$ in Case 1, $\widetilde{\mathbb{C}}_{\lambda,\nu}^{i,i-1}$ in Case 2, $\widetilde{\mathbb{C}}_{i,i+1}^{i,i+1}$ in Case 3, $\widetilde{\mathbb{C}}_{\lambda,1}^{0,1}$ in Case 3′, $\widetilde{\mathbb{C}}_{n-i,n-i+1}^{i,i-2}$ in Case 4, and $\widetilde{\mathbb{C}}_{\lambda,1}^{n,n-2}$ in Case 4′.*

5. Matrix-valued factorization identities

Suppose that $T_X : I(i, \lambda')_\delta \to I(i, \lambda)_\delta$ or $T_Y : J(j, \nu)_\varepsilon \to J(j, \nu')_\varepsilon$ are G- or G'-intertwining operators, respectively. Then the composition $T_Y \circ D_{X \to Y}$ or $D_{X \to Y} \circ T_X$ of a symmetry breaking operator $D_{X \to Y} : I(i, \lambda)_\delta \to J(j, \nu)_\varepsilon$ gives another symmetry breaking operator:

$$\begin{tikzcd}
I(i,\lambda)_\delta \arrow[r, "D_{X\to Y}"] \arrow[dr, dashed] & J(j,\nu)_\varepsilon \arrow[d, "T_Y"] \\
I(i,\lambda')_\delta \arrow[u, "T_X"] \arrow[ur, dashed] & J(j,\nu')_\varepsilon
\end{tikzcd}$$

The multiplicity-free property (see Theorem 3 (ii)) assures the existence of matrix-valued factorization identities for differential SBOs, namely, $D_{X\to Y}\circ T_X$ must be a scalar multiple of $\widetilde{\mathbb{C}}^{i,j}_{\lambda',\nu}$, and $T_Y\circ D_{X\to Y}$ must be a scalar multiple of $\widetilde{\mathbb{C}}^{i,j}_{\lambda,\nu'}$. We shall determine these constants explicitly when T_X or T_Y are Branson's conformally covariant operators [1] defined below. Let $0\le i\le n$. For $\ell\in\mathbb{N}_+$, we set

$$\mathcal{T}^{(i)}_{2\ell} := ((\frac{n}{2}-i-\ell)d_{\mathbb{R}^n}d^*_{\mathbb{R}^n}+(\frac{n}{2}-i+\ell)d^*_{\mathbb{R}^n}d_{\mathbb{R}^n})\Delta^{\ell-1}_{\mathbb{R}^n} = (-2\ell\, d_{\mathbb{R}^n}d^*_{\mathbb{R}^n}-(\frac{n}{2}-i+\ell)\Delta_{\mathbb{R}^n})\Delta^{\ell-1}_{\mathbb{R}^n}.$$

Then the differential operator $\mathcal{T}^{(i)}_{2\ell}:\mathcal{E}^i(\mathbb{R}^n)\longrightarrow\mathcal{E}^i(\mathbb{R}^n)$ induces a nonzero $O(n+1,1)$-intertwining operator, to be denoted by the same letter $\mathcal{T}^{(i)}_{2\ell}$, from $I\left(i,\frac{n}{2}-\ell\right)_\delta$ to $I\left(i,\frac{n}{2}+\ell\right)_\delta$, for $\delta\in\mathbb{Z}/2\mathbb{Z}$. Similarly, we define a G'-intertwining operator $\mathcal{T}'^{(j)}_{2\ell}: J\left(j,\frac{n-1}{2}-\ell\right)_\varepsilon\longrightarrow J\left(j,\frac{n-1}{2}+\ell\right)_\varepsilon$ for $0\le j\le n-1$ and $\varepsilon\in\mathbb{Z}/2\mathbb{Z}$ as the lift of the differential operator $\mathcal{T}'^{(j)}_{2\ell}$: $\mathcal{E}^j(\mathbb{R}^{n-1})\longrightarrow\mathcal{E}^j(\mathbb{R}^{n-1})$, which is given by

$$\mathcal{T}'^{(j)}_{2\ell} = ((\frac{n-1}{2}-j-\ell)d_{\mathbb{R}^{n-1}}d^*_{\mathbb{R}^{n-1}}+(\frac{n-1}{2}-j+\ell)d^*_{\mathbb{R}^{n-1}}d_{\mathbb{R}^{n-1}})\Delta^{\ell-1}_{\mathbb{R}^{n-1}}.$$

