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Instantons in six dimensions and twistors

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Abstract

Recently, conformal field theories in six dimensions were discussed from the twistorial point of view. In particular, it was demonstrated that the twistor transform between chiral zero-rest-mass fields and cohomology classes on twistor space can be generalized from four to six dimensions. On the other hand, the possibility of generalizing the correspondence between instanton gauge fields and holomorphic bundles over twistor space is questionable. It was shown by Sämann and Wolf that holomorphic line bundles over the canonical twistor space $\mathrm{Tw}(X)$ (defined as a bundle of almost complex structures over the six-dimensional manifold X) correspond to pure-gauge Maxwell potentials, i.e. the twistor transform fails. On the example of $X = \mathbb{C}P^3$ we show that there exists a twistor correspondence between Abelian or non-Abelian Yang–Mills instantons on $\mathbb{C}P^3$ and holomorphic bundles over complex submanifolds of $\mathrm{Tw}(\mathbb{C}P^3)$, but it is not so efficient as in the four-dimensional case because the twistor transform does not parametrize instantons by unconstrained holomorphic data as it does in four dimensions.

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1. Introduction and summary

Let us consider an oriented real four-manifold X^4 with a Riemannian metric g and the principal bundle $P(X^4, SO(4))$ of orthonormal frames over X^4. The (metric) twistor space $\mathrm{Tw}(X^4)$ of X^4 can be defined as an associated bundle [1]

$$\mathrm{Tw}\big(X^4\big) = P \times_{\mathrm{SO}(4)} \mathrm{SO}(4)/\mathrm{U}(2) \tag{1.1}$$

with the canonical projection $\mathrm{Tw}(X^4) \to X^4$. This space parametrizes the almost complex structures on X^4 compatible with the metric g (almost Hermitian structures). It was shown in [1,2] that if the Weyl tensor of (X^4, g) is anti-self-dual then the almost complex structure on the twistor space $\mathrm{Tw}(X^4)$ is integrable. Furthermore, it was proven that the rank r complex vector bundle E over X^4 with an anti-self-dual gauge potential A over such X^4 lifts to a holomorphic bundle $\hat{E}$ over complex twistor space $\mathrm{Tw}(X^4)$ [1,3].

The essence of the canonical twistor approach is to establish a correspondence between four-dimensional space X^4 (or its complex version) and complex twistor space $\mathrm{Tw}(X^4)$ of X^4. Using this correspondence, one transfers data given on X^4 to data on $\mathrm{Tw}(X^4)$ and vice versa. In twistor theory one considers *holomorphic* objects h on $\mathrm{Tw}(X^4)$ (Čech cohomology classes, holomorphic vector bundles, etc.) and transforms them to objects f on X^4 which are constrained by some differential equations [1–4]. Thus, the main idea of twistor theory is to encode solutions of some differential equations on X^4 in holomorphic data on the complex twistor space $\mathrm{Tw}(X^4)$ of X^4.

The twistor approach was recently extended to maximally supersymmetric Yang–Mills theory on $\mathbb{C}^6$ [5]. It was also generalized to Abelian [6,7] and non-Abelian [8] holomorphic principal 2-bundles over the twistor space $Q_6 \subset \mathbb{C}P^7 \setminus \mathbb{C}P^3$, corresponding to self-dual Lie-algebra-valued 3-forms on $\mathbb{C}^6$. These forms are the most important objects needed for constructing $(2,0)$ superconformal field theories in six dimensions, which are believed to describe stacks of M5-branes in the low-energy limit of M-theory [9]. Thus, it is worthwhile to analyze the twistor transform in six dimensions in more detail.

We point out that there are some problems in generalizing the twistor approach to higher dimensions. Namely, let X^{2n} be a Riemannian manifold of dimension $2n$. The metric twistor space of X^{2n} is defined as the bundle $\mathrm{Tw}(X^{2n}) \to X^{2n}$ of almost Hermitian structures on X^{2n} associated with the principal bundle of orthonormal frames of X^{2n}, i.e.

$$\mathrm{Tw}\big(X^{2n}\big) := P\big(X^{2n}, \mathrm{SO}(2n)\big) \times_{\mathrm{SO}(2n)} \mathrm{SO}(2n)/\mathrm{U}(n). \tag{1.2}$$

It is well known that $\mathrm{Tw}(X^{2n})$ can be endowed with an almost complex structure $\mathcal{J}$, which is integrable if and only if the Weyl tensor of X^{2n} vanishes when $n > 2$ [10]. This is a strong restriction on the geometry of X^{2n} allowing only conformally flat spaces, e.g. flat spaces and spheres, which may be not so interesting. The restriction can be overcome if the manifold X^{2n} has a G-structure (not necessary integrable). In this case one can find a submanifold $\mathcal{Z}$ of $\mathrm{Tw}(X^{2n})$ associated with the G-structure bundle $P(X^{2n}, G)$ for $G \subset \mathrm{SO}(2n)$, such that an induced almost complex structure (also called $\mathcal{J}$) on $\mathcal{Z}$ is integrable. Many examples were studied in [10–14]. Further problems can appear when considering the twistor transform of holomorphic objects on $\mathrm{Tw}(X^{2n})$ or on $\mathcal{Z} \hookrightarrow \mathrm{Tw}(X^{2n})$ to solutions of differential equations on X^{2n}. We will discuss this for the case of $n = 3$.

The papers [6,7] (see also references therein) show that twistor methods can be useful in describing conformally invariant massless fields on the flat space $\mathbb{R}^6 \cong \mathbb{C}^3$ and its complexification $\mathbb{C}^6$ with the twistor space

$$\mathrm{Tw}\big(\mathbb{R}^6\big) = Q_6 \cong \mathbb{R}^6 \times \mathbb{C}P^3. \tag{1.3}$$

On the other hand, Sämann and Wolf have shown [6] that holomorphic line bundles over $\mathrm{Tw}(\mathbb{R}^6)$ trivial on all $\mathbb{C}P^3_x \hookrightarrow \mathrm{Tw}(\mathbb{R}^6)$ correspond to pure-gauge Maxwell potentials on $\mathbb{R}^6$, i.e. the twistor transform fails for the metric twistor space $\mathrm{Tw}(\mathbb{R}^6)$. This was partially cured in [15] where it was shown that instantons on the six-sphere $S^6 = \mathbb{R}^6 \cup \{\infty\}$ correspond to complex vector bundles over the reduced twistor space $\hat{\mathcal{Z}} = G_2/\mathrm{U}(2) \hookrightarrow \mathrm{Tw}(S^6)$ with flat partial connections, where

$$\mathrm{Tw}\big(S^6\big) = \mathrm{Spin}(7)/\mathrm{U}(3) \tag{1.4}$$

is a compactification of the twistor space (1.3). For the definition of the instanton equations in dimensions higher than four and for some instanton solutions see e.g. [16–23]. Hence, constructing instanton configurations in six dimensions is a task more complicated than one might expect.

Instanton equations on the six-sphere S^6 are not quite standard since S^6 is a nearly Kähler space with a nonintegrable almost complex structure. In fact, instantons on S^6 are connections on *pseudo-holomorphic* bundles satisfying the Donaldson–Uhlenbeck–Yau (DUY) equations [17]. Hence, for checking the power of the twistor approach it is worthwhile to consider a Kähler 6-manifold. We choose the complex projective space $\mathbb{C}P^3$ which can be considered as yet another compactification of $\mathbb{R}^6 \cong \mathbb{C}^3$.

On $\mathbb{C}P^3$ the DUY equations are the standard Hermitian Yang–Mills (HYM) equations [17]. They are SU(3) invariant but not invariant under the SO(6) Lorentz-type rotations of orthonormal frames. Therefore, one should describe them with *reduced* twistor spaces. The DUY equations are well defined on six-dimensional Kähler manifolds X (as well as on nearly Kähler spaces [24–26]), and their solutions are natural connections $\mathcal{A}$ on holomorphic vector bundles $\mathcal{E} \to X$ [17]. As reduced twistor spaces of $\mathbb{C}P^3$ one can consider

$$\mathrm{SU}(4)/\mathrm{U}(2) \times \mathrm{U}(1) =: \mathcal{Z} \to \mathbb{C}P^3 \cong \mathrm{SU}(4)/\mathrm{U}(3) \tag{1.5}$$

or

$$\mathrm{Sp}(2)/\mathrm{U}(1) \times \mathrm{U}(1) =: \mathcal{Z}' \to \mathbb{C}P^3 \cong \mathrm{Sp}(2)/\mathrm{Sp}(1)\times\mathrm{U}(1) \tag{1.6}$$

which both are complex submanifolds of $\mathrm{Tw}(\mathbb{C}P^3)$, with fibres $\mathbb{C}P^2$ and $\mathbb{C}P^1$, respectively. We will show that bundles $(\mathcal{E}, \mathcal{A})$ over $\mathbb{C}P^3$ with HYM connections $\mathcal{A}$ are pulled back to holomorphic vector bundles $(\tilde{\mathcal{E}}, \tilde{\mathcal{A}})$ over the reduced twistor spaces (1.5) or (1.6), depending on the choice for $\mathbb{C}P^3$, being trivial along the fibres of the fibrations (1.5) or (1.6), with a Hermitian Yang–Mills connection $\tilde{\mathcal{A}}$ on $\tilde{\mathcal{E}}$. Thus, contrary to the four-dimensional case, the twistor transform in six dimensions does not parametrize instantons by unconstrained holomorphic data on the twistor space, since the corresponding holomorphic bundles over $\mathcal{Z}$ and $\mathcal{Z}'$ have to be polystable. In other words, in four dimensions the twistor transform establishes a correspondence between solutions of the instanton equations in $d = 4$ and solutions of holomorphic Chern–Simons theory on $d = 6$ twistor space, but in six dimensions the twistor transform establishes a correspondence between solutions of the instanton (HYM) equations in $d = 6$ and solutions of the HYM equations on the twistor space. The latter does not facilitate solving the $d = 6$ instanton equations. This is the outcome of our study of instantons in six dimensions.

