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Optimal detection of two counterfeit coins with two-arms balance ☆

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Abstract

We consider the following *coin-weighing problem*: suppose among the given n coins there are two counterfeit coins, which are either heavier or lighter than other $n-2$ good coins, this is not known beforehand. The weighing device is a two-arms balance. Let $\mathscr{N}_{\mathscr{A}}(k)$ be the number of coins from which k weighings suffice to identify the two counterfeit coins by algorithm $\mathscr{A}$ and $U(k)=\max\{n \mid n(n-1) \leqslant 3^k\}$ be the *information-theoretic upper bound* of the number of coins then $\mathscr{N}_{\mathscr{A}}(k) \leqslant U(k)$. We establish a new method of reducing the above original problem to another identity problem of more simple configurations. It is proved that the information-theoretic upper bound $U(k)$ are always achievable for all even integer $k \geqslant 1$. For odd integer $k \geqslant 1$, our general results can be used to approximate arbitrarily the information-theoretic upper bound. The ideas and techniques of this paper can be easily employed to settle other models of two counterfeit coins.

Keywords: Combinatorial search theory; Worst-case sequential algorithm; Two counterfeit coins problem; Information-theoretic bounds

1. Introduction

The problem of locating d counterfeit coins out of a set of n coins, $n-d$ of which are good coins having the same weight, is a classical problem in the area of Combinatorial

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Search Theory and has received considerable attention. Many papers have been devoted to it and two classes of weighing device, test-type device and comparison-type device, have been considered. Du [5] and Hwang [8] contain many results on various models of the test-type device. The most popular comparison-type device is the *two-arms balance scale* with which to compare the weights of two equally sized subsets of coins. The aim is to find an optimal algorithm which identifies these d counterfeit coins using as few weighings as possible (optimal identity problem). Two measures are commonly utilized to estimate the efficiency of an algorithm: The worst-case number of weighings and the average number of weighings needed to locate these d counterfeit coins. Moreover, two classes of algorithms are usually considered: *Sequential* (*adaptive*) *algorithm* and *predetermined* (*non-adaptive*) *algorithm* (For more details, see [1]).

Some better results on case $d=1$ have been obtained by many authors [1,3,4,10,11,13]. Case $d=2$ yields the following five models under the assumptions: All good coins are of the same weight; if two counterfeit coins are heavier (or lighter) than good coin then they are of the same weight; if one counterfeit coin is heavier and another coin is lighter then $w_{\rm h}+w_{\rm l}=2w_{\rm g}$, where $w_{\rm h}, w_{\rm l}, w_{\rm g}$ denotes the weight of heavier, lighter, good coin respectively. We give each model a notation for easy reference.

$G_{\rm hh}(n)$: G has two heavier counterfeit coins.

$G_{\rm hl}(n)$: G has two counterfeit coins. One is heavier, another is lighter.

$G_{\rm hh,ll}(n)$: G has either two heavier or two lighter counterfeit coins, this is not known beforehand.

$G_{\rm hh,hl}(n)$: G has two heavier counterfeit coins or G has one heavier and one lighter counterfeit coin.

$G_{[2]}(n)$: G has two counterfeit coins. No information on the weights of counterfeit coins beforehand.

A model is said to be symmetric to another if it can be obtained from the other by interchanging h and l. Clearly, all results for one model can be translated into results for the other. Therefore, we need only to discuss one model from every pair of symmetric model. In this sense we can ignore models $G_{\rm ll}(n)$ and $G_{\rm ll,hl}(n)$ since they are symmetric to $G_{\rm hh}(n)$ and $G_{\rm hh,hl}(n)$, respectively. We can list the information-theoretic lower bound $\mathscr{L}(M)$ of the number of weighings for model M:

$$\begin{array}{lllll} \text{Model} & G_{\rm hh}(n) & G_{\rm hl}(n), G_{\rm hh,ll}(n) & G_{\rm hh,hl}(n) & G_{[2]}(n) \\ \mathscr{L}(M) & \lceil \log_3 \frac{n(n-1)}{2} \rceil & \lceil \log_3\{n(n-1)\} \rceil & \lceil \log_3 \frac{3n(n-1)}{2} \rceil & \lceil \log_3\{2n(n-1)\} \rceil \end{array}. \tag{1}$$

Let $\mathscr{W}_{\mathscr{A}}(M)$ be the number of weighings of algorithm $\mathscr{A}$ finding the solution of M, $\mathscr{W}(M)=\min_{\mathscr{A}}\{\mathscr{W}_{\mathscr{A}}(M)\}$ be the least number of weighings required to find the solution of the model M, it is evident that

$$\mathscr{L}(M) \leqslant \mathscr{W}(M) \leqslant \mathscr{W}_{\mathscr{A}}(M). \tag{2}$$

One is concerned with the question: *Is the information-theoretic lower bound $\mathscr{L}(M)$ achievable*? i.e., $\mathscr{W}(M)=\mathscr{L}(M)$? It is useful to introduce another *criteria of the number of coins* which is equivalent to the above *criteria of the number of weighings* but enlarges its details. Let $\mathscr{N}_{\mathscr{A}}(M;k)$ be the number of coins from which k weighings suffice to identify the solution of M by algorithm $\mathscr{A}$, $\mathscr{N}(M;k)=\max_{\mathscr{A}}\{\mathscr{N}_{\mathscr{A}}(M;k)\}$, $U(M;k)$ be the *information-theoretic upper bound* of coin number thus for $k \geqslant 1$,

$$\mathscr{N}_{\mathscr{A}}(M;k) \leqslant \mathscr{N}(M;k) \leqslant U(M;k). \tag{3}$$

Similarly, one is concerned with the question: *Is the information-theoretic upper bound* $U(M;k)$ *achievable for* $k \geqslant 1$? i.e., $\mathscr{N}(M;k)=U(M;k)$ *for* $k \geqslant 1$?

Many papers have been devoted to the model $G_{\mathrm{hh}}(n)$. Bellmam and Glass [2] studied this model and wrote "A small amount of analysis discloses the enormous difference in complexity between the one- and two-coin problem". It is well-known that $\mathscr{L}(G_{\mathrm{hh}}(n)) \leqslant \mathscr{W}(G_{\mathrm{hh}}(n)) \leqslant \mathscr{L}(G_{\mathrm{hh}}(n))+1$. Guy and Nowakowski [7] proposed the question: *In which cases is* $\mathscr{W}(G_{\mathrm{hh}}(n))=\lceil log_3 \frac{n(n-1)}{2}\rceil+1$?, *Is* $n=13$ *the first*? Aigner [1] surmised that $\mathscr{W}(G_{\mathrm{hh}}(n))=\mathscr{L}(G_{\mathrm{hh}}(n))$ for all $n \geqslant 3$. Tošić [12] gave a worst-case sequential algorithm $\mathscr{P}$ such that $\mathscr{N}_{\mathscr{P}}(G_{\mathrm{hh}};2k)=3^k$ and $\mathscr{N}_{\mathscr{P}}(G_{\mathrm{hh}};2k+1)=2\times 3^k$ thus $\mathscr{W}(G_{\mathrm{hh}}(n))=\mathscr{L}(G_{\mathrm{hh}}(n))$ for $n\in[\sqrt{2}\times 3^{\ell}+1, 2\times 3^{\ell}]\cup[\sqrt{6}\times 3^{\ell}+1, 3^{\ell+1}]$. Aigner [1] improved Tošić's result: There exists a worst-case sequential algorithm $\mathscr{P}$ such that $\mathscr{N}_{\mathscr{P}}(G_{\mathrm{hh}};2k)=3^k+3^{k-1}$ and $\mathscr{N}_{\mathscr{P}}(G_{\mathrm{hh}};2k+1)=2\times 3^k+3^{k-1}$. Li [9] proved that $\mathscr{W}(G_{\mathrm{hh}}(n))=\mathscr{L}(G_{\mathrm{hh}}(n))$ for $n\in[3^{\ell}, 12.5\times 3^{\ell-2}]\cup[2\times 3^{\ell}, 21.5\times 3^{\ell-2}]$.

This paper is devoted to the model $G_{\mathrm{hh,ll}}(n)$, which is the generalization of model $G_{\mathrm{h,l}}(n)$ and $G_{\mathrm{hh}}(n)$. The following three theorems summarize the main results of this paper (to simplify the notations, by $U(k)$, $\mathscr{N}_{\mathscr{A}}(k)$, $\mathscr{N}(k)$ we denote $U(G_{\mathrm{hh,ll}};k)$, $\mathscr{N}_{\mathscr{A}}(G_{\mathrm{hh,ll}};k)$, $\mathscr{N}(G_{\mathrm{hh,ll}};k)$ respectively).

Theorem 1. *Let* n_q *be an integer with* $3^{q-1} < n_q^2 \leqslant 3^q$ *and* A, B, C *are sets of* $n_q \geqslant 3$ *coins and pairwise disjoint. If the following conditions hold then* $2K+q$ *weighings can identify the solution of* $S=G_{\mathrm{hh,ll}}(n_q\cdot 3^K)$ *for integer* K *and* n_q.

(1) *q weighings can identify the solution of* $\sigma_1(n_q)=A_{\mathrm{h}}\times B_{\mathrm{h}}$.
(2) *q weighings can split* $\sigma_2(n_q)=A_{\mathrm{h}}\times B_{\mathrm{h}}+A_{\mathrm{l}}\times B_{\mathrm{l}}$ *into configurations with cardinality* $\leqslant 2$.
(3) *q weighings can identify the solution of* $\sigma_3(n_q)=C_{\mathrm{hh,ll}}(n_q)$.
(4) $q+1$ *weighings can identify the solution of* $\sigma_4(n_q)=A_{\mathrm{h}}\times C_{\mathrm{h}}+A_{\mathrm{hh}}+B_{\mathrm{l}}\times C_{\mathrm{l}}+B_{\mathrm{ll}}$.

Theorem 2. *There exists a worst-case sequential algorithm* $\mathscr{A}_1$ *such that* $\mathscr{N}_{\mathscr{A}_1}(2k)=U(2k)$ *for all integer* $k \geqslant 1$. *In other words, the information-theoretic upper bound* $U(k)$ *are achievable for all even integer* $k \geqslant 1$.

Theorem 3. *There exists a worst-case sequential algorithm* $\mathscr{A}_2$ *such that* $\mathscr{N}_{\mathscr{A}_2}(2k+1)=5\times 3^{k-1} < U(2k+1)$ *for all integer* $k \geqslant 1$ *and the gap* $U(2k+1)-\mathscr{N}_{\mathscr{A}_2}(2k+1)$ *is very small.*

Theorem 1 is a more general result and it can be used to approximate arbitrarily the information-theoretic upper bound $U(2k+1)$. By letting $q=2$ and $n_q=3$ ($q=3$ and $n_3=5$) in Theorem 1, Theorem 2 (Theorem 3) is proved by verifying that all conditions of Theorem 1 hold, the corresponding values of $\mathscr{N}(k)$ are listed in the third row of Table 1; The fourth (fifth) row of Table 1 are the corresponding values

Table 1
Comparison of the values of $U(k)$ and $\mathcal{N}(k)$

k	2	3	4	5	6	7	8	9	10	11	12	13	14	15	...
$U(k)$	3	5	9	16	27	47	81	140	243	421	729	1263	2187	3788	...
$\mathcal{N}(k)$	3	5	9	15	27	45	81	135	243	405	729	1215	2187	3645	...
$\mathcal{N}(k)$	3		9		27	46	81	138	243	414	729	1242	2187	3726	...
$\mathcal{N}(k)$	3		9		27		81	140	243	420	729	1260	2187	3780	...

of $\mathcal{N}(k)$ obtained by letting $q=2$ and $n_2=3$, $q=7$ and $n_7=46$ ($q=2$ and $n_2=3$, $q=9$ and $n_9=140$) in Theorem 1. It is evident that $\mathcal{N}(k)$ in the fifth row are closer to $U(k)$ than those in the third and the fourth rows. Verifying the conditions of Theorem 1 is much easier than constructing a whole searching procedure. In this sense, Theorem 1 is more important than those results on some fixed value of n_q. Furthermore, the ideas and techniques of Theorem 1, transforming the original problem into another simple problem, can be used to solve other models ($G_{\text{hh}}(n)$, $G_{\text{hl}}(n)$, $G_{\text{hh,hl}}(n)$ and $G_{[2]}(n)$). Thus, two counterfeit coin problem can be settled in the frame of Theorem 1. The outline of this paper is as follows. Section 2 gives some notations, terminologies and the Structure Theorem, Isomorphism Theorem, Isomorphism Class Theorem of the search domains. Theorems 1–3 are proved in Section 3. Then Section 4 contains the proofs of three key lemmas needed in proving Theorem 1.

2. Structure and isomorphism of search domains

Theorem 4. *Let* $U(k)=\max\{n\,|\,n(n-1)\leqslant 3^k\}$, $D(k)=3^k-U(k)(U(k)-1)$. *We have*
(i) $U(2k)=3^k$ *for integer* $k\geqslant 1$. (ii) *If* k *is an odd positive integer*, *then* $U(1)=2$ *and*

$$U(k+2)=\begin{cases} 3U(k)-1 & if\ \ 0<D(k)<\dfrac{2U(k)}{3},\\[2mm] 3U(k) & if\ \ \dfrac{2U(k)}{3}\leqslant D(k)<\dfrac{4U(k)}{3},\\[2mm] 3U(k)+1 & if\ \ \dfrac{4U(k)}{3}\leqslant D(k)<2U(k). \end{cases}$$

Proof. (i) We notice that $3^k(3^k-1)=3^{2k}-3^k<3^{2k}<3^{2k}+3^k=(3^k+1)3^k$, so $U(2k)=3^k$.

(ii) It follows from the definitions of $U(k)$ and $D(k)$ that

$$U(k)(U(k)-1)\leqslant 3^k<(U(k)+1)U(k), \tag{4}$$

$$0\leqslant D(k)<(U(k)+1)U(k)-U(k)(U(k)-1)=2U(k). \tag{5}$$

Since $U(k)(U(k)-1)\neq 3^k$ for $k\geqslant 1$, so $D(k)>0$ and $U(k)(U(k)-1)<3^k$. Thus,

$$(3U(k)-1)(3U(k)-2)=9U(k)(U(k)-1)+2\leqslant 3^{k+2}, \tag{6}$$

$$(3U(k)+2)(3U(k)+1) = 9(U(k)+1)U(k)+2 > 3^{k+2}. \tag{7}$$

Eqs. (6) and (7) imply that $3U(k)-1 \leqslant U(k+2) \leqslant 3U(k)+1$. Furthermore,

$$3U(k)(3U(k)-1) = 3^{k+2} + 3U(k)\left(2 - \frac{3D(k)}{U(k)}\right), \tag{8}$$

$$(3U(k)+1)(3U(k)) = 3^{h+2} + 3U(k)\left(4 - \frac{3D(k)}{U(k)}\right). \tag{9}$$

(a) If $0 < D(k) < 2U(k)/3$, Eqs. (8) and (6) imply that $U(k+2) = 3U(k)-1$; (b) If $2U(k)/3 \leqslant D(k) < 4U(k)/3$, Eqs. (8) and (9) imply that $U(k+2) = 3U(k)$; (c) If $4U(k)/3 \leqslant D(k) < 2U(k)$, Eqs. (9) and (7) imply that $U(k+2) = 3U(k)+1$. □

We now establish some basic notations and terminologies. Let $G = \{1,2,\ldots,n\}$ be the *initial set of n coins*, for any $j \in G$, j_{h} (j_{l}) means that coin j is a heavier (lighter) coin. Let $G_{\mathrm{hh}} \triangleq \{(i_{\mathrm{h}}, j_{\mathrm{h}}) \mid i<j; i,j \in G\}$ and $G_{\mathrm{ll}} \triangleq \{(i_{\mathrm{l}}, j_{\mathrm{l}}) \mid i<j; i,j \in G\}$ then $G_{\mathrm{hh,ll}} = G_{\mathrm{hh}} + G_{\mathrm{ll}}$ is the set of all possible solutions of two counterfeit coins in G. Call *initial search domain*. For two disjoint sets $A, B \subseteq G$, let $A_{\mathrm{h}} \times B_{\mathrm{h}} \triangleq \{(i_{\mathrm{h}}, j_{\mathrm{h}}) \mid i \in A, j \in B\}$ be the search domain that each one of A and B contains exactly one heavier coin; $A_{\mathrm{l}} \times B_{\mathrm{l}} \triangleq \{(i_{\mathrm{l}}, j_{\mathrm{l}}) \mid i \in A, j \in B\}$ be the search domain that each one of A and B contains exactly one lighter coin. It is obvious that $|A_{\mathrm{hh}}| = |A_{\mathrm{ll}}| = |A|\cdot(|A|-1)/2$ and $|A_{\mathrm{h}} \times B_{\mathrm{h}}| = |A_{\mathrm{l}} \times B_{\mathrm{l}}| = |A| \cdot |B|$.