Consider the following diagrams for $j=i$ and $j=i-1$:

$$\begin{tikzcd}
I\left(i,\frac{n}{2}-\ell\right)_\delta \arrow[d, "\mathcal{T}^{(i)}_{2\ell}"'] \arrow[dr, "\widetilde{\mathbb{C}}^{i,j}_{\frac{n}{2}-\ell,\frac{n}{2}+a+\ell}"] & \\
I\left(i,\frac{n}{2}+\ell\right)_\delta \arrow[r, "\widetilde{\mathbb{C}}^{i,j}_{\frac{n}{2}+\ell,\frac{n}{2}+a+\ell}"'] & J\left(j,\frac{n}{2}+a+\ell\right)_\varepsilon,
\end{tikzcd}
\qquad
\begin{tikzcd}
I\left(i,\frac{n-1}{2}-a-\ell\right)_\delta \arrow[r, "\widetilde{\mathbb{C}}^{i,j}_{\frac{n-1}{2}-a-\ell,\frac{n-1}{2}-\ell}"] \arrow[dr, "\widetilde{\mathbb{C}}^{i,j}_{\frac{n-1}{2}-a-\ell,\frac{n-1}{2}+\ell}"'] & J\left(j,\frac{n-1}{2}-\ell\right)_\varepsilon \arrow[d, "\mathcal{T}'^{(j)}_{2\ell}"] \\
& J\left(j,\frac{n-1}{2}+\ell\right)_\varepsilon,
\end{tikzcd}$$

where parameters δ and $\varepsilon\in\mathbb{Z}/2\mathbb{Z}$ are chosen according to Theorem 3 (iii). In what follows, we put

$$p_\pm=\begin{cases} i\pm\ell-\dfrac{n}{2} & \text{if } a\neq 0\\ \pm 2 & \text{if } a=0\end{cases},\quad q=\begin{cases} i+\ell-\dfrac{n-1}{2} & \text{if } i\neq 0, a\neq 0\\ -2 & \text{if } i\neq 0, a=0\\ -\left(\ell+\dfrac{n-1}{2}\right) & \text{if } i=0\end{cases},\quad r=\begin{cases} i-\ell-\dfrac{n+1}{2} & \text{if } i\neq n, a\neq 0\\ 2 & \text{if } i\neq n, a=0\\ -\left(\ell+\dfrac{n+1}{2}\right) & \text{if } i=n\end{cases},$$

$$K_{\ell,a}:=\prod_{k=1}^{\ell}\left(\left[\frac{a}{2}\right]+k\right).$$

Then the factorization identities for differential SBOs $\widetilde{\mathbb{C}}^{i,j}_{\lambda,\nu}$ for $j\in\{i-1,i\}$ and Branson's conformally covariant operators $\mathcal{T}^{(i)}_{2\ell}$ or $\mathcal{T}'^{(j)}_{2\ell}$ are given as follows.

Theorem 5. *Suppose $0\le i\le n-1$, $a\in\mathbb{N}$ and $\ell\in\mathbb{N}_+$. Then*

(1) $\widetilde{\mathbb{C}}^{i,i}_{\frac{n}{2}+\ell,a+\ell+\frac{n}{2}}\circ\mathcal{T}^{(i)}_{2\ell}=p_-K_{\ell,a}\widetilde{\mathbb{C}}^{i,i}_{\frac{n}{2}-\ell,a+\ell+\frac{n}{2}}$.

(2) $\mathcal{T}'^{(i)}_{2\ell}\circ\widetilde{\mathbb{C}}^{i,i}_{\frac{n-1}{2}-a-\ell,\frac{n-1}{2}-\ell}=qK_{\ell,a}\widetilde{\mathbb{C}}^{i,i}_{\frac{n-1}{2}-a-\ell,\frac{n-1}{2}+\ell}$.

Theorem 6. *Suppose $1\le i\le n$, $a\in\mathbb{N}$ and $\ell\in\mathbb{N}_+$. Then*

(1) $\widetilde{\mathbb{C}}^{i,i-1}_{\frac{n}{2}+\ell,a+\ell+\frac{n}{2}}\circ\mathcal{T}^{(i)}_{2\ell}=p_+K_{\ell,a}\widetilde{\mathbb{C}}^{i,i-1}_{\frac{n}{2}-\ell,a+\ell+\frac{n}{2}}$.

(2) $\mathcal{T}'^{(i-1)}_{2\ell}\circ\widetilde{\mathbb{C}}^{i,i-1}_{\frac{n-1}{2}-a-\ell,\frac{n-1}{2}-\ell}=rK_{\ell,a}\widetilde{\mathbb{C}}^{i,i-1}_{\frac{n-1}{2}-a-\ell,\frac{n-1}{2}+\ell}$.

In the case where $i=0$, $\widetilde{\mathbb{C}}^{i,i}_{\lambda,\nu}$ is a scalar-valued operator, and the corresponding factorization identities in Theorem 5 were studied in [4,8,9].

The main results are proved by using the F-method [5,6,9]. Details will appear elsewhere.

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