The structure of the remainder of this paper is as follows. In Section 2 we portray the space $\mathbb{C}P^3$ as a homogeneous space $\mathrm{SU}(4)/\mathrm{U}(3)$ and $\mathrm{Sp}(2)/\mathrm{Sp}(1) \times \mathrm{U}(1)$, with Kähler structures in both cases and allowing for the introduction of a quasi-Kähler structure in the second case. In Section 3 we describe the geometry of the twistor spaces $\mathcal{Z}$ and $\mathcal{Z}'$ for $\mathrm{SU}(4)/\mathrm{U}(3)$ and

$\mathrm{Sp}(2)/\mathrm{Sp}(1)\times\mathrm{U}(1)$. In Section 4 we study the twistor correspondence between instanton bundles over $\mathbb{C}P^3$ and holomorphic bundles over the proper twistor spaces.

2. Kähler and quasi-Kähler structure on $\mathbb{C}P^3$

In this section we describe the geometry of the space $\mathbb{C}P^3$ as a homogeneous manifold $\mathcal{M}=\mathrm{Sp}(2)/\mathrm{Sp}(1)\times\mathrm{U}(1)$ fibred over the four-sphere S^4. We find it useful to describe orthonormal coframes on S^4, S^2 and $\mathcal{M}$ in local coordinates. First, we choose a representative element $Q\in\mathrm{Sp}(2)$ of the coset space $S^4=\mathrm{Sp}(2)/\mathrm{Sp}(1)\times\mathrm{Sp}(1)$. Then, expanding the flat connection $\mathcal{A}_0=Q^{-1}\,\mathrm{d}Q$ into a basis of the Lie algebra $sp(2)$, we obtain local $(1,0)$-forms θ^1 and θ^2 on an open subset U of S^4 as well as self-dual and anti-self-dual connections (A^+ resp. A^-) on $\mathrm{Sp}(1)$-bundles over S^4. Using a representative element $g\in\mathrm{SU}(2)$ of the coset space $S^2=\mathrm{SU}(2)/\mathrm{U}(1)\cong\mathrm{Sp}(1)/\mathrm{U}(1)$, we get a local $(1,0)$-form θ^3 on S^2 and the monopole connection a on the Hopf bundle $S^3\to S^2$. After this, we combine Q and g into a representative $\hat{Q}$ of the coset $\mathcal{M}$ and arrive at local $(1,0)$-forms $\hat{\theta}^i$ on this coset, together with their Maurer–Cartan relations (2.25). Finally, changing an almost complex structure on $\mathcal{M}$ via (2.29), we find a quasi-Kähler structure on the considered coset space.

Coset representation of S^4. Let us consider the group $\mathrm{Sp}(2)$ fibred over $S^4=\mathrm{Sp}(2)/\mathrm{Sp}(1)\times\mathrm{Sp}(1)$,

$$\mathrm{Sp}(2)\to S^4 \tag{2.1}$$

i.e. consider $\mathrm{Sp}(2)$ as the fibre bundle $P(S^4,\mathrm{Sp}(1)\times\mathrm{Sp}(1))$ with the structure group $\mathrm{Sp}(1)\times\mathrm{Sp}(1)$. Local sections of the fibrations (2.1) can be chosen as 4×4 matrices

$$Q:=f^{-\frac{1}{2}}\begin{pmatrix}\mathbf{1}_2 & -x\\ x^\dagger & \mathbf{1}_2\end{pmatrix}\quad\text{and}\quad Q^{-1}=Q^\dagger=f^{-\frac{1}{2}}\begin{pmatrix}\mathbf{1}_2 & x\\ -x^\dagger & \mathbf{1}_2\end{pmatrix}\in\mathrm{Sp}(2)\subset\mathrm{SU}(4), \tag{2.2}$$

where

$$x=x^\mu\tau_\mu,\qquad x^\dagger=x^\mu\tau_\mu^\dagger,\qquad f:=1+x^\dagger x=1+r^2=1+\delta_{\mu\nu}x^\mu x^\nu, \tag{2.3}$$

and the matrices

$$(\tau_\mu)=(-\mathrm{i}\sigma_i,\mathbf{1}_2)\quad\text{and}\quad\left(\tau_\mu^\dagger\right)=(\mathrm{i}\sigma_i,\mathbf{1}_2) \tag{2.4}$$

obey

$$\begin{aligned}
&\tau_\mu^\dagger\tau_\nu=\delta_{\mu\nu}\cdot\mathbf{1}_2+\eta^i_{\mu\nu}\mathrm{i}\sigma_i=:\delta_{\mu\nu}\cdot\mathbf{1}_2+\eta_{\mu\nu},\\
&\left\{\eta^i_{\mu\nu}\right\}=\left\{-\eta^i_{\nu\mu}\right\}=\left\{\varepsilon^i_{jk},\mu=j,\nu=k;\delta^i_j,\mu=j,\nu=4\right\},\\
&\tau_\mu\tau_\nu^\dagger=\delta_{\mu\nu}\cdot\mathbf{1}_2+\bar{\eta}^i_{\mu\nu}\mathrm{i}\sigma_i=:\delta_{\mu\nu}\cdot\mathbf{1}_2+\bar{\eta}_{\mu\nu},\\
&\left\{\bar{\eta}^i_{\mu\nu}\right\}=\left\{-\bar{\eta}^i_{\nu\mu}\right\}=\left\{\varepsilon^i_{jk},\mu=j,\nu=k;-\delta^i_j,\mu=j,\nu=4\right\}.
\end{aligned} \tag{2.5}$$

Here $\{x^\mu\}$ are local coordinates on an open set $\mathcal{U}\subset S^4$. The matrices (2.2) are representative elements for the coset space $S^4=\mathrm{Sp}(2)/\mathrm{Sp}(1)\times\mathrm{Sp}(1)$.

Flat connection on S^4. Consider a flat connection $\mathcal{A}_0$ on the trivial vector bundle $S^4\times\mathbb{C}^4\to S^4$ given by the one-form

$$\mathcal{A}_0=Q^{-1}\,\mathrm{d}Q=:\begin{pmatrix}A^- & -\phi\\ \phi^\dagger & A^+\end{pmatrix}, \tag{2.6}$$

where from (2.2) we obtain

$$A^-=\frac{1}{f}\bar\eta_{\mu\nu}x^\mu\,\mathrm{d}x^\nu=:\begin{pmatrix}\alpha_- & -\bar\beta_-\\ \beta_- & -\alpha_-\end{pmatrix}\in su(2), \tag{2.7}$$

$$A^+=\frac{1}{f}\eta_{\mu\nu}x^\mu\,\mathrm{d}x^\nu=:\begin{pmatrix}\alpha_+ & -\bar\beta_+\\ \beta_+ & -\alpha_+\end{pmatrix}\in su(2), \tag{2.8}$$

$$\begin{aligned}\phi&=\frac{1}{f}\,\mathrm{d}x=-\frac{\mathrm{i}}{f}\begin{pmatrix}\mathrm{d}x^3+\mathrm{i}\,\mathrm{d}x^4 & \mathrm{d}x^1-\mathrm{i}\,\mathrm{d}x^2\\ \mathrm{d}x^1+\mathrm{i}\,\mathrm{d}x^2 & -(\mathrm{d}x^3-\mathrm{i}\,\mathrm{d}x^4)\end{pmatrix}=-\frac{\mathrm{i}}{f}\begin{pmatrix}\mathrm{d}z & \mathrm{d}\bar y\\ \mathrm{d}y & -\mathrm{d}\bar z\end{pmatrix}\\ &=:\begin{pmatrix}\theta^2 & \theta^{\bar 1}\\ -\theta^1 & \theta^{\bar 2}\end{pmatrix},\end{aligned} \tag{2.9}$$

with

$$\alpha_+=\frac{1}{2f}(\bar y\,\mathrm{d}y+\bar z\,\mathrm{d}z-y\,\mathrm{d}\bar y-z\,\mathrm{d}\bar z),\qquad \beta_+=\frac{1}{f}(y\,\mathrm{d}z-z\,\mathrm{d}y), \tag{2.10}$$

$$\alpha_-=\frac{1}{2f}(\bar y\,\mathrm{d}y+z\mathrm{d}\bar z-y\,\mathrm{d}\bar y-\bar z\,\mathrm{d}z),\qquad \beta_-=\frac{1}{f}(y\,\mathrm{d}\bar z-\bar z\,\mathrm{d}y), \tag{2.11}$$

$$\theta^1:=\frac{\mathrm{i}\,\mathrm{d}y}{1+r^2},\qquad \theta^2:=-\frac{\mathrm{i}\,\mathrm{d}z}{1+r^2}\quad\text{and}\quad \theta^{\bar 1}:=-\frac{\mathrm{i}\,\mathrm{d}\bar y}{1+r^2},\qquad \theta^{\bar 2}:=\frac{\mathrm{i}\,\mathrm{d}\bar z}{1+r^2}. \tag{2.12}$$

Here, the bar denotes complex conjugation.