Two sets $L, R \subset G$, $L : R$ is called to be a *test-set* if $L \cap R = \emptyset$ and $|L| = |R|$ (no information can be obtained by weighing two unequal sized sets). A weighing $L : R$ means that we perform the weighing of L against R and L, R is placed on the left, the right pan of the two-arms balance. L, R and $N \triangleq G - L - R$ is called to be the *left test-set*, the *right test-set* and the *remaining set* (N is not placed on any pans of the balance) of this weighing respectively. When we perform one weighing $L : R$, the outcome of this weighing must be one of the three possible *feedbacks*: "left-heavy", "right-heavy" and "balance", denoted by $f = -1, 1, 0$ respectively. If we receive a feedback $f \in \{-1, 0, 1\}$, the search domain being consistent with the feedback f can be determined uniquely, denoted by S^f. Generally, for any integer $\ell \geqslant 1$, $S^{f_1 f_2 \cdots f_\ell}(f_j \in \{-1,0,1\}, j = 1,2,\ldots,\ell)$ denotes the search domain determined by the feedback sequence $f_1 f_2 \cdots f_\ell$ of these ℓ weighings. A search domain $S^{f_1 f_2 \cdots f_\ell}$ is called to be *final* if $|S^{f_1 f_2 \cdots f_\ell}| = 1$. We call a tree *ternary* if each node has at most three sons. Following [4], a worst-case sequential algorithm can be represented by a *ternary* tree T whose root corresponds to the initial search domain and whose *leaves* correspond to the *final* search domains; each *internal node* corresponds to a search domain $S^{f_1 f_2 \cdots f_\ell}$. If $S^{f_1 f_2 \cdots f_\ell} \neq \emptyset$ then the tree T contains a node labelled by $S^{f_1 f_2 \cdots f_\ell}$ whose *f-son search domain* exists (labelled by $S^{f_1 f_2 \cdots f_\ell f}$) if $S^{f_1 f_2 \cdots f_\ell f} \neq \emptyset$, it does not exist otherwise. It is obvious that $S = S^{-1} + S^0 + S^1$ for any search domain S and a worst-case sequential algorithm $\mathscr{A}$ identifies the solution of S by $k = \lceil \log_3 |S| \rceil$ weighings if and only if (1) $|L^\ell| = |R^\ell| (\ell = 1,2,\ldots,k)$, where $L^\ell : R^\ell$ is the test-set of the ℓth weighing; and (2) $|S^{f_1 f_2 \cdots f_\ell}| \leqslant 3^{k-\ell}$ for $1 \leqslant \ell \leqslant k$.

Suppose a weighing $L:R$ is performed, feedback $f=0$ implies that "L,R has one heavier coin respectively" or "N has the two heavier coins" or "L,R has one lighter coin respectively" or "N has the two lighter coins"; $f=-1$ implies that "L,N has one heavier coin respectively" or "L has the two heavier coins" or "R,N has one lighter coin respectively" or "R has the two lighter coins"; $f=1$ implies that "R,N has one heavier coin respectively" or "R has the two heavier coins" or "L,N has one lighter coin respectively" or "L has the two lighter coins". We write this fact as

$$\mathscr{F}^0(L,N,R) = L_{\rm h}\times R_{\rm h} + N_{\rm hh} + L_{\rm l}\times R_{\rm l} + N_{\rm ll},$$

$$\mathscr{F}^{-1}(L,N,R) = L_{\rm h}\times N_{\rm h} + L_{\rm hh} + R_{\rm l}\times N_{\rm l} + R_{\rm ll},$$

$$\mathscr{F}^1(L,N,R) = R_{\rm h}\times N_{\rm h} + R_{\rm hh} + L_{\rm l}\times N_{\rm l} + L_{\rm ll}. \tag{10}$$

Suppose we have chosen the test-set $L(S):R(S)$ for a given search domain S, $N(S)$ be the remaining set, it is evident that for $f\in\{-1,0,1\}$,

$$S^f = \mathscr{F}^f(L(S),N(S),R(S))\cap S. \tag{11}$$

We note that there are 3^k possible search domains after k weighings, as our aim is to establish some results which hold for all integer $k\geqslant 1$, thus we really have to face infinite search domains. The above aim is impossible to realize by giving the test-sets for each search domain case by case, one must find an universal method to determine all test-sets (we establish it in Definition 2). The Isomorphism Theorem (Theorem 5) states that we need only to discuss one from every pair of isomorphic (or symmetrically isomorphic) search domains if the isomorphic and symmetrically isomorphic concepts between two search domains are introduced (Definition 1). Naturally, we should concentrate attention to classify the 3^k search domains into a number of groups such that all search domains in one group are mutually isomorphic. Thus, we need only to discuss one candidate of each group. The Isomorphism Class Theorem (Theorem 8) shows that such a classification exists and there are exactly $k+1$ isomorphism classes. More concisely, for all integer k, $S^{f_1f_2\cdots f_k}$ is isomorphic to a special search domain $S^{\mathbf{0}_r\mathbf{1}_s}$ which is obtained when the feedbacks of the former r weighings are 0 and the feedbacks of the last s weighings are 1, where r, s are the number of zero feedback, non-zero feedback of $f_1f_2\cdots f_k$. i.e., $r=|\{j\mid f_j=0, 1\leqslant j\leqslant k\}|$, $s=|\{j\mid f_j\neq 0, 1\leqslant j\leqslant k\}|$ and $r+s=k$ (e.g., for $k=3$, Theorem 8 states that S^{-1-1-1}, S^{-1-11}, S^{-11-1}, S^{1-1-1}, S^{-111}, S^{1-11}, S^{11-1} are isomorphic to S^{111}; S^{-1-10}, S^{-10-1}, S^{0-1-1}, S^{-110}, S^{-101}, S^{0-11}, S^{1-10}, S^{10-1}, S^{01-1}, S^{110}, S^{101} are isomorphic to S^{011}; S^{100}, S^{010}, S^{-100}, S^{0-10}, S^{00-1} are isomorphic to S^{001}. The $k+1=4$ candidates are S^{000}, S^{001}, S^{011}, S^{111}). How to prove this general Isomorphism Class Theorem? It seems impossible if we cannot represent the structure of search domains $S^{f_1f_2\cdots f_k}$ by an explicit formula. Fortunately, the Structure Theorem gives the desired explicit representation of the structure of all search domains and based on it, the Isomorphism Class Theorem can be proved strictly.

Definition 1. For any search domain S, A is called to be a *vertex-set* of S (in symbols $A\in S$) if A appears in the representation of S. By $V(S)$ we denote the union of all vertex-sets of S. Suppose S,σ be two search domains and $\varphi: V(\sigma)\mapsto V(S)$ is a bijection. For any vertex-set $A \in \sigma$, let $\varphi(A) = \{\varphi(j) \mid j \in A\}$ be the *image-set* of A. By $\varphi(\sigma)$ we denote the search domain obtained from σ by changing each $A \in \sigma$ into $\varphi(A)$. By $\bar{\sigma}$ we denote the search domain obtained from σ by changing "h", "l" into "l", "h" respectively. S is called to be *isomorphic to* σ (in symbols $S \cong \sigma$) if $S = \varphi(\sigma)$, *symmetrically isomorphic to* σ (in symbols $S \bar{\cong} \sigma$) if $S \cong \bar{\sigma}$.

Theorem 5 (Isomorphism Theorem). *Suppose $S \cong \sigma$ or $S \bar{\cong} \sigma$. If k weighings can identify the solution of σ then k weighings can also identify the solution of S.*

Proof. (1) $S \cong \sigma$ implies that there exists a bijection $\varphi : V(\sigma) \mapsto V(S)$ with $\varphi(\sigma) = S$. The test-set of S is determined by the following *image method*: if we have chosen a test-set $L(\sigma) : R(\sigma)$ for σ then we choose $L(S) : R(S) = \varphi(L(\sigma)) : \varphi(R(\sigma))$. It suffices to prove that $S^f \cong \sigma^f$ for $f \in \{-1, 0, 1\}$, $S^{f_1 f_2 \cdots f_k} \cong \sigma^{f_1 f_2 \cdots f_k}$ can be obtained easily by induction on k. We notice that $N(S) = V(S) - L(S) - R(S) = \varphi(V(\sigma)) - \varphi(L(\sigma)) - \varphi(R(\sigma)) = \varphi(N(\sigma))$. It follows from Eqs. (10) and (11) that $\sigma^1 = \{R_{\mathrm{h}}(\sigma) \times N_{\mathrm{h}}(\sigma) + R_{\mathrm{hh}}(\sigma) + L_{\mathrm{l}}(\sigma) \times N_{\mathrm{l}}(\sigma) + L_{\mathrm{ll}}(\sigma)\} \cap \sigma$.

$$\begin{aligned}\varphi(\sigma^1) &= \{R_{\mathrm{h}}(S) \times N_{\mathrm{h}}(S) + R_{\mathrm{hh}}(S) + L_{\mathrm{l}}(S) \times N_{\mathrm{l}}(S) + L_{\mathrm{ll}}(S)\} \cap \varphi(\sigma)\\ &= \mathscr{F}^1(L(S), N(S), R(S)) \cap S = S^1.\end{aligned}$$

Therefore $S^1 \cong \sigma^1$. Similarly we have $S^0 \cong \sigma^0$ and $S^{-1} \cong \sigma^{-1}$.

(2) $S \bar{\cong} \sigma$ implies that there exists a bijection $\varphi : V(\sigma) \mapsto V(S)$ with $S = \varphi(\bar{\sigma})$. The test-set of S is determined by the following *symmetric image method*: if we have chosen a test-set $L(\sigma) : R(\sigma)$ for σ then we choose $L(S) : R(S) = \varphi(R(\sigma)) : \varphi(L(\sigma))$. It suffices to prove that $S^f \bar{\cong} \sigma^f$ for $f \in \{-1, 0, 1\}$. In fact, we have $N(S) = V(S) - L(S) - R(S) = \varphi(V(\sigma)) - \varphi(R(\sigma)) - \varphi(L(\sigma)) = \varphi(N(\sigma))$ and $\overline{\sigma^1} = \{R_{\mathrm{l}}(\sigma) \times N_{\mathrm{l}}(\sigma) + R_{\mathrm{ll}}(\sigma) + L_{\mathrm{h}}(\sigma) \times N_{\mathrm{h}}(\sigma) + L_{\mathrm{hh}}(\sigma)\} \cap \bar{\sigma}$.

$$\begin{aligned}\varphi(\overline{\sigma^1}) &= \{L_{\mathrm{l}}(S) \times N_{\mathrm{l}}(S) + L_{\mathrm{ll}}(S) + R_{\mathrm{h}}(S) \times N_{\mathrm{h}}(S) + R_{\mathrm{hh}}(S)\} \cap \varphi(\bar{\sigma})\\ &= \mathscr{F}^1(L(S), N(S), R(S)) \cap S = S^1.\end{aligned}$$

Therefore $S^1 \bar{\cong} \sigma^1$. Similarly we have $S^0 \bar{\cong} \sigma^0$ and $S^{-1} \bar{\cong} \sigma^{-1}$. □

Definition 2. (1) For a set $A \subseteq G = \{1, 2, \ldots, n\}$ with $|A| = 3m$, suppose that $A = \{i_1, i_2, i_3, \ldots, i_{3m}\}$ with $i_1 < i_2 < i_3 < \cdots < i_{3m}$. Let $A^{-1} = \{i_1, i_2, \ldots, i_m\}$, $A^0 = \{i_{m+1}, i_{m+2}, \ldots, i_{2m}\}$, $A^1 = \{i_{2m+1}, i_{2m+2}, \ldots, i_{3m}\}$. The ternary set (A^{-1}, A^0, A^1) is called to be the *ternary ordered partition* of A. Call A^{-1}, A^1 the *left-partition set*, the *right-partition set* of A respectively.

(2) Let $L(S)$ be the union of the left-partition set of any vertex-set $A \in S$, $R(S)$ be the union of the right-partition set of any vertex-set $A \in S$, i.e.,

$$L(S) \triangleq \bigcup_{A \in S} A^{-1}, \quad R(S) \triangleq \bigcup_{A \in S} A^1, \quad N(S) = \bigcup_{A \in S} A^0. \tag{12}$$

The method of choosing $L(S) : R(S)$ as the test-set of the search domain S is called to be the *Ternary Ordered Partition Method* (denoted by *TOP*).

(3) We define operation $\oplus$ over $\{-1,0,1\}$ by

$$(-1)\oplus(-1)=-1,\quad 0\oplus 0=0,\quad 1\oplus 1=1,$$

$$(-1)\oplus 0=-1,\quad (-1)\oplus 1=0,\quad 1\oplus 0=1,$$

$$0\oplus(-1)=-1,\quad 1\oplus(-1)=0,\quad 0\oplus 1=1.$$

(4) By $\{-1,0,1\}^\ell$ we denote the collection of all feedback sequences of length ℓ, i.e., $\{-1,0,1\}^\ell=\{f_1f_2\cdots f_\ell \mid f_j\in\{-1,0,1\}, 1\leqslant j\leqslant \ell\}$. $\{-1,0\}^\ell$ and $\{0,1\}^\ell$ are defined similarly. Let $\mathbf{I}_\ell=i_1i_2\cdots i_\ell$, $\mathbf{J}_\ell=j_1j_2\cdots j_\ell$ be two feedback sequences of length ℓ, we define $\mathbf{I}_\ell\oplus\mathbf{J}_\ell=z_1z_2\cdots z_\ell$, where $z_k=i_k\oplus j_k$ for $k=1,2,\ldots,\ell$; $\mathbf{I}_\ell=\mathbf{J}_\ell$ if and only if $i_k=j_k$ for $k=1,2,\ldots,\ell$; $\mathbf{I}_\ell>\mathbf{J}_\ell$ if and only if there exists $k\in\{1,2,\ldots,\ell\}$ such that $i_k>j_k$ and $i_m=j_m$ for $m=1,2,\ldots,k-1$; $\bar{\mathbf{I}}_\ell=z_1z_2\cdots z_\ell$, where $z_k=-i_k$ for $k=1,2,\ldots,\ell$. By $\mathbf{0}_\ell$, $-\mathbf{1}_\ell$, $\mathbf{1}_\ell$ we denote the feedback sequence of length ℓ and all feedbacks are 0, -1, 1 respectively.

It follows from Definition 2 that the ternary ordered partition of a given set A is unique and $A=A^{-1}\cup A^0\cup A^1$, $|A^{-1}|=|A^0|=|A^1|=|A|/3$ and A^{-1}, A^0, A^1 are pairwise disjoint. Generally, by $G^{\mathbf{F}_\ell}$ we denote the unique set determined by **TOP** and the feedback sequence $\mathbf{F}_\ell\in\{-1,0,1\}^\ell$. It is evident that if $|G|=n_q\cdot 3^k$ then $|G^{\mathbf{F}_\ell}|=n_q\cdot 3^{k-\ell}$ for $1\leqslant\ell\leqslant k$; $G^{\mathbf{F}_\ell}\cap G^{\mathbf{F}'_\ell}=\emptyset$ for $\mathbf{F}_\ell\neq\mathbf{F}'_\ell$; there always exists a bijection $\varphi: G^{\mathbf{F}_\ell}\mapsto G^{\mathbf{F}'_\ell}$ as $|G^{\mathbf{F}_\ell}|=|G^{\mathbf{F}'_\ell}|$ for any $\mathbf{F}_\ell,\mathbf{F}'_\ell\in\{-1,0,1\}^\ell$. If no confusion occurs, this bijection φ will not be given explicitly. It follows from Eq. (10) that the general structure of search domain is related to four configurations: $A_{\mathrm{h}}\times B_{\mathrm{h}}$, $A_{\mathrm{l}}\times B_{\mathrm{l}}$, A_{hh} and A_{ll}. Eqs. (10) and (11) can be used to obtain the three son-search domain, but the following lemma 6 gives an equivalent more convenient method. To simplify the representation of search domain, if necessary, we use abbreviated notation $A_1\times A_2\times\cdots\times A_m\triangleq\sum_{i=1}^{m-1}A_i\times A_{i+1}$.

Lemma 6. *Suppose $\mathscr{F}^f(S)$ be the search domain obtained from S after one weighing determined by* **TOP** *and the feedback of this weighing is $f\in\{-1,0,1\}$. We have*

$$\mathscr{F}^0(A_{\mathrm{hh}})=A_{\mathrm{h}}^1\times A_{\mathrm{h}}^{-1}+A_{\mathrm{hh}}^0;\quad \mathscr{F}^1(A_{\mathrm{hh}})=A_{\mathrm{h}}^1\times A_{\mathrm{h}}^0+A_{\mathrm{hh}}^1;$$
$$\mathscr{F}^{-1}(A_{\mathrm{hh}})=A_{\mathrm{h}}^0\times A_{\mathrm{h}}^{-1}+A_{\mathrm{hh}}^{-1},\tag{13}$$

$$\mathscr{F}^0(A_{\mathrm{ll}})=A_{\mathrm{l}}^{-1}\times A_{\mathrm{l}}^1+A_{\mathrm{ll}}^0;\quad \mathscr{F}^1(A_{\mathrm{ll}})=A_{\mathrm{l}}^{-1}\times A_{\mathrm{l}}^0+A_{\mathrm{ll}}^{-1};$$
$$\mathscr{F}^{-1}(A_{\mathrm{ll}})=A_{\mathrm{l}}^0\times A_{\mathrm{l}}^1+A_{\mathrm{ll}}^1,\tag{14}$$

$$\mathscr{F}^f(A_{\mathrm{h}}\times B_{\mathrm{h}})=\sum_{i\oplus j=f}A_{\mathrm{h}}^i\times B_{\mathrm{h}}^j;\quad \mathscr{F}^f(A_{\mathrm{l}}\times B_{\mathrm{l}})=\sum_{i\oplus j=f}A_{\mathrm{l}}^{\bar{i}}\times B_{\mathrm{l}}^{\bar{j}}.\tag{15}$$

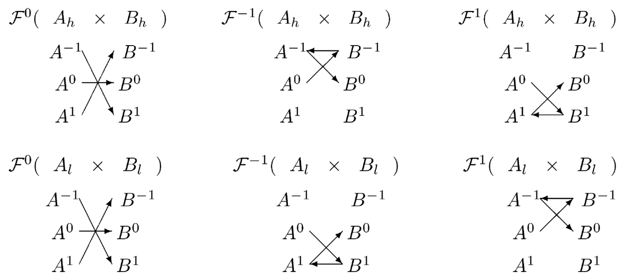
Proof. Eqs. (13) and (14) are the easy consequence of Eqs. (10) and (11). For $S=A_{\mathrm{h}}\times B_{\mathrm{h}}$, let (A^{-1},A^0,A^1), (B^{-1},B^0,B^1) be the ternary ordered partition of A, B. By the definition of *TOP*, the test-set of $A_{\mathrm{h}}\times B_{\mathrm{h}}$ is $L:R=A^{-1}\cup B^{-1}:A^1\cup B^1$ and $N=A^0\cup B^0$.