Coset representation of S^2. Let us consider the Hopf bundle

$$S^3\to S^2 \tag{2.13}$$

over the Riemann sphere $S^2\cong\mathbb{C}P^1$ and the one-monopole connection a on the bundle (2.13) having in the local complex coordinate $\zeta\in\mathbb{C}P^1$ the form

$$a=\frac{1}{2(1+\zeta\bar\zeta)}(\bar\zeta\,\mathrm{d}\zeta-\zeta\,\mathrm{d}\bar\zeta). \tag{2.14}$$

Consider a local section of the bundle (2.13) given by the matrix

$$g=\frac{1}{(1+\zeta\bar\zeta)^{\frac12}}\begin{pmatrix}1 & -\bar\zeta\\ \zeta & 1\end{pmatrix}\in \mathrm{SU}(2)\cong S^3 \tag{2.15}$$

and introduce the $su(2)$-valued one-form (flat connection)

$$g^{-1}\,\mathrm{d}g=:\begin{pmatrix}a & -\theta^{\bar 3}\\ \theta^3 & -a\end{pmatrix} \tag{2.16}$$

on the bundle $S^2\times\mathbb{C}^2\to S^2$, where

$$\theta^3=\frac{\mathrm{d}\zeta}{1+\zeta\bar\zeta}\quad\text{and}\quad \theta^{\bar 3}=\frac{\mathrm{d}\bar\zeta}{1+\zeta\bar\zeta} \tag{2.17}$$

are the forms of type $(1,0)$ and $(0,1)$ on $\mathbb{C}P^1$ and a is the one-monopole gauge potential (2.14).

Twistor space $\mathrm{Tw}(S^4)$. Let us introduce 4×4 matrices

$$G=\begin{pmatrix}\mathbf{1}_2 & 0\\ 0 & g\end{pmatrix}\quad\text{and}\quad \hat{Q}=QG\in \mathrm{Sp}(2)\subset \mathrm{SU}(4), \tag{2.18}$$

where Q and g are given in (2.2) and (2.15). The matrix $\hat{Q}$ is a local section of the bundle

$$\mathrm{Sp}(2)\to \mathrm{Sp}(2)/\mathrm{Sp}(1)\times \mathrm{U}(1)=:\mathcal{M}. \tag{2.19}$$

Let us consider a trivial complex vector bundle $\mathcal{M}\times\mathbb{C}^4\to\mathcal{M}$ with the flat connection

$$\hat{\mathcal{A}}_0=\hat{Q}^{-1}\,\mathrm{d}\hat{Q}=G^{-1}\mathcal{A}_0G+G^{-1}\,\mathrm{d}G=:\begin{pmatrix}\hat{A}^- & -\hat{\phi}\\ \hat{\phi}^\dagger & \hat{A}^+\end{pmatrix}, \tag{2.20}$$

where

$$\hat{\phi}=\phi g=:\begin{pmatrix}\hat{\theta}^2 & \hat{\theta}^{\bar{1}}\\ -\hat{\theta}^1 & \hat{\theta}^{\bar{2}}\end{pmatrix},\qquad \hat{A}^-=A^-=\begin{pmatrix}\alpha_- & -\bar{\beta}_-\\ \beta_- & -\alpha_-\end{pmatrix},\qquad \hat{A}^+=:\begin{pmatrix}\hat{\alpha}_+ & -\hat{\theta}^{\bar{3}}\\ \hat{\theta}^3 & -\hat{\alpha}_+\end{pmatrix}, \tag{2.21}$$

with α_-, β_- given in (2.11) and

$$\hat{\alpha}_+:=\frac{1}{1+\zeta\bar{\zeta}}\left\{(1-\zeta\bar{\zeta})\alpha_++\bar{\zeta}\beta_+-\zeta\bar{\beta}_++\frac{1}{2}(\bar{\zeta}\,\mathrm{d}\zeta-\zeta\,\mathrm{d}\bar{\zeta})\right\}, \tag{2.22}$$

$$\hat{\theta}^1:=\frac{1}{(1+\zeta\bar{\zeta})^{\frac12}}\big(\theta^1-\zeta\theta^{\bar{2}}\big),\qquad \hat{\theta}^2:=\frac{1}{(1+\zeta\bar{\zeta})^{\frac12}}\big(\theta^2+\zeta\theta^{\bar{1}}\big), \tag{2.23}$$

$$\hat{\theta}^3:=\frac{1}{(1+\zeta\bar{\zeta})}\big(\mathrm{d}\zeta+\beta_+-2\zeta\alpha_++\zeta^2\bar{\beta}_+\big). \tag{2.24}$$

From the flatness of the connection (2.20), $\mathrm{d}\hat{\mathcal{A}}_0+\hat{\mathcal{A}}_0\wedge\hat{\mathcal{A}}_0=0$, we obtain the equations

$$\mathrm{d}\begin{pmatrix}\hat{\theta}^1\\ \hat{\theta}^2\\ \hat{\theta}^3\end{pmatrix}+\begin{pmatrix}-\hat{\alpha}_+-\alpha_- & \beta_- & -\frac{1}{2R}\hat{\theta}^{\bar{2}}\\ -\bar{\beta}_- & -\hat{\alpha}_++\alpha_- & \frac{1}{2R}\hat{\theta}^{\bar{1}}\\ \frac{R}{2\Lambda^2}\hat{\theta}^2 & -\frac{R}{2\Lambda^2}\hat{\theta}^1 & -2\hat{\alpha}_+\end{pmatrix}\wedge\begin{pmatrix}\hat{\theta}^1\\ \hat{\theta}^2\\ \hat{\theta}^3\end{pmatrix}=0, \tag{2.25}$$

where we rescaled our one-forms $\hat{\theta}$'s as

$$\hat{\theta}^1\to\frac{1}{2\Lambda}\hat{\theta}^1,\qquad \hat{\theta}^2\to\frac{1}{2\Lambda}\hat{\theta}^2\quad\text{and}\quad \hat{\theta}^3\to\frac{1}{2R}\hat{\theta}^3. \tag{2.26}$$

We see that (2.25) defines the Levi-Civita connection with U(3) holonomy group (Kähler structure) on $\mathcal{M}$ if $R=\Lambda$, where R is the radius of S^2 and Λ is the radius of S^4.

Note that the forms $\hat{\theta}^i$ define on $\mathcal{M}$ an integrable almost complex structure $\mathcal{J}_+$ [1] such that

$$\mathcal{J}_+\hat{\theta}^i=\mathrm{i}\hat{\theta}^i \tag{2.27}$$

with $i=1,2,3$. In other words, the $\hat{\theta}^i$'s are $(1,0)$-forms with respect to $\mathcal{J}_+$ and the manifold $\mathcal{M}$ with such a complex structure can be identified with the Kähler manifold $\mathbb{C}P^3=\mathrm{SU}(4)/\mathrm{U}(3)$ with the Kähler form

$$\hat{\omega}:=\frac{\mathrm{i}}{2}\big(\hat{\theta}^1\wedge\hat{\theta}^{\bar{1}}+\hat{\theta}^2\wedge\hat{\theta}^{\bar{2}}+\hat{\theta}^3\wedge\hat{\theta}^{\bar{3}}\big). \tag{2.28}$$