It follows from Eqs. (10) and (11) that

$$\mathscr{F}^0(A_{\mathrm{h}}\times B_{\mathrm{h}}) = \mathscr{F}^0(L,N,R)\cap(A^{-1}\cup A^0\cup A^1)_{\mathrm{h}}\times(B^{-1}\cup B^0\cup B^1)_{\mathrm{h}}$$
$$= A_{\mathrm{h}}^1\times B_{\mathrm{h}}^{-1} + A_{\mathrm{h}}^0\times B_{\mathrm{h}}^0 + A_{\mathrm{h}}^{-1}\times B_{\mathrm{h}}^1,$$
$$\mathscr{F}^{-1}(A_{\mathrm{h}}\times B_{\mathrm{h}}) = \mathscr{F}^{-1}(L,N,R)\cap(A^{-1}\cup A^0\cup A^1)_{\mathrm{h}}\times(B^{-1}\cup B^0\cup B^1)_{\mathrm{h}}$$
$$= A_{\mathrm{h}}^{-1}\times B_{\mathrm{h}}^{-1} + A_{\mathrm{h}}^0\times B_{\mathrm{h}}^{-1} + A_{\mathrm{h}}^{-1}\times B_{\mathrm{h}}^0 = A_{\mathrm{h}}^0\times B_{\mathrm{h}}^{-1}\times A_{\mathrm{h}}^{-1}\times B_{\mathrm{h}}^0,$$
$$\mathscr{F}^1(A_{\mathrm{h}}\times B_{\mathrm{h}}) = \mathscr{F}^1(L,N,R)\cap(A^{-1}\cup A^0\cup A^1)_{\mathrm{h}}\times(B^{-1}\cup B^0\cup B^1)_{\mathrm{h}}$$
$$= A_{\mathrm{h}}^1\times B_{\mathrm{h}}^1 + A_{\mathrm{h}}^0\times B_{\mathrm{h}}^1 + A_{\mathrm{h}}^1\times B_{\mathrm{h}}^0 = A_{\mathrm{h}}^0\times B_{\mathrm{h}}^1\times A_{\mathrm{h}}^1\times B_{\mathrm{h}}^0.$$

The formula of $\mathscr{F}^f(A_{\mathrm{l}}\times B_{\mathrm{l}})$ can be obtained similarly. □

We will use Eq. (15) frequently. For better understanding, $\mathscr{F}^f(A_{\mathrm{h}}\times B_{\mathrm{h}})$ and $\mathscr{F}^f(A_{\mathrm{l}}\times B_{\mathrm{l}})$ can be described figuratively by the following graphs.



Theorem 7 (Structure Theorem). *Suppose $S^{\mathbf{F}_\ell}$ be the search domain obtained from $S=G_{\mathrm{hh,ll}}(n_q\cdot 3^K)$ by using* **TOP** *and the feedback sequence of these ℓ weighings be $\mathbf{F}_\ell$, $\ell\leqslant K$. Then*

$$S^{\mathbf{F}_\ell} = \sum_{\substack{\mathbf{I}_\ell\oplus\mathbf{J}_\ell=\mathbf{F}_\ell\\ \mathbf{I}_\ell>\mathbf{J}_\ell}} (G_{\mathrm{h}}^{\mathbf{I}_\ell}\times G_{\mathrm{h}}^{\mathbf{J}_\ell} + G_{\mathrm{l}}^{\bar{\mathbf{I}}_\ell}\times G_{\mathrm{l}}^{\bar{\mathbf{J}}_\ell}) + G_{\mathrm{hh}}^{\mathbf{F}_\ell} + G_{\mathrm{ll}}^{\bar{\mathbf{F}}_\ell}. \qquad (16)$$

Proof. We prove it by induction on ℓ. For $S=G_{\mathrm{hh,ll}}=G_{\mathrm{hh}}+G_{\mathrm{ll}}$, $S^f=\mathscr{F}^f(S)=\mathscr{F}^f(G_{\mathrm{hh}})+\mathscr{F}^f(G_{\mathrm{ll}})$. By virtue of Eqs. (13) and (14), $S^f=\sum_{\substack{i\oplus j=f\\ i>j}}(G_{\mathrm{h}}^i\times G_{\mathrm{h}}^j+G_{\mathrm{l}}^{\bar{i}}\times G_{\mathrm{l}}^{\bar{j}})+G_{\mathrm{hh}}^f+G_{\mathrm{ll}}^{\bar{f}}$ can be verified easily ($i\oplus j=0, i>j \Leftrightarrow (i,j)=(1,-1)$; $i\oplus j=1, i>j\Leftrightarrow(i,j)=(1,0)$; $i\oplus j=-1, i>j\Leftrightarrow(i,j)=(0,-1)$). Suppose Eq. (16) is true for $\ell\geqslant 2$, i.e., $S^{\mathbf{F}_\ell}=\sum_{\substack{\mathbf{I}_\ell\oplus\mathbf{J}_\ell=\mathbf{F}_\ell\\ \mathbf{I}_\ell>\mathbf{J}_\ell}}(G_{\mathrm{h}}^{\mathbf{I}_\ell}\times G_{\mathrm{h}}^{\mathbf{J}_\ell}+G_{\mathrm{l}}^{\bar{\mathbf{I}}_\ell}\times G_{\mathrm{l}}^{\bar{\mathbf{J}}_\ell})+G_{\mathrm{hh}}^{\mathbf{F}_\ell}+G_{\mathrm{ll}}^{\bar{\mathbf{F}}_\ell}$. It follows from Eqs. (13)–(15) that

$$S^{\mathbf{F}_\ell(-1)} = \mathscr{F}^{-1}(S^{\mathbf{F}_\ell})$$
$$= \sum_{\substack{\mathbf{I}_\ell\oplus\mathbf{J}_\ell=\mathbf{F}_\ell\\ \mathbf{I}_\ell>\mathbf{J}_\ell}} (\mathscr{F}^{-1}(G_{\mathrm{h}}^{\mathbf{I}_\ell}\times G_{\mathrm{h}}^{\mathbf{J}_\ell}) + \mathscr{F}^{-1}(G_{\mathrm{l}}^{\bar{\mathbf{I}}_\ell}\times G_{\mathrm{l}}^{\bar{\mathbf{J}}_\ell}))$$

$$+\mathscr{F}^{-1}(G_{\rm hh}^{\mathbf{F}_\ell}) + \mathscr{F}^{-1}(G_{\rm ll}^{\bar{\mathbf{F}}_\ell})$$

$$= \sum_{\substack{\mathbf{I}_\ell \oplus \mathbf{J}_\ell = \mathbf{F}_\ell \\ \mathbf{I}_\ell > \mathbf{J}_\ell}} \sum_{i \oplus j = -1} (G_{\rm h}^{\mathbf{I}_\ell i} \times G_{\rm h}^{\mathbf{J}_\ell j} + G_{\rm l}^{\bar{\mathbf{I}}_\ell \bar{i}} \times G_{\rm l}^{\bar{\mathbf{J}}_\ell \bar{j}})$$

$$+ G_{\rm h}^{\mathbf{F}_\ell 0} \times G_{\rm h}^{\mathbf{F}_\ell -1} + G_{\rm hh}^{\mathbf{F}_\ell -1} + G_{\rm l}^{\bar{\mathbf{F}}_\ell 0} \times G_{\rm l}^{\bar{\mathbf{F}}_\ell 1} + G_{\rm ll}^{\bar{\mathbf{F}}_\ell 1}$$

$$= \sum_{\substack{\mathbf{I}_{\ell+1} \oplus \mathbf{J}_{\ell+1} = \mathbf{F}_\ell(-1) \\ \mathbf{I}_{\ell+1} > \mathbf{J}_{\ell+1}}} (G_{\rm h}^{\mathbf{I}_{\ell+1}} \times G_{\rm h}^{\mathbf{J}_{\ell+1}} + G_{\rm l}^{\bar{\mathbf{I}}_{\ell+1}} \times G_{\rm l}^{\bar{\mathbf{J}}_{\ell+1}}) + G_{\rm hh}^{\mathbf{F}_\ell -1} + G_{\rm ll}^{\overline{\mathbf{F}_\ell -1}}, \quad (17)$$

where we use the following fact in the last equality: Let $\mathbf{I}_{\ell+1} = \mathbf{I}_\ell i$ and $\mathbf{J}_{\ell+1} = \mathbf{J}_\ell j$,

$$\{\mathbf{I}_{\ell+1} \oplus \mathbf{J}_{\ell+1} = \mathbf{F}_\ell(-1); \mathbf{I}_{\ell+1} > \mathbf{J}_{\ell+1}\}$$
$$= \{\mathbf{I}_\ell i \oplus \mathbf{J}_\ell j = \mathbf{F}_\ell(-1); \mathbf{I}_\ell i > \mathbf{J}_\ell j\}$$
$$= \{\mathbf{I}_\ell \oplus \mathbf{J}_\ell = \mathbf{F}_\ell, i \oplus j = -1; \mathbf{I}_\ell > \mathbf{J}_\ell\}$$
$$\cup \{\mathbf{I}_\ell \oplus \mathbf{J}_\ell = \mathbf{F}_\ell, i \oplus j = -1; \mathbf{I}_\ell = \mathbf{J}_\ell, i > j\}$$
$$= \{\mathbf{I}_\ell \oplus \mathbf{J}_\ell = \mathbf{F}_\ell, \mathbf{I}_\ell > \mathbf{J}_\ell; i \oplus j = -1\} \cup \{\mathbf{I}_\ell = \mathbf{J}_\ell = \mathbf{F}_\ell; (i,j) = (0,-1)\}.$$

Similarly, $S^{\mathbf{F}_\ell f} = \sum_{\substack{\mathbf{I}_{\ell+1} \oplus \mathbf{J}_{\ell+1} = \mathbf{F}_\ell f \\ \mathbf{I}_{\ell+1} > \mathbf{J}_{\ell+1}}} (G_{\rm h}^{\mathbf{I}_{\ell+1}} \times G_{\rm h}^{\mathbf{J}_{\ell+1}} + G_{\rm l}^{\bar{\mathbf{I}}_{\ell+1}} \times G_{\rm l}^{\bar{\mathbf{J}}_{\ell+1}}) + G_{\rm hh}^{\mathbf{F}_\ell f} + G_{\rm ll}^{\overline{\mathbf{F}_\ell f}}$ for $f = 0, 1$. □

Theorem 8 (The Isomorphism Class Theorem). *Suppose $S^{\mathbf{F}_\ell}$ be the search domain obtained from $S = G_{\rm hh,ll}(n_q \cdot 3^K)$ by using TOP, $\mathbf{F}_\ell = f_1 f_2 \cdots f_\ell \in \{-1,0,1\}^\ell, \ell \leqslant K$. Then $S^{\mathbf{F}_\ell}$ is isomorphic to $S^{\mathbf{0_r 1_s}}$, where $r = |\{j \mid f_j = 0, 1 \leqslant j \leqslant \ell\}|$, $s = |\{j | f_j \neq 0, 1 \leqslant j \leqslant \ell\}|$ and $r + s = \ell$.*

Proof. If the following facts are proved: (1) $S^{\mathbf{F}_\ell -1} \cong S^{\mathbf{F}_\ell 1}$; (2) If $S^{\mathbf{F}'_\ell} \cong S^{\mathbf{F}_\ell}$ then $S^{\mathbf{F}'_\ell f} \cong S^{\mathbf{F}_\ell f}$ for $f \in \{-1,0,1\}$; (3) $S^{\mathbf{F}_\ell 10} \cong S^{\mathbf{F}_\ell 01}$, then the proof of Isomorphism Class Theorem is an easy consequence of the above facts (1)–(3). We note that (1) and (2) imply that $S^{\mathbf{F}_\ell} \cong S^{\mathbf{F}'_\ell}$, where $\mathbf{F}'_\ell$ is obtained from $\mathbf{F}_\ell$ by changing each feedback -1 into 1; (3) and (2) imply that $S^{\mathbf{F}'_\ell} \cong S^{\mathbf{0_r 1_s}}$. For example, $S^{01-10-11} \cong S^{001111}$ is true: $S^{01-10-11} \overset{(2)}{\cong} S^{0110-11}$ (as $S^{01-1} \overset{(1)}{\cong} S^{011}$) $\overset{(2)}{\cong} S^{011011}$ (as $S^{0110-1} \overset{(1)}{\cong} S^{01101}$), i.e., $S^{01-10-11} \cong S^{011011}$. On the other hand, $S^{011011} \overset{(2)}{\cong} S^{010111}$ (as $S^{0110} \overset{(3)}{\cong} S^{0101}$) $\overset{(2)}{\cong} S^{001111}$ (as $S^{010} \overset{(3)}{\cong} S^{001}$).

Proof of fact (1). By Eqs. (17), (13)–(15), we have

$$S^{\mathbf{F}_\ell -1} = \sum_{\substack{\mathbf{I}_\ell \oplus \mathbf{J}_\ell = \mathbf{F}_\ell \\ \mathbf{I}_\ell > \mathbf{J}_\ell}} (G_{\rm h}^{\mathbf{I}_\ell 0} \times G_{\rm h}^{\mathbf{J}_\ell -1} \times G_{\rm h}^{\mathbf{I}_\ell -1} \times G_{\rm h}^{\mathbf{J}_\ell 0} + G_{\rm l}^{\bar{\mathbf{I}}_\ell 0} \times G_{\rm l}^{\bar{\mathbf{J}}_\ell 1} \times G_{\rm l}^{\bar{\mathbf{I}}_\ell 1} \times G_{\rm l}^{\bar{\mathbf{J}}_\ell 0})$$

$$+ G_{\rm h}^{\mathbf{F}_\ell 0} \times G_{\rm h}^{\mathbf{F}_\ell -1} + G_{\rm hh}^{\mathbf{F}_\ell -1} + G_{\rm l}^{\bar{\mathbf{F}}_\ell 0} \times G_{\rm l}^{\bar{\mathbf{F}}_\ell 1} + G_{\rm ll}^{\bar{\mathbf{F}}_\ell 1},$$

$$S^{\mathbf{F}_\ell 1} = \sum_{\substack{\mathbf{I}_\ell \oplus \mathbf{J}_\ell = \mathbf{F}_\ell \\ \mathbf{I}_\ell > \mathbf{J}_\ell}} (G_{\mathrm{h}}^{\mathbf{I}_\ell 0} \times G_{\mathrm{h}}^{\mathbf{J}_\ell 1} \times G_{\mathrm{h}}^{\mathbf{I}_\ell 1} \times G_{\mathrm{h}}^{\mathbf{J}_\ell 0} + G_{\mathrm{l}}^{\bar{\mathbf{I}}_\ell 0} \times G_{\mathrm{l}}^{\bar{\mathbf{J}}_\ell -1} \times G_{\mathrm{l}}^{\bar{\mathbf{I}}_\ell -1} \times G_{\mathrm{l}}^{\bar{\mathbf{J}}_\ell 0})$$

$$+ G_{\mathrm{h}}^{\mathbf{F}_\ell 0} \times G_{\mathrm{h}}^{\mathbf{F}_\ell 1} + G_{\mathrm{hh}}^{\mathbf{F}_\ell 1} + G_{\mathrm{l}}^{\bar{\mathbf{F}}_\ell 0} \times G_{\mathrm{l}}^{\bar{\mathbf{F}}_\ell -1} + G_{\mathrm{ll}}^{\bar{\mathbf{F}}_\ell -1}. \tag{18}$$

Since $|G^{\mathbf{J}_\ell \bar{j}}| = |G^{\mathbf{J}_\ell j}| = n_q \cdot 3^{K-\ell-1}$ for any $\mathbf{J}_\ell \in \{-1,0,1\}^\ell$, $j \in \{-1,0,1\}$, by virtue of above two equalities, the bijection $\varphi : V(S^{\mathbf{F}_\ell 1}) \mapsto V(S^{\mathbf{F}_\ell -1})$ with $S^{\mathbf{F}_\ell -1} = \varphi(S^{\mathbf{F}_\ell 1})$ can be given by $\varphi(G^{\mathbf{J}_\ell j}) = G^{\mathbf{J}_\ell \bar{j}}$ for each $G^{\mathbf{J}_\ell j} \in S^{\mathbf{F}_\ell 1}$.