Quasi-Kähler structure on $\mathcal{M}$. Recall that on the same manifold $\mathcal{M}$ one can introduce the forms

$$\Theta^1 := \hat\theta^1, \qquad \Theta^2 := \hat\theta^2 \quad \text{and} \quad \Theta^3 := \hat\theta^{\bar 3}, \tag{2.29}$$

which are forms of type $(1,0)$ with respect to an almost complex structure $\mathcal{J}_-$ [27], $\mathcal{J}_-\Theta^i = \mathrm{i}\Theta^i$, which is a never integrable almost complex structure. For Θ^i with the rescaling (2.26) we have

$$\mathrm{d}\begin{pmatrix}\Theta^1\\ \Theta^2\\ \Theta^3\end{pmatrix} + \begin{pmatrix}-\hat\alpha_+ - \alpha_- & \beta_- & 0\\ -\bar\beta_- & -\hat\alpha_+ + \alpha_- & 0\\ 0 & 0 & 2\hat\alpha_+\end{pmatrix}\wedge\begin{pmatrix}\Theta^1\\ \Theta^2\\ \Theta^3\end{pmatrix} = \frac{1}{2R}\begin{pmatrix}\Theta^{\bar 2}\wedge\Theta^{\bar 3}\\ \Theta^{\bar 3}\wedge\Theta^{\bar 1}\\ \frac{2R^2}{\Lambda^2}\Theta^{\bar 1}\wedge\Theta^{\bar 2}\end{pmatrix}. \tag{2.30}$$

The manifold $(\mathcal{M}, \mathcal{J}_-)$ is a quasi-Kähler manifold. Recall that an almost Hermitian $2n$-manifold with the fundamental $(1,1)$-form ω is called quasi-Kähler if only $(3,0)+(0,3)$ components of $\mathrm{d}\omega$ are non-vanishing [12,25]. In our case

$$\omega := \frac{\mathrm{i}}{2}\left(\Theta^1\wedge\Theta^{\bar 1} + \Theta^2\wedge\Theta^{\bar 2} + \Theta^3\wedge\Theta^{\bar 3}\right). \tag{2.31}$$

One can check that for arbitrary ratio Λ/R the $(1,2)$ part of $\mathrm{d}\omega$ vanishes and therefore $\mathcal{M}$ is quasi-Kähler [24,27].

From (2.30) one sees that the manifold $\mathcal{M} = \mathrm{Sp}(2)/\mathrm{Sp}(1)\times\mathrm{U}(1)$ with an almost complex structure $\mathcal{J}_-$ becomes a nearly Kähler manifold if $\Lambda^2 = 2R^2$. Recall that a six-manifold is called nearly Kähler if [12,24,25]

$$\mathrm{d}\omega = 3\rho\,\mathrm{Im}\,\Omega \quad \text{for } \Omega := \Theta^1\wedge\Theta^2\wedge\Theta^3 \quad \text{and} \quad \mathrm{d}\Omega = 2\rho\,\omega\wedge\omega, \tag{2.32}$$

where $\rho\in\mathbb{R}$ is proportional to the inverse "radius" $\Lambda = \sqrt{2}R$ of $\mathcal{M}$.

3. Twistor spaces of $\mathbb{C}P^3$

Here we describe the geometry of the twistor spaces for the cosets SU(4)/U(3) and Sp(2)/Sp(1) × U(1) by using the same approach as in Section 2. First, we choose a coset representative $V\in\mathrm{SU}(3)$ of $\mathbb{C}P^2 = \mathrm{SU}(3)/\mathrm{U}(2)$, introduce a coset representative $\tilde Q = \hat Q\tilde V \in \mathrm{SU}(4)$ of $\mathrm{SU}(4)/\mathrm{U}(2)\times\mathrm{U}(1)$ and derive the Maurer–Cartan relations (3.18) for $(1,0)$-forms $\tilde\theta^a$ on the twistor space $\mathrm{SU}(4)/\mathrm{U}(2)\times\mathrm{U}(1)$ of $\mathbb{C}P^3$. Then we do the same for the twistor space $\mathrm{Sp}(2)/\mathrm{U}(1)\times\mathrm{U}(1)$ of the coset $\mathrm{Sp}(2)/\mathrm{Sp}(1)\times\mathrm{U}(1)\cong\mathbb{C}P^3$. Namely, we choose a representative $\breve Q$ of the coset $\mathrm{Sp}(2)/\mathrm{U}(1)\times\mathrm{U}(1)$, construct $(1,0)$-forms $\breve\theta^a$ on it via expanding the flat connection $\mathcal{A}_0' = \breve Q^{-1}\mathrm{d}\breve Q$ into an $sp(2)$-basis and finally derive the Maurer–Cartan equations (3.35) for $\breve\theta^a$.

Coset representation of $\mathbb{C}P^2$. Let us consider the projection

$$\mathrm{SU}(3)\to\mathrm{SU}(3)/\mathrm{U}(2) = \mathbb{C}P^2. \tag{3.1}$$

One can choose as a coset representative of $\mathbb{C}P^2$ a local section of the bundle (3.1) given by the matrix

$$V = \frac{1}{\gamma}\begin{pmatrix}1 & Y^\dagger\\ -Y & W\end{pmatrix} := \frac{1}{\gamma}\begin{pmatrix}1 & \bar\lambda^{\bar 1} & \bar\lambda^{\bar 2}\\ -\lambda^1 & W_{11} & W_{12}\\ -\lambda^2 & W_{21} & W_{22}\end{pmatrix}\in\mathrm{SU}(3), \tag{3.2}$$

where

$$\gamma^2 := 1 + Y^\dagger Y = 1 + \lambda^1\bar\lambda^{\bar 1} + \lambda^2\bar\lambda^{\bar 2} \quad \text{and} \quad W = W^\dagger = \gamma \cdot \mathbf{1}_2 - \frac{1}{\gamma+1} Y Y^\dagger. \tag{3.3}$$

Here λ^1 and λ^2 are local complex coordinates on a patch of $\mathbb{C}P^2$. From (3.2) and (3.3) it is easy to see that

$$WY = Y \quad \text{and} \quad W^2 = \gamma^2 - YY^\dagger \quad \Leftrightarrow \quad V^\dagger V = \mathbf{1}_3 = VV^\dagger. \tag{3.4}$$

Twistor space of SU(4)/U(3). Consider the coset space

$$\mathcal{Z} := \mathrm{SU}(4)/\mathrm{U}(2) \times \mathrm{U}(1) \tag{3.5}$$

and the projection

$$\pi : \mathrm{SU}(4)/\mathrm{U}(2) \times \mathrm{U}(1) \to \mathrm{SU}(4)/\mathrm{U}(3) \cong \mathbb{C}P^3 \tag{3.6}$$

with fibres $\mathbb{C}P^2$. Using the group element (3.2) to parametrize the typical $\mathbb{C}P^2$-fibre in (3.6), we introduce a flat connection $\tilde{\mathcal{A}}_0$ on the trivial bundle $\mathcal{Z} \times \mathbb{C}^4 \to \mathcal{Z}$ as

$$\tilde{\mathcal{A}}_0 = \tilde{Q}^{-1}\,\mathrm{d}\tilde{Q} = \tilde{V}^\dagger \hat{\mathcal{A}}_0 \tilde{V} + \tilde{V}^\dagger\,\mathrm{d}\tilde{V}, \tag{3.7}$$

where

$$\tilde{Q} = \hat{Q}\tilde{V} \in \mathrm{SU}(4) \quad \text{and} \quad \tilde{V} := \begin{pmatrix} V & 0 \\ 0 & 1 \end{pmatrix} \quad \text{with } V \in \mathrm{SU}(3). \tag{3.8}$$

The flat connection $\hat{\mathcal{A}}_0$ is given in (2.20) but here we write it as

$$\hat{\mathcal{A}}_0 = \begin{pmatrix} \alpha_- & -\bar\beta_- & -\hat\theta^{\bar 2} & -\hat\theta^{\bar 1} \\ \beta_- & -\alpha_- & \hat\theta^1 & -\hat\theta^{\bar 2} \\ \hat\theta^{\bar 2} & -\hat\theta^{\bar 1} & \hat\alpha_+ & -\hat\theta^{\bar 3} \\ \hat\theta^1 & \hat\theta^2 & \hat\theta^3 & -\hat\alpha_+ \end{pmatrix} =: \begin{pmatrix} B & -T \\ T^\dagger & -\hat\alpha_+ \end{pmatrix}, \tag{3.9}$$

where

$$B = \begin{pmatrix} \alpha_- & -\bar\beta_- & -\hat\theta^{\bar 2} \\ \beta_- & -\alpha_- & \hat\theta^1 \\ \hat\theta^{\bar 2} & -\hat\theta^{\bar 1} & \hat\alpha_+ \end{pmatrix}, \qquad T := \begin{pmatrix} \hat\theta^{\bar 1} \\ \hat\theta^{\bar 2} \\ \hat\theta^{\bar 3} \end{pmatrix} \quad \text{and} \quad T^\dagger = \left(\hat\theta^1 \hat\theta^2 \hat\theta^3\right). \tag{3.10}$$