Proof of fact (2). $S^{\mathbf{F}'_\ell} \cong S^{\mathbf{F}_\ell}$ implies that there exists a bijection $\varphi : V(S^{\mathbf{F}}_\ell) \mapsto V(S^{\mathbf{F}'_\ell})$ such that $S^{\mathbf{F}'_\ell} = \varphi(S^{\mathbf{F}_\ell})$. Let $A^{\mathbf{J}_\ell}$ be the image-set of $G^{\mathbf{J}_\ell}$, i.e., $\varphi(G^{\mathbf{J}_\ell}) = A^{\mathbf{J}_\ell}$ for each vertex-set $G^{\mathbf{J}_\ell} \in S^{\mathbf{F}_\ell}$. By Eq. (16) $S^{\mathbf{F}'_\ell} = \sum_{\substack{\mathbf{I}_\ell \oplus \mathbf{J}_\ell = \mathbf{F}_\ell \\ \mathbf{I}_\ell > \mathbf{J}_\ell}} (A_{\mathrm{h}}^{\mathbf{I}_\ell} \times A_{\mathrm{h}}^{\mathbf{J}_\ell} + A_{\mathrm{l}}^{\bar{\mathbf{I}}_\ell} \times A_{\mathrm{l}}^{\bar{\mathbf{J}}_\ell}) + A_{\mathrm{hh}}^{\mathbf{F}_\ell} + A_{\mathrm{ll}}^{\bar{\mathbf{F}}_\ell}$. By the same arguments as the derivation of Eq. (18),

$$S^{\mathbf{F}'_\ell 1} = \sum_{\substack{\mathbf{I}_\ell \oplus \mathbf{J}_\ell = \mathbf{F}_\ell \\ \mathbf{I}_\ell > \mathbf{J}_\ell}} (A_{\mathrm{h}}^{\mathbf{I}_\ell 0} \times A_{\mathrm{h}}^{\mathbf{J}_\ell 1} \times A_{\mathrm{h}}^{\mathbf{I}_\ell 1} \times A_{\mathrm{h}}^{\mathbf{J}_\ell 0} + A_{\mathrm{l}}^{\bar{\mathbf{I}}_\ell 0} \times A_{\mathrm{l}}^{\bar{\mathbf{J}}_\ell -1} \times A_{\mathrm{l}}^{\bar{\mathbf{I}}_\ell -1} \times A_{\mathrm{l}}^{\bar{\mathbf{J}}_\ell 0})$$

$$+ A_{\mathrm{h}}^{\mathbf{F}_\ell 0} \times A_{\mathrm{h}}^{\mathbf{F}_\ell 1} + A_{\mathrm{hh}}^{\mathbf{F}_\ell 1} + A_{\mathrm{l}}^{\bar{\mathbf{F}}_\ell 0} \times A_{\mathrm{l}}^{\bar{\mathbf{F}}_\ell -1} + A_{\mathrm{ll}}^{\bar{\mathbf{F}}_\ell -1}. \tag{19}$$

Combining Eqs. (19) and (18), it is evident that the bijection $\varphi_1 : V(S^{\mathbf{F}_\ell 1}) \mapsto V(S^{\mathbf{F}'_\ell 1})$ with $S^{\mathbf{F}'_\ell 1} = \varphi_1(S^{\mathbf{F}_\ell 1})$ can be given by $\varphi_1(G^{\mathbf{J}_\ell j}) = A^{\mathbf{J}_\ell j}$ for any $G^{\mathbf{J}_\ell j} \in S^{\mathbf{F}_\ell 1}$ ($|G^{\mathbf{J}_\ell j}| = |A^{\mathbf{J}_\ell j}|$ as $|G^{\mathbf{J}_\ell}| = |A^{\mathbf{J}_\ell}|$). Therefore $S^{\mathbf{F}'_\ell 1} \cong S^{\mathbf{F}_\ell 1}$. Similarly $S^{\mathbf{F}'_\ell f} \cong S^{\mathbf{F}_\ell f}$ is true for $f \in \{0,-1\}$.

Proof of (3). (3) By the Structure Theorem, we have

$$S^{\mathbf{F}_\ell 01} = \sum_{\substack{\mathbf{I}_\ell i_1 i_2 \oplus \mathbf{J}_\ell j_1 j_2 = \mathbf{F}_\ell 01 \\ \mathbf{I}_\ell i_1 i_2 > \mathbf{J}_\ell j_1 j_2}} (G_{\mathrm{h}}^{\mathbf{I}_\ell i_1 i_2} \times G_{\mathrm{h}}^{\mathbf{J}_\ell j_1 j_2} + G_{\mathrm{l}}^{\bar{\mathbf{I}}_\ell \bar{i}_1 \bar{i}_2} \times G_{\mathrm{l}}^{\bar{\mathbf{J}}_\ell \bar{j}_1 \bar{j}_2}) + G_{\mathrm{hh}}^{\mathbf{F}_\ell 01} + G_{\mathrm{ll}}^{\bar{\mathbf{F}}_\ell 0-1}.$$

We notice that the constraint $\mathbf{I}_\ell i_1 i_2 \oplus \mathbf{J}_\ell j_1 j_2 = \mathbf{F}_\ell 01, \mathbf{I}_\ell i_1 i_2 > \mathbf{J}_\ell j_1 j_2 \Leftrightarrow \mathbf{I}_\ell \oplus \mathbf{J}_\ell = \mathbf{F}_\ell, \mathbf{I}_\ell > \mathbf{J}_\ell, i_1 i_2 \oplus j_1 j_2 = 01$ or $\mathbf{I}_\ell = \mathbf{J}_\ell = \mathbf{F}_\ell, i_1 i_2 \oplus j_1 j_2 = 01, i_1 i_2 > j_1 j_2$. Meanwhile, the constraint $i_1 i_2 \oplus j_1 j_2 = 01, i_1 i_2 > j_1 j_2 \Leftrightarrow \{(i_1 i_2, j_1 j_2) \mid (11,-11),(11,-10),(10,-11),(01,00)\}$. Thus, $S^{\mathbf{F}_\ell 01}$ can be rewritten as

$$S^{\mathbf{F}_\ell 01} = \sum_{\substack{\mathbf{I}_\ell i_1 i_2 \oplus \mathbf{J}_\ell j_1 j_2 = \mathbf{F}_\ell 01 \\ \mathbf{I}_\ell > \mathbf{J}_\ell}} (G_{\mathrm{h}}^{\mathbf{I}_\ell i_1 i_2} \times G_{\mathrm{h}}^{\mathbf{J}_\ell j_1 j_2} + G_{\mathrm{l}}^{\bar{\mathbf{I}}_\ell \bar{i}_1 \bar{i}_2} \times G_{\mathrm{l}}^{\bar{\mathbf{J}}_\ell \bar{j}_1 \bar{j}_2})$$

$$+ G_{\mathrm{h}}^{\mathbf{F}_\ell 10} \times G_{\mathrm{h}}^{\mathbf{F}_\ell -11} \times G_{\mathrm{h}}^{\mathbf{F}_\ell 11} \times G_{\mathrm{h}}^{\mathbf{F}_\ell -10} + G_{\mathrm{h}}^{\mathbf{F}_\ell 01} \times G_{\mathrm{h}}^{\mathbf{F}_\ell 00} + G_{\mathrm{hh}}^{\mathbf{F}_\ell 01}$$

$$+ G_{\mathrm{l}}^{\bar{\mathbf{F}}_\ell -10} \times G_{\mathrm{l}}^{\bar{\mathbf{F}}_\ell 1-1} \times G_{\mathrm{l}}^{\bar{\mathbf{F}}_\ell -1-1} \times G_{\mathrm{l}}^{\bar{\mathbf{F}}_\ell 10} + G_{\mathrm{l}}^{\bar{\mathbf{F}}_\ell 0-1} \times G_{\mathrm{l}}^{\bar{\mathbf{F}}_\ell 00} + G_{\mathrm{ll}}^{\bar{\mathbf{F}}_\ell 0-1}.$$

By the same way, we have

$$S^{\mathbf{F}_\ell 10} = \sum_{\substack{\mathbf{I}_\ell i_2 i_1 \oplus \mathbf{J}_\ell j_2 j_1 = \mathbf{F}_\ell 10 \\ \mathbf{I}_\ell i_2 i_1 > \mathbf{J}_\ell j_2 j_1}} (G_{\mathrm{h}}^{\mathbf{I}_\ell i_2 i_1} \times G_{\mathrm{h}}^{\mathbf{J}_\ell j_2 j_1} + G_{\mathrm{l}}^{\bar{\mathbf{I}}_\ell \bar{i}_2 \bar{i}_1} \times G_{\mathrm{l}}^{\bar{\mathbf{J}}_\ell \bar{j}_2 \bar{j}_1}) + G_{\mathrm{hh}}^{\mathbf{F}_\ell 10} + G_{\mathrm{ll}}^{\bar{\mathbf{F}}_\ell -10}$$

$$= \sum_{\substack{\mathbf{I}_\ell i_2 i_1 \oplus \mathbf{J}_\ell j_2 j_1 = \mathbf{F}_\ell 10 \\ \mathbf{I}_\ell > \mathbf{J}_\ell}} (G_{\mathrm{h}}^{\mathbf{I}_\ell i_2 i_1} \times G_{\mathrm{h}}^{\mathbf{J}_\ell j_2 j_1} + G_{\mathrm{l}}^{\bar{\mathbf{I}}_\ell \bar{i}_2 \bar{i}_1} \times G_{\mathrm{l}}^{\bar{\mathbf{J}}_\ell \bar{j}_2 \bar{j}_1})$$

$$+ G_{\mathrm{h}}^{\mathbf{F}_\ell 01} \times G_{\mathrm{h}}^{\mathbf{F}_\ell 1-1} \times G_{\mathrm{h}}^{\mathbf{F}_\ell 11} \times G_{\mathrm{h}}^{\mathbf{F}_\ell 0-1} + G_{\mathrm{h}}^{\mathbf{F}_\ell 10} \times G_{\mathrm{h}}^{\mathbf{F}_\ell 00} + G_{\mathrm{hh}}^{\mathbf{F}_\ell 10}$$

$$+ G_{\mathrm{l}}^{\bar{\mathbf{F}}_\ell 0-1} \times G_{\mathrm{l}}^{\bar{\mathbf{F}}_\ell -11} \times G_{\mathrm{l}}^{\bar{\mathbf{F}}_\ell -1-1} \times G_{\mathrm{l}}^{\bar{\mathbf{F}}_\ell 01} + G_{\mathrm{l}}^{\bar{\mathbf{F}}_\ell -10} \times G_{\mathrm{l}}^{\bar{\mathbf{F}}_\ell 00} + G_{\mathrm{ll}}^{\bar{\mathbf{F}}_\ell -10}.$$

The bijection $\varphi: V(S^{\mathbf{F}_\ell 01}) \mapsto V(S^{\mathbf{F}_\ell 10})$ with $\varphi(S^{\mathbf{F}_\ell 01}) = S^{\mathbf{F}_\ell 10}$ can be given by $\varphi(G^{\mathbf{I}_\ell i_1 i_2}) = G^{\mathbf{I}_\ell i_2 i_1}$ for all $G^{\mathbf{I}_\ell i_1 i_2} \in S^{\mathbf{F}_\ell 01}$. □

3. Proofs of the main theorems

Proof of Theorem 1. $3^{q-1} < n_q^2 \leqslant 3^q$ implies that $n_q 3^K (n_q 3^K - 1) \leqslant 3^{2K+q} (K \geqslant 0)$ thus the information-theoretic lower bound $\mathscr{L}(G_{\mathrm{hh,ll}}(n_q \cdot 3^K)) \leqslant 2K + q$. In order to prove Theorem 1, it suffices to construct an algorithm $\mathscr{A}$ by which the solution of $S = G_{\mathrm{hh,ll}}(n_q \cdot 3^K)$ can be identified in $2K + q$ weighings. The test-sets of the former K weighings of our algorithm $\mathscr{A}$ are always chosen by **TOP**; By Isomorphism Class Theorem and Isomorphism Theorem, we need only to give the test-sets of the later $K+q$ weighings for the search domains $S^{\mathbf{0}_\mathrm{r} \mathbf{1}_\mathrm{s}}$ $(r+s=K)$. It is a common knowledge that the constraint $|L|=|R|$ on each weighing $L:R$ may influence the efficiency of searching procedure. After ℓ weighings, some coins are known to be good (the coins which are not contained in $S^{\mathbf{F}_\ell}$ must be good), thus constructing the test-sets becomes more easy by using these good coins to balance the two pans of the scale. To determine the number of good coins after the former K weighings, $S^{\mathbf{0}_\mathrm{r} \mathbf{1}_\mathrm{s}}$ can be rewritten as follow:

$$S^{\mathbf{0}_\mathrm{r} \mathbf{1}_\mathrm{s}} = \sum_{\substack{\mathbf{I}_\mathrm{r} \mathbf{I}'_\mathrm{s} \oplus \mathbf{J}_\mathrm{r} \mathbf{J}'_\mathrm{s} = \mathbf{0}_\mathrm{r} \mathbf{1}_\mathrm{s} \\ \mathbf{I}_\mathrm{r} \mathbf{I}'_\mathrm{s} > \mathbf{J}_\mathrm{r} \mathbf{J}'_\mathrm{s}}} (G_{\mathrm{h}}^{\mathbf{I}_\mathrm{r} \mathbf{I}'_\mathrm{s}} \times G_{\mathrm{h}}^{\mathbf{J}_\mathrm{r} \mathbf{J}'_\mathrm{s}} + G_{\mathrm{l}}^{\bar{\mathbf{I}}_\mathrm{r} \overline{\mathbf{I}'}_\mathrm{s}} \times G_{\mathrm{l}}^{\bar{\mathbf{J}}_\mathrm{r} \overline{\mathbf{J}'}_\mathrm{s}}) + G_{\mathrm{hh}}^{\mathbf{0}_\mathrm{r} \mathbf{1}_\mathrm{s}} + G_{\mathrm{ll}}^{\mathbf{0}_\mathrm{r} (-\mathbf{1}_\mathrm{s})}$$

$$= \sum_{\mathbf{I}_\mathrm{r} > \mathbf{0}_\mathrm{r}} \sum_{\mathbf{I}'_\mathrm{s} \oplus \mathbf{J}'_\mathrm{s} = \mathbf{1}_\mathrm{s}} (G_{\mathrm{h}}^{\mathbf{I}_\mathrm{r} \mathbf{I}'_\mathrm{s}} \times G_{\mathrm{h}}^{\bar{\mathbf{I}}_\mathrm{r} \mathbf{J}'_\mathrm{s}} + G_{\mathrm{l}}^{\bar{\mathbf{I}}_\mathrm{r} \overline{\mathbf{I}'}_\mathrm{s}} \times G_{\mathrm{l}}^{\mathbf{I}_\mathrm{r} \overline{\mathbf{J}'}_\mathrm{s}})$$

$$+ \sum_{\substack{\mathbf{I}'_\mathrm{s} \oplus \mathbf{J}'_\mathrm{s} = \mathbf{1}_\mathrm{s} \\ \mathbf{I}'_\mathrm{s} > \mathbf{J}'_\mathrm{s}}} (G_{\mathrm{h}}^{\mathbf{0}_\mathrm{r} \mathbf{I}'_\mathrm{s}} \times G_{\mathrm{h}}^{\mathbf{0}_\mathrm{r} \mathbf{J}'_\mathrm{s}} + G_{\mathrm{l}}^{\mathbf{0}_\mathrm{r} \overline{\mathbf{I}'}_\mathrm{s}} \times G_{\mathrm{l}}^{\mathbf{0}_\mathrm{r} \overline{\mathbf{J}'}_\mathrm{s}}) + G_{\mathrm{hh}}^{\mathbf{0}_\mathrm{r} \mathbf{1}_\mathrm{s}} + G_{\mathrm{ll}}^{\mathbf{0}_\mathrm{r} (-\mathbf{1}_\mathrm{s})}, \quad (20)$$

where, we use the following equivalent representation in the last equality $\{\mathbf{I}_\mathrm{r} \mathbf{I}'_\mathrm{s} \oplus \mathbf{J}_\mathrm{r} \mathbf{J}'_\mathrm{s} = \mathbf{0}_\mathrm{r} \mathbf{1}_\mathrm{s}, \mathbf{I}_\mathrm{r} \mathbf{I}'_\mathrm{s} > \mathbf{J}_\mathrm{r} \mathbf{J}'_\mathrm{s}\} \Leftrightarrow \{\mathbf{I}_\mathrm{r} \oplus \mathbf{J}_\mathrm{r} = \mathbf{0}_\mathrm{r}, \mathbf{I}_\mathrm{r} > \mathbf{J}_\mathrm{r}, \mathbf{I}'_\mathrm{s} \oplus \mathbf{J}'_\mathrm{s} = \mathbf{1}_\mathrm{s}\} \cup \{\mathbf{I}_\mathrm{r} = \mathbf{J}_\mathrm{r} = \mathbf{0}_\mathrm{r}, \mathbf{I}'_\mathrm{s} \oplus \mathbf{J}'_\mathrm{s} = \mathbf{1}_\mathrm{s}, \mathbf{I}'_\mathrm{s} > \mathbf{J}'_\mathrm{s}\}$ and $\{(\mathbf{I}_\mathrm{r}, \mathbf{J}_\mathrm{r}) \mid \mathbf{I}_\mathrm{r} \oplus \mathbf{J}_\mathrm{r} = \mathbf{0}_\mathrm{r}; \mathbf{I}_\mathrm{r} > \mathbf{J}_\mathrm{r}\} \Leftrightarrow \{(\mathbf{I}_\mathrm{r}, \bar{\mathbf{I}}_\mathrm{r}) \mid \mathbf{I}_\mathrm{r} > \mathbf{0}_\mathrm{r}\}$. Clearly, $V(S^{\mathbf{0}_\mathrm{r} \mathbf{1}_\mathrm{s}}) = \{G^{\mathbf{I}_\mathrm{r} \mathbf{I}'_\mathrm{s}} \mid \mathbf{I}_\mathrm{r} \in \{-1, 0, 1\}^\mathrm{r}; \mathbf{I}'_\mathrm{s} \in \{0, 1\}^\mathrm{s} \cup \{0, -1\}^\mathrm{s}\}$ ($\mathbf{I}'_\mathrm{s} \oplus \mathbf{J}'_\mathrm{s} = \mathbf{1}_\mathrm{s}$ implies that $\mathbf{I}'_\mathrm{s}, \mathbf{J}'_\mathrm{s} \in \{0, 1\}^\mathrm{s}$

and $\overline{\mathbf{I}'}_{\mathbf{s}}, \overline{\mathbf{J}'}_{\mathbf{s}} \in \{0,-1\}^{\mathbf{s}}$) and each $G^{\mathbf{I}_r\mathbf{I}'_s} \in V(S^{\mathbf{0}_r\mathbf{1}_s})$ contains $n_q \cdot 3^{K-(r+s)} = n_q$ coins, i.e., $S^{\mathbf{0}_r\mathbf{1}_s}$ contains $n_q \cdot 3^r \cdot (2^s + 2^s) - n_q \cdot 3^r = n_q \cdot 3^r \cdot (2^{s+1} - 1)$ coins ($\mathbf{I}'_\mathbf{s} = \mathbf{0}_\mathbf{s}$ belongs to both $\{0,1\}^s$ and $\{0,-1\}^s$). This means that we have found $g(s) \triangleq n_q \cdot 3^K - n_q \cdot 3^r \cdot (2^{s+1} - 1) = n_q \cdot 3^r \cdot (3^s - 2^{s+1} + 1)$ good coins after $K = r + s$ weighings. We notice that $g(s) = 0$ if $s = 0, 1$ and $g(s) \geqslant 2n_q \cdot 3^r$ if $s \geqslant 2$, so we classify $K+1$ search domains $S^{\mathbf{0}_r\mathbf{1}_s}$ $(r+s=K)$ into the following three cases: (1) $S^{\mathbf{0}_\mathbf{K}}$; (2) $S^{\mathbf{0}_{\mathbf{K}-1}1}$; (3) $S^{\mathbf{0}_r\mathbf{1}_s}$ $(2 \leqslant s \leqslant K, r = K-s)$. The following Lemma 9 (Lemmas 10 and 11) states that $K+q$ weighings can identify the solution of case (1) (case (2), case (3)). They will be proved in Section 4.