Using (3.7), we obtain the connection

$$\tilde{\mathcal{A}}_0 = \begin{pmatrix} V^\dagger B V + V^\dagger \mathrm{d}V & -V^\dagger T \\ T^\dagger V & -\hat\alpha_+ \end{pmatrix} =: \begin{pmatrix} \tilde{B} & -\tilde{T} \\ \tilde{T}^\dagger & -\hat\alpha_+ \end{pmatrix} \quad \text{with } \tilde{T} = \begin{pmatrix} \tilde\theta^{\bar 1} \\ \tilde\theta^{\bar 2} \\ \tilde\theta^{\bar 3} \end{pmatrix} \tag{3.11}$$

and for the curvature $\tilde{\mathcal{F}}_0 = \mathrm{d}\tilde{\mathcal{A}}_0 + \tilde{\mathcal{A}}_0 \wedge \tilde{\mathcal{A}}_0$ we get

$$\tilde{\mathcal{F}}_0 = \begin{pmatrix} \mathrm{d}\tilde{B} + \tilde{B}\wedge\tilde{B} - \tilde{T}\wedge\tilde{T}^\dagger & -\mathrm{d}\tilde{T} - (\tilde{B} + \hat\alpha_+ \cdot \mathbf{1}_3)\wedge\tilde{T} \\ \mathrm{d}\tilde{T}^\dagger + \tilde{T}^\dagger \wedge (\tilde{B} + \hat\alpha_+ \cdot \mathbf{1}_3) & -\mathrm{d}\hat\alpha_+ - \tilde{T}^\dagger\wedge\tilde{T} \end{pmatrix}. \tag{3.12}$$

We have

$$\tilde{B} = V^\dagger B V + V^\dagger\,\mathrm{d}V =: \begin{pmatrix} \tilde\alpha_- & \Upsilon^\dagger \\ -\Upsilon & \Sigma \end{pmatrix} \tag{3.13}$$

with

$$\Sigma =: \begin{pmatrix} \tilde{a} - \tilde{\alpha}_- & -\bar{b} \\ b & -\tilde{a} + \hat{\alpha}_+ \end{pmatrix} \quad \text{and} \quad \Upsilon =: \begin{pmatrix} \tilde{\theta}^4 \\ \tilde{\theta}^5 \end{pmatrix}. \tag{3.14}$$

Flatness $\tilde{\mathcal{F}}_0 = 0$ of the connection (3.11) yields

$$\mathrm{d}\begin{pmatrix} \tilde{\theta}^1 \\ \tilde{\theta}^2 \\ \tilde{\theta}^3 \end{pmatrix} + \begin{pmatrix} -\tilde{\alpha}_- - \hat{\alpha}_+ & 0 & 0 \\ 0 & -\tilde{a} + \tilde{\alpha}_- - \hat{\alpha}_+ & -b \\ 0 & \bar{b} & \tilde{a} - 2\hat{\alpha}_+ \end{pmatrix} \wedge \begin{pmatrix} \tilde{\theta}^1 \\ \tilde{\theta}^2 \\ \tilde{\theta}^3 \end{pmatrix} = \begin{pmatrix} \tilde{\theta}^{24} + \tilde{\theta}^{35} \\ -\tilde{\theta}^{1\bar{4}} \\ -\tilde{\theta}^{1\bar{5}} \end{pmatrix}. \tag{3.15}$$

From

$$\mathrm{d}\tilde{B} + \tilde{B} \wedge \tilde{B} - \tilde{T} \wedge \tilde{T}^\dagger = 0 \tag{3.16}$$

it follows that

$$\mathrm{d}\begin{pmatrix} \tilde{\theta}^4 \\ \tilde{\theta}^5 \end{pmatrix} + \begin{pmatrix} \tilde{a} - 2\tilde{\alpha}_- & -\bar{b} \\ b & -\tilde{a} + \hat{\alpha}_+ - \tilde{\alpha}_- \end{pmatrix} \wedge \begin{pmatrix} \tilde{\theta}^4 \\ \tilde{\theta}^5 \end{pmatrix} = \begin{pmatrix} \tilde{\theta}^{1\bar{2}} \\ \tilde{\theta}^{1\bar{3}} \end{pmatrix}. \tag{3.17}$$

We obtain

$$\mathrm{d}\begin{pmatrix} \tilde{\theta}^1 \\ \tilde{\theta}^2 \\ \tilde{\theta}^3 \\ \tilde{\theta}^4 \\ \tilde{\theta}^5 \end{pmatrix} + \begin{pmatrix} -\tilde{\alpha}_- - \hat{\alpha}_+ & 0 & 0 & 0 & 0 \\ 0 & -\tilde{a} + \tilde{\alpha}_- - \hat{\alpha}_+ & -b & 0 & 0 \\ 0 & \bar{b} & \tilde{a} - 2\hat{\alpha}_+ & 0 & 0 \\ 0 & 0 & 0 & \tilde{a} - 2\tilde{\alpha}_- & -\bar{b} \\ 0 & 0 & 0 & b & -\tilde{a} - \tilde{\alpha}_- + \hat{\alpha}_+ \end{pmatrix}$$
$$\wedge \begin{pmatrix} \tilde{\theta}^1 \\ \tilde{\theta}^2 \\ \tilde{\theta}^3 \\ \tilde{\theta}^4 \\ \tilde{\theta}^5 \end{pmatrix} = \begin{pmatrix} \tilde{\theta}^{24} + \frac{\Lambda}{R}\tilde{\theta}^{35} \\ -\tilde{\theta}^{1\bar{4}} \\ -\frac{R}{\Lambda}\tilde{\theta}^{1\bar{5}} \\ \frac{1}{4\Lambda^2}\tilde{\theta}^{1\bar{2}} \\ \frac{1}{4\Lambda R}\tilde{\theta}^{1\bar{3}} \end{pmatrix}, \tag{3.18}$$

where we rescaled our $\tilde{\theta}^a$ with $a = 1, \ldots, 5$ as in (2.26):

$$\tilde{\theta}^1 \to \frac{1}{2\Lambda}\tilde{\theta}^1, \qquad \tilde{\theta}^2 \to \frac{1}{2\Lambda}\tilde{\theta}^2, \qquad \tilde{\theta}^3 \to \frac{1}{2R}\tilde{\theta}^3, \qquad \tilde{\theta}^4 \to \tilde{\theta}^4 \quad \text{and} \quad \tilde{\theta}^5 \to \tilde{\theta}^5. \tag{3.19}$$

The manifold $\mathrm{SU}(4)/\mathrm{U}(2) \times \mathrm{U}(1)$ is the twistor space for the Kähler space $\mathbb{C}P^3 = \mathrm{SU}(4)/\mathrm{U}(3)$ for $\Lambda^2 = R^2$. Forms $\tilde{\theta}^a$ define on $\mathrm{SU}(4)/\mathrm{U}(2) \times \mathrm{U}(1)$ an integrable almost complex structure $\tilde{\mathcal{J}}_+$ such that

$$\tilde{\mathcal{J}}_+\tilde{\theta}^a = \mathrm{i}\tilde{\theta}^a. \tag{3.20}$$

In the Kähler case we choose $\Lambda = R = \frac{1}{2}$.

Twistor space of $\mathrm{Sp}(2)/\mathrm{Sp}(1) \times \mathrm{U}(1)$. Consider the coset space

$$\mathcal{Z}' := \mathrm{Sp}(2)/\mathrm{U}(1) \times \mathrm{U}(1) \tag{3.21}$$

and the projection

$$\pi' : \mathrm{Sp}(2)/\mathrm{U}(1) \times \mathrm{U}(1) \to \mathrm{Sp}(2)/\mathrm{Sp}(1) \times \mathrm{U}(1) \cong \mathbb{C}P^3 \tag{3.22}$$

with fibres $\mathbb{C}P^1 \cong \mathrm{Sp}(1)/\mathrm{U}(1)$. We choose the group element

$$\hat{g} = \frac{1}{(1+\lambda\bar{\lambda})^{\frac{1}{2}}} \begin{pmatrix} 1 & -\bar{\lambda} \\ \lambda & 1 \end{pmatrix} \in \mathrm{SU}(2) \cong \mathrm{Sp}(1) \tag{3.23}$$

to parametrize the typical $\mathbb{C}P^1$-fibre in (3.22), where λ is a local complex coordinate on the Riemann sphere $\mathbb{C}P^1$. By the formula

$$\hat{g}^{-1}\,\mathrm{d}\hat{g} =: \begin{pmatrix} \hat{a} & -\theta^{\bar{4}} \\ \theta^4 & -\hat{a} \end{pmatrix}, \tag{3.24}$$

where

$$\hat{a} := \frac{1}{2(1+\lambda\bar{\lambda})}(\bar{\lambda}\,\mathrm{d}\lambda - \lambda\,\mathrm{d}\bar{\lambda}), \tag{3.25}$$

we introduce on $\mathbb{C}P^1$ the forms

$$\theta^4 = \frac{\mathrm{d}\lambda}{1+\lambda\bar{\lambda}} \quad \text{and} \quad \theta^{\bar{4}} = \frac{\mathrm{d}\bar{\lambda}}{1+\lambda\bar{\lambda}} \tag{3.26}$$

of type $(1,0)$ and $(0,1)$, respectively.

Using the group element (3.23), we introduce a flat connection $\mathcal{A}'_0$ on the trivial bundle $\mathcal{Z}' \times \mathbb{C}^4 \to \mathcal{Z}'$ as

$$\mathcal{A}'_0 = \breve{Q}^{-1}\,\mathrm{d}\breve{Q} = \hat{G}^\dagger \hat{\mathcal{A}}_0 \hat{G} + \hat{G}^\dagger\,\mathrm{d}\hat{G}, \tag{3.27}$$

where

$$\breve{Q} = \hat{Q}\hat{G} \in \mathrm{Sp}(2) \quad \text{and} \quad \hat{G} := \begin{pmatrix} \hat{g} & 0 \\ 0 & \mathbf{1}_2 \end{pmatrix} \in \mathrm{Sp}(1) \subset \mathrm{Sp}(2). \tag{3.28}$$

The flat connection $\hat{\mathcal{A}}_0$ is given in (2.20) and (3.9). Using (3.27), we obtain the connection

$$\mathcal{A}'_0 = \begin{pmatrix} \hat{g}^\dagger \hat{A}^- \hat{g} + \hat{g}^\dagger\,\mathrm{d}\hat{g} & -\hat{g}^\dagger\hat{\phi} \\ \hat{\phi}^\dagger\hat{g} & \hat{A}^+ \end{pmatrix} =: \begin{pmatrix} \breve{A}^- & -\breve{\phi} \\ \breve{\phi}^\dagger & \breve{A}^+ \end{pmatrix} \tag{3.29}$$

with

$$\breve{\phi} = \hat{g}^\dagger\hat{\phi} = \frac{1}{(1+\lambda\bar{\lambda})^{1/2}} \begin{pmatrix} \hat{\theta}^2 - \bar{\lambda}\hat{\theta}^1 & \hat{\theta}^{\bar{1}} + \bar{\lambda}\hat{\theta}^{\bar{2}} \\ -\hat{\theta}^1 - \lambda\hat{\theta}^2 & \hat{\theta}^{\bar{2}} - \lambda\hat{\theta}^{\bar{1}} \end{pmatrix} =: \begin{pmatrix} \breve{\theta}^2 & \breve{\theta}^{\bar{1}} \\ -\breve{\theta}^1 & \breve{\theta}^{\bar{2}} \end{pmatrix}, \tag{3.30}$$

$$\breve{A}^+ := \begin{pmatrix} \breve{\alpha}_+ & -\breve{\theta}^{\bar{3}} \\ \breve{\theta}^3 & -\breve{\alpha}_+ \end{pmatrix} = \begin{pmatrix} \hat{\alpha}_+ & -\hat{\theta}^{\bar{3}} \\ \hat{\theta}^3 & -\hat{\alpha}_+ \end{pmatrix} = \hat{A}^+ \quad \text{and} \quad \breve{A}^- := \begin{pmatrix} \breve{\alpha}_- & -\breve{\theta}^{\bar{4}} \\ \breve{\theta}^4 & -\breve{\alpha}_- \end{pmatrix}, \tag{3.31}$$

where

$$\breve{\alpha}_- = \frac{1}{1+\lambda\bar{\lambda}}\left\{(1-\lambda\bar{\lambda})\alpha_- + \bar{\lambda}\beta_- - \lambda\bar{\beta}_- + \frac{1}{2}(\bar{\lambda}\,\mathrm{d}\lambda - \lambda\,\mathrm{d}\bar{\lambda})\right\}, \tag{3.32}$$

$$\breve{\theta}^4 = \frac{1}{1+\lambda\bar{\lambda}}\left\{\mathrm{d}\lambda + \beta_- - 2\lambda\alpha_- + \lambda^2\bar{\beta}_-\right\}, \qquad \breve{\theta}^{\bar{4}} := \overline{\breve{\theta}^4}. \tag{3.33}$$

For the curvature $\mathcal{F}'_0 = \mathrm{d}\mathcal{A}'_0 + \mathcal{A}'_0 \wedge \mathcal{A}'_0$ we get

$$\mathcal{F}'_0 = \begin{pmatrix} \mathrm{d}\breve{A}^- + \breve{A}^- \wedge \breve{A}^- - \breve{\phi} \wedge \breve{\phi}^\dagger & -\mathrm{d}\breve{\phi} - \breve{A}^- \wedge \breve{\phi} - \breve{\phi} \wedge \breve{A}^+ \\ \mathrm{d}\breve{\phi}^\dagger + \breve{\phi}^\dagger \wedge \breve{A}^- + \breve{A}^+ \wedge \breve{\phi}^\dagger & \mathrm{d}\breve{A}^+ + \breve{A}^+ \wedge \breve{A}^+ - \breve{\phi}^\dagger \wedge \breve{\phi} \end{pmatrix}. \tag{3.34}$$

From the flatness $\mathcal{F}'_0 = 0$ of the connection (3.29) we obtain the Maurer–Cartan equations

$$\mathrm{d}\begin{pmatrix}\breve{\theta}^1\\ \breve{\theta}^2\\ \breve{\theta}^3\\ \breve{\theta}^4\end{pmatrix}+\begin{pmatrix}-\breve{\alpha}_- - \breve{\alpha}_+ & 0 & 0 & 0\\ 0 & \breve{\alpha}_- - \breve{\alpha}_+ & 0 & 0\\ 0 & 0 & -2\breve{\alpha}_+ & 0\\ 0 & 0 & 0 & -2\breve{\alpha}_-\end{pmatrix}\wedge\begin{pmatrix}\breve{\theta}^1\\ \breve{\theta}^2\\ \breve{\theta}^3\\ \breve{\theta}^4\end{pmatrix}=\begin{pmatrix}-\breve{\theta}^{24}-\breve{\theta}^{3\bar{2}}\\ \breve{\theta}^{3\bar{1}}+\breve{\theta}^{1\bar{4}}\\ 2\breve{\theta}^{12}\\ -2\breve{\theta}^{1\bar{2}}\end{pmatrix}, \tag{3.35}$$

which define the $u(1)\oplus u(1)$ torsionful connection on the twistor space $\mathcal{Z}'=\mathrm{Sp}(2)/\mathrm{U}(1)\times\mathrm{U}(1)$. The forms $\breve{\theta}^a$ in (3.35) with $a=1,\dots,4$ define on $\mathcal{Z}'$ an integrable almost complex structure I'_+ such that

$$I'_+\breve{\theta}^a=\mathrm{i}\breve{\theta}^a. \tag{3.36}$$

Its integrability follows from the vanishing (0, 2)-type components of the torsion on the right hand side of (3.35).

4. Twistor description of instanton bundles over $\mathbb{C}P^3$

Instanton bundles over $\mathbb{C}P^3$. Consider a complex vector bundle $\mathcal{E}$ over $\mathbb{C}P^3$ with a connection one-form $\mathcal{A}$ having the curvature $\mathcal{F}$. Recall that $(\mathcal{E},\mathcal{A})$ is called an instanton bundle if $\mathcal{A}$ satisfies the Hermitian Yang–Mills equations,[1] which on $\mathbb{C}P^3$ can be written in the form

$$\mathcal{F}^{0,2}=0=\mathcal{F}^{2,0}\quad\Leftrightarrow\quad\hat{\Omega}\wedge\mathcal{F}=0, \tag{4.1}$$

$$\hat{\omega}\lrcorner\mathcal{F}=0\quad\Leftrightarrow\quad\hat{\omega}\wedge\hat{\omega}\wedge\mathcal{F}=0, \tag{4.2}$$

where the notation $\hat{\omega}\lrcorner$ exploits the underlying Riemannian metric $g=\delta_{\hat{a}\hat{b}}e^{\hat{a}}e^{\hat{b}}$ on $\mathbb{C}P^3$, $\hat{a},\hat{b},\ldots=1,\dots,6$. Here, $\hat{\omega}$ given in (2.28) is a (1, 1)-form, and $\hat{\Omega}:=\hat{\theta}^1\wedge\hat{\theta}^2\wedge\hat{\theta}^3$ is a locally defined (3, 0)-form on $\mathbb{C}P^3$. Recall that, from the point of view of algebraic geometry, (4.1) means that the bundle $\mathcal{E}\to\mathbb{C}P^3$ is holomorphic and (4.2) means that $\mathcal{E}$ is a polystable vector bundle [17]. In fact, in the right hand side of (4.2) one can add the term $\beta\hat{\omega}\wedge\hat{\omega}\wedge\hat{\omega}$ with β proportional to the first Chern number $c_1(\mathcal{E})$, but we assume $c_1(\mathcal{E})=0$ since for a bundle with field strength $\mathcal{F}$ of non-zero degree one can obtain a degree-zero bundle by considering $\check{\mathcal{F}}=\mathcal{F}-\frac{1}{r}(\mathrm{tr}\mathcal{F})\cdot\mathbf{1}_r$, where $r=\mathrm{rank}\,\mathcal{E}$.