Lemma 9. *Suppose $A \cap B = \emptyset$ and A, B are sets of $n_q \geqslant 3$ coins. If* (1) *q weighings can identify the solution of $\sigma_a(n_q) = A_{\mathrm{hh}} + A_{\mathrm{ll}}$ and* (2) *q weighings can split $\sigma_b(n_q) = A_\mathrm{h} \times B_\mathrm{h} + A_\mathrm{l} \times B_\mathrm{l}$ into configurations with cardinality $\leqslant 2$ then $K+q$ weighings can identify the solution of $S^{\mathbf{0}_\mathbf{K}}$.*

Lemma 10. *Suppose the following two conditions hold*:

(1) *$q+1$ weighings can identify the solution of $\sigma_c(n_q) = A_\mathrm{h} \times C_\mathrm{h} + B_\mathrm{l} \times C_\mathrm{l} + A_{\mathrm{hh}} + B_{\mathrm{ll}}$, where A, B, C are sets of $n_q \geqslant 3$ coins and pairwise disjoint.*

(2) *$q+1$ weighings can split $\sigma_d(n_q) = A_\mathrm{h} \times B_\mathrm{h} \times C_\mathrm{h} \times D_\mathrm{h} + A_\mathrm{l} \times U_\mathrm{l} \times V_\mathrm{l} \times D_\mathrm{l}$ into configurations with cardinality $\leqslant 2$, where A, B, C, D, U, V are sets of $n_q \geqslant 3$ coins and pairwise disjoint.*

Then $K+q$ weighings can identify the solution of $S^{\mathbf{0}_{\mathbf{K}-1}1}$ $(K \geqslant 1)$.

Lemma 11. *Suppose $2 \leqslant s \leqslant K$, $r = K-s$, $A \cap B = \emptyset$ and A, B are sets of $n_q \geqslant 3$ coins. If q weighings can identify the solution of $\sigma_e(n_q) = A_\mathrm{h} \times B_\mathrm{h}$ and $\sigma_f(n_q) = A_{\mathrm{hh}} + B_{\mathrm{ll}}$ respectively then $K+q$ weighings can identify the solution of $S^{\mathbf{0}_r\mathbf{1}_s}$.*

Comparing six conditions in Lemmas 9–11 with the four conditions in Theorem 1, it suffices to prove that $q+1$ weighings suffice to split $\sigma_d(n_q)$ into configurations with cardinality $\leqslant 2$ and q weighings suffice to identify the solution of $\sigma_f(n_q)$ under the given assumptions of Theorem 1. In fact, $\sigma_d \triangleq A_\mathrm{h} \times B_\mathrm{h} \times C_\mathrm{h} \times D_\mathrm{h} + A_\mathrm{l} \times U_\mathrm{l} \times V_\mathrm{l} \times D_\mathrm{l}$, by performing $L : R = B \cup U : C \cup V$ then

$$\sigma_d^0 = B_\mathrm{h} \times C_\mathrm{h} + U_\mathrm{l} \times V_\mathrm{l}, \qquad \sigma_d^{-1}(n_q) = A_\mathrm{h} \times B_\mathrm{h} + V_\mathrm{l} \times D_\mathrm{l},$$

$$\sigma_d^1(n_q) = C_\mathrm{h} \times D_\mathrm{h} + A_\mathrm{l} \times U_\mathrm{l}.$$

We observe that $\sigma_d^{-1} \cong \sigma_d^1 \cong \sigma_d^0$. It is enough to prove that q weighings can split σ_d^0 into configurations with cardinality $\leqslant 2$ by the following *super-coin construction method*: Let $B = \{b^1, b^2, \ldots, b^{n_q}\}$, $C = \{c^1, c^2, \ldots, c^{n_q}\}$, $U = \{u^1, u^2, \ldots, u^{n_q}\}$, $V = \{v^1, v^2, \ldots, v^{n_q}\}$. Constructing super-coin $z^j = \{b^j, u^j\}$, $w^j = \{c^j, v^j\}$ for $j = 1, 2, \ldots, n_q$ so that each super-coin contains two coins. Let $Z = \{z^j \mid 1 \leqslant j \leqslant n_q\}$, $W = \{w^j \mid 1 \leqslant j \leqslant n_q\}$ thus Z, W are sets of n_q super-coins and $S \triangleq Z_\mathrm{h} \times W_\mathrm{h} + Z_\mathrm{l} \times W_\mathrm{l} \cong \sigma_2(n_q)$. By condition (2) of Theorem 1 and the Isomorphism Theorem, q weighings suffice to split S into configurations with cardinality $\leqslant 2$. We note that a solution $z^{j_1} \times w^{j_2}$ of S corresponds to an unique solution of σ_d^0 ($\{b^{j_1}, u^{j_1}\}_\mathrm{h} \times \{c^{j_2}, v^{j_2}\}_\mathrm{h}$ corresponds to $b_\mathrm{h}^{j_1} \times c_\mathrm{h}^{j_2}$;

$\{b^{j_1}, u^{j_1}\}_{\rm l} \times \{c^{j_2}, v^{j_2}\}_{\rm l}$ corresponds to $u_{\rm l}^{j_1} \times v_{\rm l}^{j_2}$) thus q weighings suffice to split σ_d^0 into configurations with cardinality $\leqslant 2$. For $\sigma_f(n_q) = A_{\rm hh} + B_{\rm ll}$, let $A = \{a^1, a^2, \ldots, a^{n_q}\}$, $B = \{b^1, b^2, \ldots, b^{n_q}\}$ and $c^j = \{a^j, b^j\}$ $(j = 1, 2, \ldots, n_q)$ and $C = \{c^j \mid j = 1, 2, \ldots, n_q\}$ then C is a set of n_q super-coins (each super-coin contains two coins). Condition (3) of Theorem 1 implies that q weighings suffice to identify the solution of $S \triangleq C_{\rm hh} + C_{\rm ll}$. We note that a solution $c^{j_1} \times c^{j_2}$ of S corresponds to an unique solution of $\sigma_f(n_q)$ $(\{a^{j_1}, b^{j_1}\}_{\rm h} \times \{a^{j_2}, b^{j_2}\}_{\rm h}$ corresponds to $a_{\rm h}^{j_1} \times a_{\rm h}^{j_2}$ and $\{a^{j_1}, b^{j_1}\}_{\rm l} \times \{a^{j_2}, b^{j_2}\}_{\rm l}$ corresponds to $b_{\rm l}^{j_1} \times b_{\rm l}^{j_2}$). The proof of Theorem 1 is completed. □

Proof of Theorem 2. It suffices to prove that $2K + q$ weighings can identify the solution of $G_{\rm hh,ll}$ for $|G| = n_q \cdot 3^K$ and $q = 2, n_q = 3, K \geqslant 0$. We need only to verify that the four conditions of Theorem 1 hold for $q = 2, n_q = 3$.

(1) It is well known that $\lceil \log_3 n \rceil$ weighings can identify one heavy coin from a set of n coins (Ref. [1]). Thus, 2 weighings can identify the solution of $\sigma_1(3) = A_{\rm h} \times B_{\rm h}(A \cap B = \emptyset)$ by applying above result to A, B respectively.

(2) For $\sigma_2(3) = A_{\rm h} \times B_{\rm h} + A_{\rm l} \times B_{\rm l}$, let $A = \{a^1, a^2, a^3\}$, $B = \{b^1, b^2, b^3\}$ then $(A^{-1}, A^0, A^1) = (a^1, a^2, a^3)$, $(B^{-1}, B^0, B^1) = (b^1, b^2, b^3)$ are the ternary ordered partition of A, B. Let $L(\sigma_2) : R(\sigma_2) = A^{-1} \cup B^{-1} : A^1 \cup B^1 = \{a^1, b^1\} : \{a^3, b^3\}$. It follows from Eq. (15) that

$$\sigma_2^0 = \mathscr{F}^0(A_{\rm h} \times B_{\rm h}) + \mathscr{F}^0(A_{\rm l} \times B_{\rm l})$$
$$= a_{\rm h}^1 \times b_{\rm h}^3 + a_{\rm h}^2 \times b_{\rm h}^2 + a_{\rm h}^3 \times b_{\rm h}^1 + a_{\rm l}^1 \times b_{\rm l}^3 + a_{\rm l}^2 \times b_{\rm l}^2 + a_{\rm l}^3 \times b_{\rm l}^1,$$

$$\sigma_2^1 = \mathscr{F}^1(A_{\rm h} \times B_{\rm h}) + \mathscr{F}^1(A_{\rm l} \times B_{\rm l}) = a_{\rm h}^2 \times b_{\rm h}^3 \times a_{\rm h}^3 \times b_{\rm h}^2 + a_{\rm l}^2 \times b_{\rm l}^1 \times a_{\rm l}^1 \times b_{\rm l}^2,$$

$$\sigma_2^{-1} = \mathscr{F}^{-1}(A_{\rm h} \times B_{\rm h}) + \mathscr{F}^{-1}(A_{\rm l} \times B_{\rm l}) = a_{\rm h}^2 \times b_{\rm h}^1 \times a_{\rm h}^1 \times b_{\rm h}^2 + a_{\rm l}^2 \times b_{\rm l}^3 \times a_{\rm l}^3 \times b_{\rm l}^2.$$

We see that $\sigma_2^{-1} \tilde{\cong} \sigma_2^1$. By performing $L(\sigma_2^0) : R(\sigma_2^0) = \{a^1\} : \{a^3\}$, $L(\sigma_2^1) : R(\sigma_2^1) = \{b^1, b^3\} : \{a^1, a^3\}$, σ_2^0, σ_2^1 are split into configurations with cardinality $\leqslant 2$: $\sigma_2^{00} = a_{\rm h}^2 \times b_{\rm h}^2 + a_{\rm l}^2 \times b_{\rm l}^2$, $\sigma_2^{01} = a_{\rm h}^3 \times b_{\rm h}^1 + a_{\rm l}^1 \times b_{\rm l}^3$, $\sigma_2^{0-1} = a_{\rm h}^1 \times b_{\rm h}^3 + a_{\rm l}^3 \times b_{\rm l}^1$ and $\sigma_2^{10} = a_{\rm h}^3 \times b_{\rm h}^3 + a_{\rm l}^1 \times b_{\rm l}^1$, $\sigma_2^{11} = a_{\rm h}^3 \times b_{\rm h}^2 + a_{\rm l}^2 \times b_{\rm l}^1$, $\sigma_2^{1-1} = a_{\rm h}^2 \times b_{\rm h}^3 + a_{\rm l}^1 \times b_{\rm l}^2$.

(3) $\sigma_3(3) = C_{\rm hh} + C_{\rm ll}$. Let $C = \{c^1, c^2, c^3\}$ and $L(\sigma_3) : R(\sigma_3) = \{c^1\} : \{c^3\}$ then $\sigma_3^0 = c_{\rm h}^1 \times c_{\rm h}^3 + c_{\rm l}^1 \times c_{\rm l}^3$, $\sigma_3^1 = c_{\rm h}^3 \times c_{\rm h}^2 + c_{\rm l}^1 \times c_{\rm l}^2$ and $\sigma_3^{-1} = c_{\rm h}^1 \times c_{\rm h}^2 + c_{\rm l}^3 \times c_{\rm l}^2$. The second weighing are always given by $\{c^2\} : \{c^3\}$.

(4) We prove that $q + 1 = 3$ weighings can identify the solution of $\sigma_4(3) = A_{\rm h} \times C_{\rm h} + B_{\rm l} \times C_{\rm l} + A_{\rm hh} + B_{\rm ll}$. Let $A = \{a^1, a^2, a^3\}$, $B = \{b^1, b^2, b^3\}$, $C = \{c^1, c^2, c^3\}$ and $L(\sigma_4) : R(\sigma_4) = A^{-1} \cup B^{-1} \cup C^{-1} : A^1 \cup B^1 \cup C^1 = \{a^1, b^1, c^1\} : \{a^3, b^3, c^3\}$ then $N(\sigma_4) = \{a^2, b^2, c^2\}$. It follows from Eqs. (13)–(15) that

$$\sigma_4^0 = a_{\rm h}^1 \times c_{\rm h}^3 + a_{\rm h}^2 \times c_{\rm h}^2 + a_{\rm h}^3 \times c_{\rm h}^1$$
$$+ b_{\rm l}^1 \times c_{\rm l}^3 + b_{\rm l}^2 \times c_{\rm l}^2 + b_{\rm l}^3 \times c_{\rm l}^1 + a_{\rm h}^1 \times a_{\rm h}^3 + b_{\rm l}^1 \times b_{\rm l}^3,$$

$$\sigma_4^1 = a_{\rm h}^2 \times c_{\rm h}^3 \times a_{\rm h}^3 \times c_{\rm h}^2 + a_{\rm h}^3 \times a_{\rm h}^2 + b_{\rm l}^2 \times c_{\rm l}^1 \times b_{\rm l}^1 \times c_{\rm l}^2 + b_{\rm l}^1 \times b_{\rm l}^2,$$

$$\sigma_4^{-1} = a_{\rm h}^2 \times c_{\rm h}^1 \times a_{\rm h}^1 \times c_{\rm h}^2 + a_{\rm h}^1 \times a_{\rm h}^2 + b_{\rm l}^2 \times c_{\rm l}^3 \times b_{\rm l}^3 \times c_{\rm l}^2 + b_{\rm l}^3 \times b_{\rm l}^2.$$

We observe that $\sigma_4^{-1} \cong \sigma_4^1$ and $|\sigma_4^{-1}|=|\sigma_4^1|=|\sigma_4^0|=8 < 3^2$. By performing $L(\sigma_4^0):R(\sigma_4^0)=\{a^1,b^1,c^3\}:\{a^3,b^3,c^2\}$, $L(\sigma_4^1):R(\sigma_4^1)=\{a^2,b^2\}:\{a^3,b^1\}$, it follows from Eqs. (10) and (11) that

$$\sigma_4^{00} = a_{\rm h}^1 \times a_{\rm h}^3 + b_{\rm l}^1 \times b_{\rm l}^3 \quad \text{by } a^2 : a^3$$

$$\sigma_4^{10} = a_{\rm h}^3 \times a_{\rm h}^2 + b_{\rm l}^1 \times b_{\rm l}^2 \quad \text{by } a^1 : a^3$$

$$\sigma_4^{01} = a_{\rm h}^2 \times c_{\rm h}^2 + a_{\rm h}^3 \times c_{\rm h}^1 + b_{\rm l}^1 \times c_{\rm l}^3 \quad \text{by } a^2 : a^3$$

$$\sigma_4^{11} = a_{\rm h}^3 \times \{c^2,c^3\}_{\rm h} + b_{\rm l}^2 \times c_{\rm l}^1 \quad \text{by } c^2 : c^3$$

$$\sigma_4^{0-1} = a_{\rm h}^1 \times c_{\rm h}^3 + b_{\rm l}^2 \times c_{\rm l}^2 + b_{\rm l}^3 \times c_{\rm l}^1 \quad \text{by } b^2 : b^3$$

$$\sigma_4^{1-1} = a_{\rm h}^2 \times c_{\rm h}^3 + b_{\rm l}^1 \times \{c^1,c^2\}_{\rm l} \quad \text{by } c^1 : c^2. \qquad \square$$

Proof of Theorem 3. We need only to verify that the four conditions of Theorem 1 hold for $q=3, n_q=5$. All isomorphic (or symmetrically isomorphic) search domains are not listed in the search trees and all search domains can be obtained by Eqs. (10) and (11).

(1) We prove that $q=3$ weighings can identify the solution of $\sigma_1(5)=A_{\rm h}\times B_{\rm h}(A\cap B=\emptyset)$. Let $A=\{a^1,a^2,\ldots,a^5\}$, $B=\{b^1,b^2,\ldots,b^5\}$ and the first weighing $L:R=\{a^1,a^2\}\cup\{b^1,b^2\}:\{a^4,a^5\}\cup\{b^4,b^5\}$.

$$\sigma_1, \quad \text{by } L:R \begin{cases} \sigma_1^0, \text{ by } \{a^1,a^4,b^4\}:\{a^2,a^5,b^5\} \begin{cases} \sigma_1^{00}, \text{ by } a^1:a^2 \\ \sigma_1^{01}, \text{ by } b^1:b^2 \end{cases} \\ \sigma_1^1, \text{ by } \{b^4,a^4\}:\{b^5,a^5\} \begin{cases} \sigma_1^{10}, \text{ by } a^4:a^5 \\ \sigma_1^{11}, \text{ by } a^3:b^3. \end{cases} \end{cases}$$

(2) We prove that $q=3$ weighings can split $\sigma_2(5)=A_{\rm h}\times B_{\rm h}+A_{\rm l}\times B_{\rm l}$ into configurations with cardinality $\leqslant 2$. Let $A=\{a^1,a^2,\ldots,a^5\}$, $B=\{b^1,b^2,\ldots,b^5\}$ and $L:R=\{a^1,a^2\}\cup\{b^1,b^2\}:\{a^4,a^5\}\cup\{b^4,b^5\}$ then $N=\{a^3,b^3\}$. The desired search tree is given below.

$$\sigma_2, \quad \text{by } L:R \begin{cases} \sigma_2^0, \text{ by } \{a^1,a^4,b^4\}:\{a^2,a^5,b^5\} \begin{cases} \sigma_2^{00}, \text{ by } a^1:a^2 \\ \sigma_2^{01}, \text{ by } b^1:b^2 \end{cases} \\ \sigma_2^1, \ \{b^4,a^4,b^1,a^1\}:\{b^5,a^5,b^2,a^2\} \begin{cases} \sigma_2^{10}, \ \{a^1,a^4\}:\{a^2,a^5\} \\ \sigma_2^{11}, \ a^3:b^3. \end{cases} \end{cases}$$

(3) For $\sigma_3(5)=C_{\rm hh}+C_{\rm ll}$. Let $C=\{c^1,c^2,\ldots,c^5\}$ and $L:R=\{c^1,c^2\}:\{c^4,c^5\}$ then $N=\{c^3\}$.

$$\sigma_3, \quad \text{by } L:R \begin{cases} \sigma_3^0, \text{ by } \{c^1,c^4\}:\{c^2,c^3\} \begin{cases} \sigma_3^{00}, \text{ by } c^3:c^2 \\ \sigma_3^{01}, \text{ by } c^3:c^5 \end{cases} \\ \sigma_3^1, \text{ by } \{c^1,c^4\}:\{c^2,c^5\} \begin{cases} \sigma_3^{10}, \text{ by } c^3:c^5 \\ \sigma_3^{11}, \text{ by } c^2:c^3. \end{cases} \end{cases}$$

(4) It suffices to prove that $q+1=4$ weighings can identify the solution of $\sigma_4(5)=A_{\rm h}\times C_{\rm h}+B_{\rm l}\times C_{\rm l}+A_{\rm hh}+B_{\rm ll}$. Let $A=\{a^1,a^2,\ldots,a^5\}$, $B=\{b^1,b^2,\ldots,b^5\}$, $C=\{c^1,c^2,\ldots,c^5\}$ and $L:R=\{a^1,a^2\}\cup\{b^1,b^2\}\cup\{c^1,c^2\}:\{a^4,a^5\}\cup\{b^4,b^5\}\cup\{c^4,c^5\}$ then $N=\{a^3,b^3,c^3\}$.

$$\sigma_4,\quad \text{by } L:R\begin{cases}\sigma_4^0,\ \text{by } \{b^1,c^1,c^4\}:\{b^2,c^2,c^5\}\\ \begin{cases}\sigma_4^{00},\ \text{by } \{a^1,b^1\}:\{a^2,b^2\}\begin{cases}\sigma_4^{000},\ \text{by } c^2:c^3\\ \sigma_4^{001},\ \text{by } a^4:a^5\end{cases}\\ \sigma_4^{01},\ \text{by } \{a^4,b^4\}:\{a^5,b^5\}\begin{cases}\sigma_4^{010},\ \text{by } a^1:a^2\\ \sigma_4^{011},\ \text{by } b^1:c^1\end{cases}\end{cases}\\ \sigma_4^1,\ \text{by } \{c^4,a^4,c^1,b^1\}:\{c^5,a^5,c^2,b^2\}\\ \begin{cases}\sigma_4^{10},\ \text{by } \{a^5,b^2\}:\{a^4,b^1\}\begin{cases}\sigma_4^{100},\ \text{by } a^3:a^5\\ \sigma_4^{101},\ \text{by } a^3:a^4\end{cases}\\ \sigma_4^{11},\ \text{by } \{a^3,b^3\}:\{a^5,b^1\}\begin{cases}\sigma_4^{110},\ \text{by } a^3:a^4\\ \sigma_4^{111},\ \text{by } c^3:c^5.\end{cases}\end{cases}\end{cases}$$

□

4. The Proofs of Lemmas 9–11

The proofs of Lemmas 9–11 are related to the following models. For model $G_{0,\rm h,l}(n)$, Eves [6] gave the following result: we may determine the existence of a counterfeit coin and its identity among $n=(3^t-1)/2$ coins in $t=\lceil\log_3(2n+1)\rceil$ weighings if we are given 3^{t-1} good coins. There are not so many coins known to be good, we need to establish the following Proposition 1.

$G_{\rm h}(n)$: G has one heavier counterfeit coin. $G_{\rm l}(n)$ is defined similarly.

$G_{0,\rm h}(n)$: G has no counterfeit coin or has one heavier counterfeit coin. $G_{0,\rm l}(n)$ is defined similarly.

$G_{\rm h,l}(n)$: G has one heavier or lighter counterfeit coin.

$G_{0,\rm h,l}(n)$: G has no counterfeit coin or one heavier counterfeit coin or one lighter counterfeit coin.

Proposition 1. (1) $\lceil\log_3 n\rceil$ *weighings can identify the solution of* $G_{\rm h}(n)$ $(G_{\rm l}(n))$;

(2) $\lceil\log_3(n+1)\rceil$ *weighings can identify the solution of* $G_{0,\rm h}(n)$ $(G_{0,\rm l}(n))$;

(3) $\lceil\log_3(2n+2)\rceil$ *weighings can identify the solution of* $G_{\rm h,l}(n)$;

(4) $\lceil\log_3(2n+2)\rceil$ *weighings can identify the solution of* $G_{0,\rm h,l}(n)$.

(5) *We are given an extra coin known to be good.* $\lceil\log_3(2n)\rceil$ *weighings can identify the solution of* $G_{\rm h,l}(n)$; $\lceil\log_3(2n+1)\rceil$ *weighings can identify the solution of* $G_{0,\rm h,l}(n)$.

Proof. (1) See [1, p. 81].

(2) See [1, Exercise 2, p. 91].

(3) See [1, p. 87].

(4) See [1, Remark p. 91].

(5) Let $3^{t-1} < 2n \leqslant 3^t$, $n \neq (3^t-1)/2$. The former is settled by (3) as $\lceil \log_3(2n+2) \rceil = \lceil \log_3(2n) \rceil$, the later is settled by (4) as $\lceil \log_3(2n+2) \rceil = \lceil \log_3(2n+1) \rceil$. For $n=(3^t-1)/2$, Linial and Tarsi have given a predetermined algorithm

$$M_t = \begin{cases} B_t \\ (R^t) \end{cases}$$

which identifies the solution of $G_{\mathrm{h},1}(n)$ by $t=\lceil \log_3(2n) \rceil$ weighings and it is well-known that a predetermined algorithm is a special sequential algorithm. It is evident that the algorithm given by Linial and Tarsi can also identify the solution of $G_{0,\mathrm{h},1}(n)$ by $t=\lceil \log_3(2n+1) \rceil$ weighings (for more details, see [10, p. 415]). □

Proposition 2. *Suppose search domain S be one of the following configurations and* $n=(3^t-1)/2$. *Then we can identify the solution of S by t weighings.*

(1) $S=\sum_{i=1}^{n}(a_\mathrm{h}^i \times b_\mathrm{h}^i + c_\mathrm{l}^i \times d_\mathrm{l}^i)$; $S=\sum_{i=1}^{n}(a_\mathrm{h}^i \times b_\mathrm{h}^i + c_\mathrm{l}^i \times d_\mathrm{l}^i) + a_\mathrm{h}^0 \times b_\mathrm{h}^0$; $S=\sum_{i=1}^{n}(a_\mathrm{h}^i \times b_\mathrm{h}^i + c_\mathrm{l}^i \times d_\mathrm{l}^i) + a_\mathrm{l}^0 \times b_\mathrm{l}^0$; *where* $\{b^i, d^i \mid 1 \leqslant i \leqslant n\} \cup \{b^0\}$ *is a set of pairwise distinct coins and there exist two good coins beforehand.*

(2) $S=\sum_{i=1}^{n}(a_\mathrm{h}^i \times b_\mathrm{h}^i + a_\mathrm{l}^i \times b_\mathrm{l}^i)$; $S=\sum_{i=1}^{n}(a_\mathrm{h}^i \times b_\mathrm{h}^i + a_\mathrm{l}^i \times b_\mathrm{l}^i) + a_\mathrm{h}^0 \times b_\mathrm{h}^0$; $S=\sum_{i=1}^{n}(a_\mathrm{h}^i \times b_\mathrm{h}^i + a_\mathrm{l}^i \times b_\mathrm{l}^i) + a_\mathrm{l}^0 \times b_\mathrm{l}^0$; *where* $\{a^i, b^i \mid 1 \leqslant i \leqslant n\} \cup \{a^0, b^0\}$ *is a set of pairwise distinct coins and there exists a good coin beforehand.*

(3) $S=\sum_{i=1}^{n}(a_\mathrm{h}^i \times b_\mathrm{h}^i + c_\mathrm{h}^i \times d_\mathrm{h}^i)$; $S=\sum_{i=1}^{n}(a_\mathrm{h}^i \times b_\mathrm{h}^i + c_\mathrm{h}^i \times d_\mathrm{h}^i) + a_\mathrm{h}^0 \times b_\mathrm{h}^0$; $S=\sum_{i=1}^{n}(a_\mathrm{h}^i \times b_\mathrm{h}^i + c_\mathrm{h}^i \times d_\mathrm{h}^i) + a_\mathrm{l}^0 \times b_\mathrm{l}^0$; *where* $\{b^i, d^i \mid 1 \leqslant i \leqslant n\} \cup \{b^0\}$ *is a set of pairwise distinct coins.*

(4) $S=\sum_{i=1}^{n} a_\mathrm{h}^i \times b_\mathrm{h}^i$; $S=\sum_{i=1}^{n} a_\mathrm{h}^i \times b_\mathrm{h}^i + a_\mathrm{h}^0 \times b_\mathrm{h}^0$; $S=\sum_{i=1}^{n} a_\mathrm{h}^i \times b_\mathrm{h}^i + a_\mathrm{l}^0 \times b_\mathrm{l}^0$.

Proof. (1) Let $G=\{g_i=\{b^i, d^i\} \mid 1 \leqslant i \leqslant n\}$ then G is a set of n super-coins consisting of two coins. Identifying the solution of $S=\sum_{i=1}^{n}(a_\mathrm{h}^i \times b_\mathrm{h}^i + c_\mathrm{l}^i \times d_\mathrm{l}^i)$ is equivalent to identifying the solution of $G_{\mathrm{h},1}(n)$ because a solution of $G_{\mathrm{h},1}(n)$ corresponds to a unique solution of S (if the final solution of $G_{\mathrm{h},1}(n)$ is "$\{b^i, d^i\}$ is a heavy (light) super-coin" then the corresponding solution of S is $a_\mathrm{h}^i \times b_\mathrm{h}^i$ ($c_\mathrm{l}^i \times d_\mathrm{l}^i$)). By Proposition 1(5), $\lceil \log_3(2n) \rceil = t$ weighings suffice as we have a good super-coin consisting of two given good coins. Identifying the solution of $S=\sum_{i=1}^{n}(a_\mathrm{h}^i \times b_\mathrm{h}^i + c_\mathrm{l}^i \times d_\mathrm{l}^i) + a_\mathrm{h}^0 \times b_\mathrm{h}^0$ is equivalent to identifying the solution of $G_{0,\mathrm{h},1}(n)$ because a solution of $G_{0,\mathrm{h},1}(n)$ corresponds to a unique solution of S (if the final solution of $G_{0,\mathrm{h},1}(n)$ is "G has no counterfeit coin" then the corresponding solution of S is $a_\mathrm{h}^0 \times b_\mathrm{h}^0$). By Proposition 1(5), $\lceil \log_3(2n+1) \rceil = t$ weighings suffice. The third configuration is similar to the second case.

(2) Let $G=\{b^i \mid i=1,2,\ldots,n\}$ then $|G|=n$. Identifying the solution of $S=\sum_{i=1}^{n}(a_\mathrm{h}^i \times b_\mathrm{h}^i + a_\mathrm{l}^i \times b_\mathrm{l}^i)$ is equivalent to identifying the solution of $G_{\mathrm{h},1}(n)$. By Proposition 1(5), $\lceil \log_3(2n) \rceil = t$ weighings suffice as a good coin is given. Identifying the solution of $S=\sum_{i=1}^{n}(a_\mathrm{h}^i \times b_\mathrm{h}^i + a_\mathrm{l}^i \times b_\mathrm{l}^i) + a_\mathrm{h}^0 \times b_\mathrm{h}^0$ is equivalent to identifying the solution of $G_{0,\mathrm{h},1}(n)$. By Proposition 1(5), $\lceil \log_3(2n+1) \rceil = t$ weighings suffice.

(3) Let $G=\{b^i, d^i \mid i=1,2,\ldots,n\}$, then $|G|=2n$. Identifying the solution of $S=\sum_{i=1}^{n}(a_\mathrm{h}^i \times b_\mathrm{h}^i + c_\mathrm{h}^i \times d_\mathrm{h}^i)$ is equivalent to identifying the solution of $G_\mathrm{h}(2n)$ (if the final solution of $G_\mathrm{h}(2n)$ is "b^i (d^i) is a heavy coin" then the corresponding solution of S

is $a^i_{\rm h} \times b^i_{\rm h}$ ($c^i_{\rm h} \times d^i_{\rm h}$)). By Proposition 1(1), $\lceil \log_3 |G| \rceil = t$ weighings suffice. Identifying the solution of $S = \sum_{i=1}^n (a^i_{\rm h} \times b^i_{\rm h} + c^i_{\rm h} \times d^i_{\rm h}) + a^0_{\rm h} \times b^0_{\rm h}$ is equivalent to identifying the solution of $G_{0,\rm h}(2n)$. By Proposition 1(2), $\lceil \log_3(|G|+1) \rceil = t$ weighings suffice. The third configuration is similar to the second case.