Pull-back to $\mathcal{Z}$. Consider the twistor fibration (3.6). Let $(\tilde{\mathcal{E}},\tilde{\mathcal{A}})=(\pi^*\mathcal{E},\pi^*\mathcal{A})$ be the pulled-back instanton bundle over $\mathcal{Z}$ with the curvature $\tilde{\mathcal{F}}=\mathrm{d}\tilde{\mathcal{A}}+\tilde{\mathcal{A}}\wedge\tilde{\mathcal{A}}$. We have

$$\tilde{\mathcal{F}}=\frac{1}{2}\tilde{\mathcal{F}}_{ab}\tilde{\theta}^a\wedge\tilde{\theta}^b+\tilde{\mathcal{F}}_{a\bar{b}}\tilde{\theta}^a\wedge\tilde{\theta}^{\bar{b}}+\frac{1}{2}\tilde{\mathcal{F}}_{\bar{a}\bar{b}}\tilde{\theta}^{\bar{a}}\wedge\tilde{\theta}^{\bar{b}}=\pi^*\mathcal{F} \tag{4.3}$$

with $a,b,\ldots=1,\dots,5$. Using the relation between $\tilde{\theta}^a$ and $\hat{\theta}^a$ described in Section 3, we obtain

$$\tilde{\mathcal{F}}_{\bar{i}\bar{j}}=C_{\bar{i}}^{\bar{k}}C_{\bar{j}}^{\bar{l}}\mathcal{F}_{\bar{k}\bar{l}}\quad\text{and}\quad\tilde{\mathcal{F}}_{i\bar{j}}=\bar{C}_i^kC_{\bar{j}}^{\bar{l}}\mathcal{F}_{k\bar{l}}, \tag{4.4}$$

where $C=\bar{V}^\dagger$ with

$$C_{\bar{1}}^{\bar{1}}=\frac{1}{\gamma},\qquad C_{\bar{2}}^{\bar{1}}=-\frac{\lambda^1}{\gamma},\qquad C_{\bar{3}}^{\bar{1}}=-\frac{\lambda^2}{\gamma},$$

$$C_{\bar{1}}^{\bar{2}}=\frac{\bar{\lambda}^{\bar{1}}}{\gamma},\qquad C_{\bar{2}}^{\bar{2}}=\frac{\gamma+1+\lambda^2\bar{\lambda}^{\bar{2}}}{\gamma(\gamma+1)},\qquad C_{\bar{3}}^{\bar{2}}=-\frac{\lambda^2\bar{\lambda}^{\bar{1}}}{\gamma(\gamma+1)},$$

[1] These equations are also called the Donaldson–Uhlenbeck–Yau equations.

$$C^{\bar 3}_{\bar 1}=\frac{\bar\lambda^{\bar 2}}{\gamma},\qquad C^{\bar 3}_{\bar 2}=-\frac{\lambda^1\bar\lambda^{\bar 2}}{\gamma(\gamma+1)},\qquad C^{\bar 3}_{\bar 3}=\frac{\gamma+1+\lambda^1\bar\lambda^{\bar 1}}{\gamma(\gamma+1)},\tag{4.5}$$

and $\bar C$ is the complex conjugate matrix. Thus, more explicitly, we get

$$\tilde{\mathcal F}_{\bar 1\bar 2}=\frac{1}{\gamma}\left\{\frac{\gamma+1+\lambda^1\bar\lambda^{\bar 1}}{\gamma+1}\mathcal F_{\bar 1\bar 2}-\frac{\lambda^1\bar\lambda^{\bar 2}}{\gamma+1}\mathcal F_{\bar 3\bar 1}-\bar\lambda^{\bar 2}\mathcal F_{\bar 2\bar 3}\right\},\tag{4.6}$$

$$\tilde{\mathcal F}_{\bar 3\bar 1}=\frac{1}{\gamma}\left\{\frac{\gamma+1+\lambda^2\bar\lambda^{\bar 2}}{\gamma+1}\mathcal F_{\bar 3\bar 1}-\frac{\lambda^2\bar\lambda^{\bar 1}}{\gamma+1}\mathcal F_{\bar 1\bar 2}-\bar\lambda^{\bar 1}\mathcal F_{\bar 2\bar 3}\right\},\tag{4.7}$$

$$\tilde{\mathcal F}_{\bar 2\bar 3}=\frac{1}{\gamma}\left\{\mathcal F_{\bar 2\bar 3}+\lambda^1\mathcal F_{\bar 3\bar 1}+\lambda^2\mathcal F_{\bar 1\bar 2}\right\},\tag{4.8}$$

$$\tilde{\mathcal F}_{\bar\imath\bar 4}=\tilde{\mathcal F}_{\bar\imath\bar 5}=0,\tag{4.9}$$

$$\tilde{\mathcal F}_{1\bar 1}+\tilde{\mathcal F}_{2\bar 2}+\tilde{\mathcal F}_{3\bar 3}+\tilde{\mathcal F}_{4\bar 4}+\tilde{\mathcal F}_{5\bar 5}=\mathcal F_{1\bar 1}+\mathcal F_{2\bar 2}+\mathcal F_{3\bar 3}.\tag{4.10}$$

The vanishing of $\tilde{\mathcal F}_{\bar 2\bar 3}$ for all values of $(\lambda^1,\lambda^2)\in\mathbb CP^2$ is equivalent to the holomorphicity equation (4.1). In homogeneous coordinates y^i on $\mathbb CP^2$ ($\lambda^1=y^2/y^1$, $\lambda^2=y^3/y^1$, $y^1\neq 0$), this condition can be written as

$$\tilde{\mathcal F}_{\bar 2\bar 3}=0\quad\Leftrightarrow\quad y^i\varepsilon_{ijk}\mathcal F^{jk}=0,\tag{4.11}$$

where the indices $\bar\imath,\bar\jmath,\dots$ are raised with the metric $\delta^{i\bar\jmath}$. From (4.6)–(4.10) we see that solutions $\mathcal A$ of the HYM equations (4.1), (4.2) on $\mathbb CP^3$ correspond to solutions $\tilde{\mathcal A}=\pi^*\mathcal A$ of the HYM equations on the twistor space $\mathcal Z$ of $\mathbb CP^3$, and $\tilde{\mathcal A}$ are flat connections along fibres $\mathbb CP^2_x\hookrightarrow\mathcal Z$. In other words, from (4.6)–(4.9) we see that the bundle $\tilde{\mathcal E}$ is holomorphic for holomorphic $\mathcal E$ as well as polystable due to (4.2), (4.10), and it is holomorphically trivial after restricting to the fibres $\mathbb CP^2_x\hookrightarrow\mathcal Z$ of the projection π for each $x\in\mathbb CP^3$. Vice versa, polystable holomorphic bundles over $\mathcal Z$ trivial on any fibre $\mathbb CP^2_x\hookrightarrow\mathcal Z$ over $\mathbb CP^3$ correspond to solutions $\mathcal A$ of the HYM equations on $\mathbb CP^3$. The only difference from the canonical twistor correspondence is that the bundle $\tilde{\mathcal E}$ is not only holomorphic[2] but also polystable, which is equivalent to imposing on $\tilde{\mathcal A}$ the additional equation

$$\tilde{\mathcal F}_{1\bar 1}+\tilde{\mathcal F}_{2\bar 2}+\tilde{\mathcal F}_{3\bar 3}+\tilde{\mathcal F}_{4\bar 4}+\tilde{\mathcal F}_{5\bar 5}=0.\tag{4.12}$$

Hence, the twistor transform does not help in solving the instanton equations in six dimensions.

Pull-back to $\mathcal Z'$. Consider now the twistor fibration (3.22) and the pulled-back instanton bundle $(\mathcal E',\mathcal A')=(\pi'^*\mathcal E,\pi'^*\mathcal A)$ over $\mathcal Z'$ with the curvature $\mathcal F'=\mathrm d\mathcal A'+\mathcal A'\wedge\mathcal A'$. We again have the relation (4.3) with $a,b,\dots=1,\dots,4$. For the matrix C in (4.4) we now find

$$C=\begin{pmatrix}\varkappa & \varkappa\lambda & 0\\ -\varkappa\bar\lambda & \varkappa & 0\\ 0 & 0 & 1\end{pmatrix}\quad\text{with }\varkappa=(1+\lambda\bar\lambda)^{-\frac12},\tag{4.13}$$

where λ is a local complex coordinate on $\mathbb CP^1$ used in (3.23)–(3.26).

Using (4.13), we obtain

[2] Meaning it is defined by the equation $\bar\partial^2_{\tilde{\mathcal A}}=0$ of holomorphic Chern–Simons theory for $\tilde{\mathcal A}$.

$$\mathcal{F}'_{1\bar{2}} = \mathcal{F}_{1\bar{2}}, \qquad \mathcal{F}'_{3\bar{1}} = \varkappa(\mathcal{F}_{3\bar{1}} + \bar{\lambda}\mathcal{F}_{\bar{2}\bar{3}}), \qquad \mathcal{F}'_{\bar{2}\bar{3}} = \varkappa(\mathcal{F}_{\bar{2}\bar{3}} - \lambda\mathcal{F}_{3\bar{1}}), \qquad \mathcal{F}'_{\bar{\imath}\bar{4}} = 0, \tag{4.14}$$

$$\mathcal{F}'_{1\bar{1}} + \mathcal{F}'_{2\bar{2}} + \mathcal{F}'_{3\bar{3}} + \mathcal{F}'_{4\bar{4}} = \mathcal{F}_{1\bar{1}} + \mathcal{F}_{2\bar{2}} + \mathcal{F}_{3\bar{3}}. \tag{4.15}$$

Therefore, instanton bundles $(\mathcal{E}, \mathcal{A})$ over the nonsymmetric Kähler coset space $\mathrm{Sp}(2)/\mathrm{Sp}(1) \times \mathrm{U}(1) \cong \mathbb{C}P^3$ are pulled back to holomorphic polystable bundles $(\mathcal{E}', \mathcal{A}')$ over the complex twistor space $\mathcal{Z}' = \mathrm{Sp}(2)/\mathrm{U}(1) \times \mathrm{U}(1)$. Furthermore, $\mathcal{E}'$ is flat along the fibres $\mathbb{C}P^1_x$ of the bundle (3.22), and one can set the components of $\mathcal{A}'$ along the fibres equal to zero. Thus, the restrictions of the vector bundle $\mathcal{E}'$ to fibres $\mathbb{C}P^1_x \hookrightarrow \mathcal{Z}'$ of the projection π' are holomorphically trivial for each $x \in \mathrm{Sp}(2)/\mathrm{Sp}(1) \times \mathrm{U}(1) \cong \mathbb{C}P^3$. Note that (4.14) and (4.15) can be obtained from (4.6)–(4.10) by putting $\lambda^1 = -\lambda$ and $\lambda^2 = 0$. Then (3.11) will coincide with (3.29) after the substitution $\tilde{\theta}^{\bar{4}} \to -\breve{\theta}^{\bar{4}}$, $\tilde{\theta}^{\bar{5}} \to -\breve{\theta}^{2}$, $b \to -\breve{\theta}^{\bar{1}}$, etc. This correspondence follows from the fact that $\mathcal{Z}'$ is a complex (codimension one) submanifold of the twistor space $\mathcal{Z}$.

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References

[1] M.F. Atiyah, N.J. Hitchin, I.M. Singer, Self-duality in four-dimensional Riemannian geometry, Proc. R. Soc. Lond. A 362 (1978) 425.
[2] R. Penrose, Nonlinear gravitons and curved twistor theory, Gen. Relativ. Gravit. 7 (1976) 31.
[3] R.S. Ward, On self-dual gauge fields, Phys. Lett. A 61 (1977) 81;
M.F. Atiyah, R.S. Ward, Instantons and algebraic geometry, Commun. Math. Phys. 55 (1977) 117.
[4] R. Penrose, The twistor program, Rep. Math. Phys. 12 (1977) 65;
R.O. Wells, Complex manifolds and mathematical physics, Bull. Am. Math. Soc. 1 (1979) 296.
[5] C. Sämann, R. Wimmer, M. Wolf, A twistor description of six-dimensional $\mathcal{N} = (1,1)$ super Yang–Mills theory, J. High Energy Phys. 1205 (2012) 20, arXiv:1201.6285 [hep-th].
[6] C. Sämann, M. Wolf, On twistors and conformal field theories from six dimensions, J. Math. Phys. 54 (2013) 013507, arXiv:1111.2539 [hep-th].
[7] L.J. Mason, R.A. Reid-Edwards, A. Taghavi-Chabert, Conformal field theories in six-dimensional twistor space, J. Geom. Phys. 62 (2012) 2353, arXiv:1111.2585 [hep-th].
[8] C. Sämann, M. Wolf, Non-Abelian tensor multiplet equations from twistor space, arXiv:1205.3108 [hep-th].
[9] G.W. Moore, Applications of the six-dimensional $(2,0)$ theories to physical mathematics, in: Lecture Notes for Felix Klein Lectures, Bonn, 1–11 October, 2012, 2012, 210 pp.
[10] L. Berard Bergery, T. Ochiai, On some generalization of the construction of twistor spaces, in: T.J. Willmore, N.J. Hitchin (Eds.), Global Riemann. Geom. Symp, Ellis Horwood, 1984;
N.R. O'Brian, J.H. Rawnsley, Twistor spaces, Ann. Glob. Anal. Geom. 3 (1985) 29;
F.E. Burstall, Riemannian twistor spaces and holonomy groups, in: T.N. Bailey, R.J. Baston (Eds.), Twistors in Mathematics and Physics, in: L.M.S. Lect. Notes, vol. 156, Cambridge University Press, 1990.
[11] R.L. Bryant, Submanifolds and special structures on the octonians, J. Differ. Geom. 17 (1982) 185;
R.L. Bryant, Lie groups and twistor spaces, Duke Math. J. 52 (1985) 223.
[12] S. Salamon, Harmonic and holomorphic maps, Lect. Notes Math. 1164 (1985) 161.
[13] D.V. Alekseevsky, M.M. Graev, G-structures of twistor type and their twistor spaces, J. Geom. Phys. 3 (1993) 203.
[14] J.-B. Butruille, Twistors and 3-symmetric spaces, Proc. Lond. Math. Soc. 96 (2008) 738, arXiv:math/0604394.
[15] O. Lechtenfeld, A.D. Popov, Instantons on the six-sphere and twistors, J. Math. Phys. 53 (2012) 123506, arXiv:1206.4128 [hep-th].
[16] E. Corrigan, C. Devchand, D.B. Fairlie, J. Nuyts, First order equations for gauge fields in spaces of dimension greater than four, Nucl. Phys. B 214 (1983) 452;
R.S. Ward, Completely solvable gauge field equations in dimension greater than four, Nucl. Phys. B 236 (1984) 381.

[17] S.K. Donaldson, Anti-self-dual Yang–Mills connections on a complex algebraic surface and stable vector bundles, Proc. Lond. Math. Soc. 50 (1985) 1;
K.K. Uhlenbeck, S.-T. Yau, On the existence of Hermitian–Yang–Mills connections on stable bundles over compact Kähler manifolds, Commun. Pure Appl. Math. 39 (1986) 257.
[18] M. Mamone Capria, S.M. Salamon, Yang–Mills fields on quaternionic spaces, Nonlinearity 1 (1988) 517;
T.A. Ivanova, A.D. Popov, (Anti)self-dual gauge fields in dimension $d \geqslant 4$, Theor. Math. Phys. 94 (1993) 225;
R. Reyes Carrión, A generalization of the notion of instanton, Differ. Geom. Appl. 8 (1998) 1.
[19] S.K. Donaldson, R.P. Thomas, Gauge theory in higher dimensions, in: S.A. Huggett, et al. (Eds.), The Geometric Universe, Oxford University Press, Oxford, 1998;
S.K. Donaldson, E. Segal, Gauge theory in higher dimensions II, in: N.C. Leung, S.-T. Yau (Eds.), Surv. Differ. Geom., vol. 16, International Press, Boston, 2011, arXiv:0902.3239 [math.DG].
[20] G. Tian, Gauge theory and calibrated geometry, Ann. Math. 151 (2000) 193, arXiv:math/0010015.
[21] M. Wolf, A connection between twistors and superstring sigma models on coset superspaces, J. High Energy Phys. 0909 (2009) 071, arXiv:0907.3862 [hep-th];
M. Wolf, Contact manifolds, contact instantons, and twistor geometry, J. High Energy Phys. 1207 (2012) 074, arXiv:1203.3423 [hep-th].
[22] A.D. Popov, Non-Abelian vortices, super-Yang–Mills theory and Spin(7)-instantons, Lett. Math. Phys. 92 (2010) 253, arXiv:0908.3055 [hep-th];
A.D. Popov, R.J. Szabo, Double quiver gauge theory and nearly Kähler flux compactifications, J. High Energy Phys. 1202 (2012) 033, arXiv:1009.3208 [hep-th].
[23] D. Harland, C. Nölle, Instantons and Killing spinors, J. High Energy Phys. 1203 (2012) 082, arXiv:1109.3552 [hep-th];
T.A. Ivanova, A.D. Popov, Instantons on special holonomy manifolds, Phys. Rev. D 85 (2012) 105012, arXiv:1203.2657 [hep-th].
[24] S. Chiossi, S. Salamon, The intrinsic torsion of SU(3) and G_2 structures, in: O. Gil-Medrano, V. Miquel (Eds.), Differential Geometry, Valencia, 2001, World Sci. Publ., River Edge, NJ, 2002, arXiv:math/0202282.
[25] J.-B. Butruille, Homogeneous nearly Kähler manifolds, Ann. Glob. Anal. Geom. 27 (2005) 201, arXiv:math/0612655.
[26] R.L. Bryant, Remarks on the geometry of almost complex 6-manifolds, Asian J. Math. 10 (2006) 561, arXiv:math/0508428.
[27] J. Eells, S. Salamon, Constructions twistorielles des applications harmoniques, C.R. Acad. Sci. Paris 296 (1983) 685.