(4) Let $G = \{b^i \mid i = 1, 2, \ldots, n\}$ then $|G| = n$. Identifying the solution of $S = \sum_{i=1}^n a^i_{\rm h} \times b^i_{\rm h}$ is equivalent to identifying the solution of $G_{\rm h}(n)$. By Proposition 1(1), $\lceil \log_3 |G| \rceil \leqslant t$ weighings suffice. Identifying the solution of $S = \sum_{i=1}^n a^i_{\rm h} \times b^i_{\rm h} + a^0_{\rm h} \times b^0_{\rm h}$ is equivalent to identifying the solution of $G_{0,\rm h}(n)$. By Proposition 1(2), $\lceil \log_3(|G|+1) \rceil \leqslant t$ weighings suffice. □

Proof of Lemma 9. Letting $r = K, s = 0$ in Eq. (20), we have $S^{\mathbf{0_K}} = \sum_{\mathbf{I_K} > \mathbf{0_K}} (G^{\mathbf{I_K}}_{\rm h} \times G^{\bar{\mathbf{I}}_\mathbf{K}}_{\rm h} + G^{\mathbf{I_K}}_{\rm l} \times G^{\bar{\mathbf{I}}_\mathbf{K}}_{\rm l}) + G^{\mathbf{0_K}}_{\rm hh} + G^{\mathbf{0_K}}_{\rm ll}$. If $K = 0$, $S^{\mathbf{0_K}} = G_{\rm hh} + G_{\rm ll} \cong \sigma_a(n_q)$ thus q weighings can identify the solution of S by the given condition (1). If $K \geqslant 1$, we notice that $n \triangleq |\{\mathbf{I_K} \mid \mathbf{I_K} > \mathbf{0_K}\}| = (3^K - 1)/2$ and all vertex-sets $G^{\mathbf{I_K}}$, $G^{\bar{\mathbf{I}}_\mathbf{K}}$, $G^{\mathbf{0_K}}$ are sets of n_q coins and pairwise disjoint thus $S^{\mathbf{0_K}}$ can be rewritten as $S^{\mathbf{0_K}} = \sum_{i=1}^n (A^i_{\rm h} \times B^i_{\rm h} + A^i_{\rm l} \times B^i_{\rm l}) + A^0_{\rm hh} + A^0_{\rm ll}$, where A^i, B^i, A^0 are sets of n_q coins and pairwise disjoint. We observe that for $i = 1, 2, \ldots, n$, $S_i \triangleq A^i_{\rm h} \times B^i_{\rm h} + A^i_{\rm l} \times B^i_{\rm l} \cong \sigma_b(n_q)$ and $S_0 \triangleq A^0_{\rm hh} + A^0_{\rm ll} \cong \sigma_a(n_q)$. Thus, there exist n bijections $\varphi_i : V(\sigma_b) \mapsto V(S_i)$ such that $\varphi_i(\sigma_b) = S_i$ and a bijection $\varphi_0 : V(\sigma_a) \mapsto V(S_0)$ such that $\varphi_0(\sigma_a) = S_0$. We will prove that $K + q$ weighings can identify the solution of $S \triangleq \sum_{i=1}^n S_i + S_0 = S^{\mathbf{0_K}}$. For $1 \leqslant j \leqslant q$, the test-set $L^j(S) : R^j(S)$ of the jth weighing of S is determined by the following *union-image method*: we have had the test-set $L^j(\sigma_a) : R^j(\sigma_a)$, $L^j(\sigma_b) : R^j(\sigma_b)$ of the jth weighing of σ_a given by Condition (1), σ_b given by Condition (2) respectively; the test-sets of the jth-weighing of S_i and S_0 are given by the corresponding image-sets, i.e., $L^j(S_i) : R^j(S_i) = \varphi_i(L^j(\sigma_b)) : \varphi_i(R^j(\sigma_b))$ for $i = 1, 2, \ldots, n$ and $L^j(S_0) : R^j(S_0) = \varphi_0(L^j(\sigma_a)) : \varphi_0(R^j(\sigma_a))$; Furthermore, let $L^j(S) = \bigcup_{i=1}^n L^j(S_i) \cup L^j(S_0)$ $R^j(S) = \bigcup_{i=1}^n R^j(S_i) \cup R^j(S_0)$. It follows from Isomorphism Theorem that for $\mathbf{F_q} \in \{-1, 0, 1\}^q$, $S_i^{\mathbf{F_q}} \cong \sigma_b^{\mathbf{F_q}}$, $S_0^{\mathbf{F_q}} \cong \sigma_a^{\mathbf{F_q}}$. The fact $V(S_i)$ $(i = 1, 2, \ldots, n)$ are pairwise disjoint implies that $S^{\mathbf{F_q}} = \sum_{i=1}^n S_i^{\mathbf{F_q}} + S_0^{\mathbf{F_q}}$. Condition (1) implies that $S_0^{\mathbf{F_q}} = \emptyset$ or $S_0^{\mathbf{F_q}} = a^0_{\rm h} \times b^0_{\rm h}$ or $S_0^{\mathbf{F_q}} = a^0_{\rm l} \times b^0_{\rm l}$; condition (2) implies that $S_i^{\mathbf{F_q}}$ $(i = 1, 2, \ldots, n)$ is a configuration with cardinality $\leqslant 2$. i.e., $S_i^{\mathbf{F_q}} = a^i_{\rm h} \times b^i_{\rm h}$, or $a^i_{\rm h} \times b^i_{\rm h} + c^i_{\rm h} \times d^i_{\rm h} (b^i \neq d^i)$, or $a^i_{\rm h} \times b^i_{\rm h} + c^i_{\rm l} \times d^i_{\rm l} (b^i \neq d^i)$, or $a^i_{\rm h} \times b^i_{\rm h} + a^i_{\rm l} \times b^i_{\rm l}$ in the sense of isomorphism. Combining all possible cases, $S^{\mathbf{F_q}}$ is always one of the configurations listed in Proposition 2 and we notice that $V(S^{\mathbf{F_q}})$ contains at most $4n+2$ coins $(|\{a^i, b^i, c^i, d^i \mid i = 1, 2, \ldots, n\} \cup \{a^0, b^0\}| \leqslant 4n + 2)$, i.e., we have found at least $n_q \cdot 3^K - 4n - 2 \geqslant 2$ good coins as $n_q \geqslant 3$ and $K \geqslant 1$. Thus, next K weighings suffice to identify the solution of $S^{\mathbf{F_q}}$ by Proposition 2. Therefore, $K + q$ weighings can identify the solution of $S = S^{\mathbf{0_K}}$. □

Proof of Lemma 10. By letting $\ell = K - 1$, $\mathbf{F}_\ell = \mathbf{0_{K-1}}$ in Eq. (18) and applying $\{(\mathbf{I_{K-1}}, \mathbf{J_{K-1}}) \mid \mathbf{I_{K-1}} \oplus \mathbf{J_{K-1}} = \mathbf{0_{K-1}}; \mathbf{I_{K-1}} > \mathbf{J_{K-1}}\} \Leftrightarrow \{(\mathbf{I_{K-1}}, \bar{\mathbf{I}}_\mathbf{K-1}) \mid \mathbf{I_{K-1}} > \mathbf{0_{K-1}}\}$, we have

$$S^{\mathbf{0_{K-1}}1} = \sum_{\mathbf{I_{K-1}} > \mathbf{0_{K-1}}} (G^{\mathbf{I_{K-1}}0}_{\rm h} \times G^{\bar{\mathbf{I}}_\mathbf{K-1}1}_{\rm h} \times G^{\mathbf{I_{K-1}}1}_{\rm h} \times G^{\bar{\mathbf{I}}_\mathbf{K-1}0}_{\rm h}$$

$$+ G_{\mathrm{l}}^{\bar{\mathbf{I}}_{\mathbf{K-1}}0} \times G_{\mathrm{l}}^{\mathbf{I}_{\mathbf{K-1}}-1} \times G_{\mathrm{l}}^{\bar{\mathbf{I}}_{\mathbf{K-1}}-1} \times G_{\mathrm{l}}^{\mathbf{I}_{\mathbf{K-1}}0})$$

$$+ (G_{\mathrm{h}}^{\mathbf{0}_{\mathbf{K-1}}0} \times G_{\mathrm{h}}^{\mathbf{0}_{\mathbf{K-1}}1} + G_{\mathrm{l}}^{\mathbf{0}_{\mathbf{K-1}}0} \times G_{\mathrm{l}}^{\mathbf{0}_{\mathbf{K-1}}-1} + G_{\mathrm{hh}}^{\mathbf{0}_{\mathbf{K-1}}1} + G_{\mathrm{ll}}^{\mathbf{0}_{\mathbf{K-1}}-1}).$$

If $K = 1$, $S^1 = G_{\mathrm{h}}^0 \times G_{\mathrm{h}}^1 + G_{\mathrm{l}}^0 \times G_{\mathrm{l}}^{-1} + G_{\mathrm{hh}}^1 + G_{\mathrm{ll}}^{-1} \cong \sigma_c(n_q)$, thus $1 + q$ weighings can identify the solution of S^1 by the given condition (1). For $K \geqslant 2$, we notice that $n \triangleq |\{\mathbf{I}_{\mathbf{K-1}} \mid \mathbf{I}_{\mathbf{K-1}} > \mathbf{0}_{\mathbf{K-1}}\}| = (3^{K-1} - 1)/2$ and all vertex-sets $G^{\mathbf{I}_{\mathbf{K-1}}j}$, $G^{\bar{\mathbf{I}}_{\mathbf{K-1}}j}$, $G^{\mathbf{0}_{\mathbf{K-1}}j}$ are sets of n_q coins and pairwise disjoint thus $S^{\mathbf{0}_{\mathbf{K-1}}1}$ can be rewritten as $S^{\mathbf{0}_{\mathbf{K-1}}1} = \sum_{i=1}^{n} (A_{\mathrm{h}}^i \times B_{\mathrm{h}}^i \times C_{\mathrm{h}}^i \times D_{\mathrm{h}}^i + A_{\mathrm{l}}^i \times U_{\mathrm{l}}^i \times V_{\mathrm{l}}^i \times D_{\mathrm{l}}^i) + (A_{\mathrm{h}}^0 \times C_{\mathrm{h}}^0 + B_{\mathrm{l}}^0 \times C_{\mathrm{l}}^0 + A_{\mathrm{hh}}^0 + B_{\mathrm{ll}}^0) \triangleq \sum_{i=1}^{n} S_i + S_0$. We observe that for $i = 1, 2, \ldots, n$, $S_i \cong \sigma_d(n_q)$ and $S_0 \cong \sigma_c(n_q)$. For $j = 1, 2, \ldots, q + 1$, the test-set $L^j(S) : R^j(S)$ of the jth weighing of S is given by the *union-image method* as used in Lemma 9, we have that for $\mathbf{F}_{\mathbf{q+1}} \in \{-1, 0, 1\}^{q+1}$, $S^{\mathbf{F}_{\mathbf{q+1}}} = \sum_{i=1}^{n} S_i^{\mathbf{F}_{\mathbf{q+1}}} + S_0^{\mathbf{F}_{\mathbf{q+1}}}$. The remaining proof is similar to that of Lemma 9 and $K - 1$ weighings can identify the solution of $S^{\mathbf{F}_{\mathbf{q+1}}}$ thus $(q + 1) + (K - 1) = K + q$ weighings suffice to identify the solution of $S^{\mathbf{0}_{\mathbf{K-1}}1}$. □

We will prove Lemma 11 by induction on r and s. The idea can be described as follows: After the former $K = r + s$ weighings, we have a search domain $S^{\mathbf{0}_{\mathbf{r}}\mathbf{1}_{\mathbf{s}}} \triangleq \mathscr{S}(r; s)$, where $2 \leqslant s \leqslant K, r = K - s$; Performing the next s weighings by induction on s, $\mathscr{S}(r; s)$ can be reduced to two search domains $\mathscr{S}_b(r; 0)$ and $\mathscr{S}(r; 0)$ in the sense of isomorphism; The next r weighings by induction on r will reduce $\mathscr{S}_b(r; 0)$ and $\mathscr{S}(r; 0)$ to two search domains $\mathscr{S}_a(0; 0)$ and $\mathscr{S}(0; 0)$ in the sense of isomorphism; If q weighings can identify the solution of $\mathscr{S}_a(0; 0) \cong \sigma_e(n_q)$ and $\mathscr{S}(0; 0) \cong \sigma_f(n_q)$ then $s + r + q = K + q$ weighings can identify the solution of $\mathscr{S}(r; s) = S^{\mathbf{0}_{\mathbf{r}}\mathbf{1}_{\mathbf{s}}}$. We firstly establish Propositions 3–6 serving for Lemma 11. The relations between them can be shown as follows:

$$S^{\mathbf{0}_{\mathbf{r}}\mathbf{1}_{\mathbf{s}}} \triangleq \mathscr{S}(r; s) \xrightarrow{\text{Lemma 11, 1 weighing}} \begin{cases} \mathscr{S}_b(r; s-1) \xrightarrow{\text{Proposition 6, } s-1 \text{ weighings}} \mathscr{S}_b(r; 0) \\ \mathscr{S}(r; s-1) \xrightarrow{\text{Lemma 11, } s-1 \text{ weighings}} \begin{cases} \mathscr{S}_b(r; 0) \\ \mathscr{S}(r; 0) \end{cases} \end{cases}$$

$$\mathscr{S}(r; 0) \xrightarrow{\text{Proposition 4, 1 weighing}}$$

$$\begin{cases} \mathscr{S}_a(r-1; 0) \xrightarrow{\text{Proposition 3, } r-1 \text{ weighings}} \mathscr{S}_a(0; 0) \cong \sigma_e(n_q) \\ \mathscr{S}(r-1; 0) \xrightarrow{\text{Proposition 4, } r-1 \text{ weighings}} \begin{cases} \mathscr{S}_a(0; 0) \cong \sigma_e(n_q) \\ \mathscr{S}(0; 0) \cong \sigma_f(n_q) \end{cases} \end{cases}$$

$$\mathscr{S}_b(r; 0) \xrightarrow{\text{Proposition 5, 1 weighing}}$$

$$\begin{cases} \mathscr{S}_a(r-1; 0) \xrightarrow{\text{Proposition 3, } r-1 \text{ weighings}} \mathscr{S}_a(0; 0) \cong \sigma_e(n_q) \\ \mathscr{S}_b(r-1; 0) \xrightarrow{\text{Proposition 5, } r-1 \text{ weighings}} \begin{cases} \mathscr{S}_a(0; 0) \cong \sigma_e(n_q) \\ \mathscr{S}_b(0; 0) \cong \sigma_e(n_q). \end{cases} \end{cases}$$

Proposition 3. *Suppose* $2 \leqslant s \leqslant K$, $r = K - s$. *If* q *weighings can identify the solution of* $\sigma_e(n_q)$ *then* $m+q$ *weighings can identify the solution of* $\mathscr{S}_a(m;0)$ $(0 \leqslant m \leqslant r-1)$, *where*

$$\mathscr{S}_a(m;0) \triangleq \sum_{\mathbf{I_m}} G_{\rm h}^{\mathbf{1_{r-m}I_m1_s}} \times G_{\rm h}^{(-\mathbf{1_{r-m}})\bar{\mathbf{I}}_\mathbf{m}\mathbf{1_s}}.$$

Proof. We proceed by induction on m. For $m=0$, $\mathscr{S}_a(0;0) = G_{\rm h}^{\mathbf{1_r1_s}} \times G_{\rm h}^{(-\mathbf{1_r})\mathbf{1_s}} \cong \sigma_e(n_q)$. For $1 \leqslant m \leqslant r-1$, $\mathscr{S}_a(m;0)$ can be rewritten by the equality $\{\mathbf{I_m} \mid \mathbf{I_m} \in \{-1,0,1\}^m\} = \{i\mathbf{I_{m-1}} \mid i \in \{-1,0,1\}, \mathbf{I_{m-1}} \in \{-1,0,1\}^{m-1}\}$:

$$\begin{aligned}\mathscr{S}_a(m;0) = \sum_{\mathbf{I_{m-1}}} (&G_{\rm h}^{\mathbf{1_{r-m}}(-1)\mathbf{I_{m-1}1_s}}(*L) \times G_{\rm h}^{(-\mathbf{1_{r-m}})1\bar{\mathbf{I}}_\mathbf{m-1}\mathbf{1_s}}\\ &+ G_{\rm h}^{\mathbf{1_{r-m}}0\mathbf{I_{m-1}1_s}} \times G_{\rm h}^{(-\mathbf{1_{r-m}})0\bar{\mathbf{I}}_\mathbf{m-1}\mathbf{1_s}}\\ &+ G_{\rm h}^{\mathbf{1_{r-m}}1\mathbf{I_{m-1}1_s}}(*R) \times G_{\rm h}^{(-\mathbf{1_{r-m}})(-1)\bar{\mathbf{I}}_\mathbf{m-1}\mathbf{1_s}}).\end{aligned}$$

Let left test-set $L = \{G^{\mathbf{1_{r-m}}(-1)\mathbf{I_{m-1}1_s}} \mid \mathbf{I_{m-1}} \in \{-1,0,1\}^{m-1}\}$ which is marked by $(*L)$ in $\mathscr{S}_a(m;0)$; right test-set $R = \{G^{\mathbf{1_{r-m}}1\mathbf{I_{m-1}1_s}} \mid \mathbf{I_{m-1}} \in \{-1,0,1\}^{m-1}\}$ which is marked by $(*R)$. We note that $|L| = |R|$ and $L \cap R = \emptyset$. It follows from Eq. (10) that

$$\mathscr{S}_a^0 = \sum_{\mathbf{I_{m-1}}} G_{\rm h}^{\mathbf{1_{r-m}}0\mathbf{I_{m-1}1_s}} \times G_{\rm h}^{(-\mathbf{1_{r-m}})0\bar{\mathbf{I}}_\mathbf{m-1}\mathbf{1_s}},$$

$$\mathscr{S}_a^1 = \sum_{\mathbf{I_{m-1}}} G_{\rm h}^{\mathbf{1_{r-m}}1\mathbf{I_{m-1}1_s}} \times G_{\rm h}^{(-\mathbf{1_{r-m}})(-1)\bar{\mathbf{I}}_\mathbf{m-1}\mathbf{1_s}},$$

$$\mathscr{S}_a^{-1} = \sum_{\mathbf{I_{m-1}}} G_{\rm h}^{\mathbf{1_{r-m}}(-1)\mathbf{I_{m-1}1_s}} \times G_{\rm h}^{(-\mathbf{1_{r-m}})1\bar{\mathbf{I}}_\mathbf{m-1}\mathbf{1_s}}.$$

We see that $\mathscr{S}_a^0 \cong \mathscr{S}_a^{-1} \cong \mathscr{S}_a^1 = \mathscr{S}_a(m-1;0)$. The hypothesis and Isomorphism Theorem imply that we can identify the solution of $\mathscr{S}_a^1$, $\mathscr{S}_a^{-1}$, $\mathscr{S}_a^0$ by $(m-1)+q$ weighings. Thus, we can identify the solution of $\mathscr{S}_a(m;0)$ by $1+(m-1)+q = m+q$ weighings. □

Proposition 4. *Suppose* $2 \leqslant s \leqslant K$, $r = K - s$. *If* q *weighings can identify the solution of* $\sigma_e(n_q)$, $\sigma_f(n_q)$ *respectively then* $m+q$ *weighings can identify the solution of* $\mathscr{S}(m;0)$ $(0 \leqslant m \leqslant r)$, *where*

$$\begin{aligned}\mathscr{S}(m;0) \triangleq \sum_{\mathbf{I_m} > \mathbf{0_m}} &(G_{\rm h}^{\mathbf{0_{r-m}I_m1_s}} \times G_{\rm h}^{\mathbf{0_{r-m}}\bar{\mathbf{I}}_\mathbf{m}\mathbf{1_s}} + G_{\rm l}^{\mathbf{0_{r-m}}\bar{\mathbf{I}}_\mathbf{m}(-\mathbf{1_s})} \times G_{\rm l}^{\mathbf{0_{r-m}I_m}(-\mathbf{1_s})})\\ &+ G_{\rm hh}^{\mathbf{0_r1_s}} + G_{\rm ll}^{\mathbf{0_r}(-\mathbf{1_s})}.\end{aligned} \tag{21}$$

Proof. We proceed by induction on m. For $m = 0$, $\mathscr{S}(0;0) = G_{\rm hh}^{\mathbf{0_r1_s}} + G_{\rm ll}^{\mathbf{0_r}(-\mathbf{1_s})} \cong \sigma_f(n_q)$. By the assumption on $\sigma_f(n_q)$, q weighings suffices to identify the solution of $\mathscr{S}(0;0)$. For $1 \leqslant m \leqslant r$, using the equality $\{\mathbf{I_m} \mid \mathbf{I_m} > \mathbf{0_m}\} = \{i\mathbf{I_{m-1}} \mid i = 1, \mathbf{I_{m-1}} \in \{-1,0,1\}^{m-1}\} \cup \{i\mathbf{I_{m-1}} \mid i = 0, \mathbf{I_{m-1}} > \mathbf{0_{m-1}}\}$, $\mathscr{S}(m;0)$ can be rewritten (due to space limit, it is not listed. Really, $\mathscr{S}(m;0) = \mathscr{S}^{-1} + \mathscr{S}^{0} + \mathscr{S}^{1}$, see below). Let right test-set R be the collection of all sets which are marked by $(*R)$, left test-set $L = \{2n_q \cdot 3^{m-1}$ good coins$\}$ because the number of good coins $g(s) = n_q \cdot 3^r \cdot (3^s - 2^{s+1} + 1) \geqslant 2n_q \cdot 3^{m-1}$ for $1 \leqslant m \leqslant r$, $s \geqslant 2$. After performing the weighing $L : R$ ($|L| = |R|$, $L \cap R = \emptyset$), it follows from Eq. (10) that

$$\mathscr{S}^{1} = \sum_{\mathbf{I_{m-1}}} G_{\mathrm{h}}^{\mathbf{0_{r-m}}1\mathbf{I_{m-1}}\mathbf{1_s}}(*R) \times G_{\mathrm{h}}^{\mathbf{0_{r-m}}(-1)\bar{\mathbf{I}}_{\mathbf{m-1}}\mathbf{1_s}},$$

$$\mathscr{S}^{-1} = \sum_{\mathbf{I_{m-1}}} G_{\mathrm{l}}^{\mathbf{0_{r-m}}(-1)\bar{\mathbf{I}}_{\mathbf{m-1}}(-\mathbf{1_s})}(*R) \times G_{\mathrm{l}}^{\mathbf{0_{r-m}}1\mathbf{I_{m-1}}(-\mathbf{1_s})},$$

$$\mathscr{S}^{0} = \sum_{\mathbf{I_{m-1}} > \mathbf{0_{m-1}}} (G_{\mathrm{h}}^{\mathbf{0_{r-m}}0\mathbf{I_{m-1}}\mathbf{1_s}} \times G_{\mathrm{h}}^{\mathbf{0_{r-m}}0\bar{\mathbf{I}}_{\mathbf{m-1}}\mathbf{1_s}}$$

$$+ G_{\mathrm{l}}^{\mathbf{0_{r-m}}0\bar{\mathbf{I}}_{\mathbf{m-1}}(-\mathbf{1_s})} \times G_{\mathrm{l}}^{\mathbf{0_{r-m}}0\mathbf{I_{m-1}}(-\mathbf{1_s})}) + G_{\mathrm{hh}}^{\mathbf{0_r}\mathbf{1_s}} + G_{\mathrm{ll}}^{\mathbf{0_r}(-\mathbf{1_s})}.$$

We see that $\mathscr{S}^{-1} \bar{\cong} \mathscr{S}^{1}$ and $\mathscr{S}^{1} \cong \mathscr{S}_a(m-1;0)$; $\mathscr{S}^{0} = \mathscr{S}(m-1;0)$. Proposition 3 and Isomorphism Theorem imply that we can identify the solution of $\mathscr{S}^{-1}$, $\mathscr{S}^{1}$ by $(m-1)+q$ weighings; The hypothesis implies that we can identify the solution of $\mathscr{S}^{0}$ by $(m-1)+q$ weighings. Thus we can identify the solution of $\mathscr{S}(m;0)$ by $1+(m-1)+q = m+q$ weighings. □

Proposition 5. *Suppose $2 \leqslant s \leqslant K$, $r = K - s$. If q weighings suffice to identify the solution of $\sigma_e(n_q)$ then $m+q$ weighings can identify the solution of $\mathscr{S}_b(m;0)$ $(0 \leqslant m \leqslant r)$, where*

$$\mathscr{S}_b(m;0) \triangleq \sum_{\mathbf{I_m} > \mathbf{0_m}} (G_{\mathrm{h}}^{\mathbf{0_{r-m}}\mathbf{I_m}\mathbf{1_s}} \times G_{\mathrm{h}}^{\mathbf{0_{r-m}}\bar{\mathbf{I}}_{\mathbf{m}}\mathbf{0_s}} + G_{\mathrm{l}}^{\mathbf{0_{r-m}}\bar{\mathbf{I}}_{\mathbf{m}}(-\mathbf{1_s})} \times G_{\mathrm{l}}^{\mathbf{0_{r-m}}\mathbf{I_m}\mathbf{0_s}})$$

$$+ G_{\mathrm{h}}^{\mathbf{0_r}\mathbf{1_s}} \times G_{\mathrm{h}}^{\mathbf{0_r}\mathbf{0_s}}. \tag{22}$$

Proof. For $m = 0$, $\mathscr{S}_b(0;0) = G_{\mathrm{h}}^{\mathbf{0_r}\mathbf{1_s}} \times G_{\mathrm{h}}^{\mathbf{0_r}\mathbf{0_s}} \cong \sigma_e(n_q)$. By the assumption on $\sigma_e(n_q)$, q weighings suffices to identify the solution of $\mathscr{S}_b(0;0)$. The process of induction on m is completely similar to that of Proposition 4 and the test-set is the same one. □

Proposition 6. *Suppose $2 \leqslant s \leqslant K$, $r = K - s$. If q weighings can identify the solution of $\sigma_e(n_q)$ then $\ell + r + q$ weighings can identify the solution of $\mathscr{S}_b(r;\ell)$ $(0 \leqslant \ell \leqslant s-1)$, where*

$$\mathscr{S}_b(r;\ell) \triangleq \sum_{\mathbf{I_r} > \mathbf{0_r}} \sum_{\mathbf{I}'_\ell \oplus \mathbf{J}'_\ell = \mathbf{1}_\ell} (G_{\mathrm{h}}^{\mathbf{I_r}\mathbf{1}_{\mathbf{s}-\ell}\mathbf{I}'_\ell} \times G_{\mathrm{h}}^{\bar{\mathbf{I}}_{\mathbf{r}}\mathbf{0}_{\mathbf{s}-\ell}\mathbf{J}'_\ell} + G_{\mathrm{l}}^{\bar{\mathbf{I}}_{\mathbf{r}}(-\mathbf{1}_{\mathbf{s}-\ell})\overline{\mathbf{I}'}_\ell} \times G_{\mathrm{l}}^{\mathbf{I_r}\mathbf{0}_{\mathbf{s}-\ell}\overline{\mathbf{J}'}_\ell})$$

$$+ \sum_{\mathbf{I}'_\ell \oplus \mathbf{J}'_\ell = \mathbf{1}_\ell} G_{\mathrm{h}}^{\mathbf{0_r}\mathbf{1}_{\mathbf{s}-\ell}\mathbf{I}'_\ell} \times G_{\mathrm{h}}^{\mathbf{0_r}\mathbf{0}_{\mathbf{s}-\ell}\mathbf{J}'_\ell}. \tag{23}$$

Proof. If $\ell = 0$ then $\mathscr{S}_b(r;0) = \sum_{\mathbf{I_r} > \mathbf{0_r}} (G_{\mathrm{h}}^{\mathbf{I_r}\mathbf{1_s}} \times G_{\mathrm{h}}^{\bar{\mathbf{I}}_{\mathbf{r}}\mathbf{0_s}} + G_{\mathrm{l}}^{\bar{\mathbf{I}}_{\mathbf{r}}(-\mathbf{1_s})} \times G_{\mathrm{l}}^{\mathbf{I_r}\mathbf{0_s}}) + G_{\mathrm{h}}^{\mathbf{0_r}\mathbf{1_s}} \times G_{\mathrm{h}}^{\mathbf{0_r}\mathbf{0_s}}$. Proposition 6 is true for $\ell = 0$ by Proposition 5 (set $m = r$ in Eq. (22)). For $\ell \geqslant 1$, let $\mathbf{I}'_\ell = i'\mathbf{I}'_{\ell-1}$ and $\mathbf{J}'_\ell = j'\mathbf{J}'_{\ell-1}$ then $\mathbf{I}'_\ell \oplus \mathbf{J}'_\ell = \mathbf{1}_\ell \Leftrightarrow i' \oplus j' = 1$ and $\mathbf{I}'_{\ell-1} \oplus \mathbf{J}'_{\ell-1} = \mathbf{1}_{\ell-1}$. Thus, $\mathscr{S}_b(r;\ell)$ can be rewritten (due to space limit, it is not listed. Really, $\mathscr{S}_b(r;\ell) = \mathscr{S}_b^{-1} + \mathscr{S}_b^0 + \mathscr{S}_b^1$, see below). Let L be the collection of all sets which are marked by $(*L)$, R be the collection of all sets which are marked by $(*R)$. After performing the weighing $L:R$ ($|L| = |R|$, $L \cap R = \emptyset$), it follows from Eq. (10) that

$$\begin{aligned}
\mathscr{S}_b^0 = &\sum_{\mathbf{I}_r > \mathbf{0}_r} \sum_{\mathbf{I}'_{\ell-1} \oplus \mathbf{J}'_{\ell-1} = \mathbf{1}_{\ell-1}} (G_{\mathrm{h}}^{\mathbf{I}_r \mathbf{1}_{s-\ell} 1 \mathbf{I}'_{\ell-1}} \times G_{\mathrm{h}}^{\bar{\mathbf{I}}_r \mathbf{0}_{s-\ell} 1 \mathbf{J}'_{\ell-1}} \\
&+ G_{\mathrm{l}}^{\bar{\mathbf{I}}_r (-\mathbf{1}_{s-\ell})(-1)\overline{\mathbf{I}'}_{\ell-1}} \times G_{\mathrm{l}}^{\mathbf{I}_r \mathbf{0}_{s-\ell}(-1)\overline{\mathbf{J}'}_{\ell-1}}) \\
&+ \sum_{\mathbf{I}'_{\ell-1} \oplus \mathbf{J}'_{\ell-1} = \mathbf{1}_{\ell-1}} G_{\mathrm{h}}^{\mathbf{0}_r \mathbf{1}_{s-\ell} 1 \mathbf{I}'_{\ell} - \mathbf{1}} \times G_{\mathrm{h}}^{\mathbf{0}_r \mathbf{0}_{s-\ell} 1 \mathbf{J}'_{\ell-1}}, \\
\mathscr{S}_b^1 = &\sum_{\mathbf{I}_r > \mathbf{0}_r} \sum_{\mathbf{I}'_{\ell-1} \oplus \mathbf{J}'_{\ell-1} = \mathbf{1}_{\ell-1}} (G_{\mathrm{h}}^{\mathbf{I}_r \mathbf{1}_{s-\ell} 1 \mathbf{I}'_{\ell-1}} \times G_{\mathrm{h}}^{\bar{\mathbf{I}}_r \mathbf{0}_{s-\ell} 0 \mathbf{J}'_{\ell-1}}(*R) \\
&+ G_{\mathrm{l}}^{\bar{\mathbf{I}}_r (-\mathbf{1}_{s-\ell})(-1)\overline{\mathbf{I}'}_{\ell-1}} \times G_{\mathrm{l}}^{\mathbf{I}_r \mathbf{0}_{s-\ell} 0 \overline{\mathbf{J}'}_{\ell-1}}(*L)) \\
&+ \sum_{\mathbf{I}'_{\ell-1} \oplus \mathbf{J}'_{\ell-1} = \mathbf{1}_{\ell-1}} G_{\mathrm{h}}^{\mathbf{0}_r \mathbf{1}_{s-\ell} 1 \mathbf{I}'_{\ell-1}} \times G_{\mathrm{h}}^{\mathbf{0}_r \mathbf{0}_{s-\ell} 0 \mathbf{J}'_{\ell-1}}(*R), \\
\mathscr{S}_b^{-1} = &\sum_{\mathbf{I}_r > \mathbf{0}_r} \sum_{\mathbf{I}'_{\ell-1} \oplus \mathbf{J}'_{\ell-1} = \mathbf{1}_{\ell-1}} (G_{\mathrm{h}}^{\mathbf{I}_r \mathbf{1}_{s-\ell} 0 \mathbf{I}'_{\ell-1}}(*L) \times G_{\mathrm{h}}^{\bar{\mathbf{I}}_r \mathbf{0}_{s-\ell} 1 \mathbf{J}'_{\ell-1}} \\
&+ G_{\mathrm{l}}^{\bar{\mathbf{I}}_r (-\mathbf{1}_{s-\ell}) 0 \overline{\mathbf{I}'}_{\ell-1}}(*R) \times G_{\mathrm{l}}^{\mathbf{I}_r \mathbf{0}_{s-\ell}(-1)\overline{\mathbf{J}'}_{\ell-1}}) \\
&+ \sum_{\mathbf{I}'_{\ell-1} \oplus \mathbf{J}'_{\ell-1} = \mathbf{1}_{\ell-1}} G_{\mathrm{h}}^{\mathbf{0}_r \mathbf{1}_{s-\ell} 0 \mathbf{I}'_{\ell-1}}(*L) \times G_{\mathrm{h}}^{\mathbf{0}_r \mathbf{0}_{s-\ell} 1 \mathbf{J}'_{\ell-1}}.
\end{aligned}$$

We see that $\mathscr{S}_b^0 \cong \mathscr{S}_b^{-1} \cong \mathscr{S}_b^1 = \mathscr{S}_b(r;\ell-1)$. The hypothesis and Isomorphism Theorem imply that we can identify the solution of $\mathscr{S}_b^{-1}$, $\mathscr{S}_b^0$, $\mathscr{S}_b^1$ by $(\ell-1)+r+q$ weighings. Thus we can identify the solution of $\mathscr{S}_b(r;\ell)$ by $1+(\ell-1)+r+q = \ell+r+q$ weighings. □

Lemma 11. *Suppose $2 \leqslant s \leqslant K$, $r = K - s$, $A \cap B = \emptyset$ and A, B are sets of $n_q \geqslant 3$ coins. If q weighings can identify the solution of $\sigma_e(n_q) = A_{\mathrm{h}} \times B_{\mathrm{h}}$ and $\sigma_f(n_q) = A_{\mathrm{hh}} + B_{\mathrm{ll}}$ respectively then $\ell + r + q$ weighings can identify the solution of $\mathscr{S}(r;\ell)$, where, $0 \leqslant \ell \leqslant s$, and*

$$\begin{aligned}
\mathscr{S}(r;\ell) \triangleq &\sum_{\mathbf{I}_r > \mathbf{0}_r} \sum_{\mathbf{I}'_\ell \oplus \mathbf{J}'_\ell = \mathbf{1}_\ell} (G_{\mathrm{h}}^{\mathbf{I}_r \mathbf{1}_{s-\ell} \mathbf{I}'_\ell} \times G_{\mathrm{h}}^{\bar{\mathbf{I}}_r \mathbf{1}_{s-\ell} \mathbf{J}'_\ell} + G_{\mathrm{l}}^{\bar{\mathbf{I}}_r (-\mathbf{1}_{s-\ell}) \overline{\mathbf{I}'}_\ell} \times G_{\mathrm{l}}^{\mathbf{I}_r (-\mathbf{1}_{s-\ell}) \overline{\mathbf{J}'}_\ell}) \\
&+ \sum_{\substack{\mathbf{I}'_\ell \oplus \mathbf{J}'_\ell = \mathbf{1}_\ell \\ \mathbf{I}'_\ell > \mathbf{J}'_\ell}} (G_{\mathrm{h}}^{\mathbf{0}_r \mathbf{1}_{s-\ell} \mathbf{I}'_\ell} \times G_{\mathrm{h}}^{\mathbf{0}_r \mathbf{1}_{s-\ell} \mathbf{J}'_\ell} + G_{\mathrm{l}}^{\mathbf{0}_r (-\mathbf{1}_{s-\ell}) \overline{\mathbf{I}'}_\ell} \times G_{\mathrm{l}}^{\mathbf{0}_r (-\mathbf{1}_{s-\ell}) \overline{\mathbf{J}'}_\ell}) \\
&+ G_{\mathrm{hh}}^{\mathbf{0}_r \mathbf{1}_s} + G_{\mathrm{ll}}^{\mathbf{0}_r (-\mathbf{1}_s)}. \qquad (24)
\end{aligned}$$

In particular, $K+q$ *weighings can identify the solution of* $S^{\mathbf{0_r1_s}}$.

Proof. If $\ell=0$ then $\mathscr{S}(r;0)=\sum_{\mathbf{I_r}>\mathbf{0_r}}(G_{\mathrm{h}}^{\mathbf{I_r1_s}}\times G_{\mathrm{h}}^{\bar{\mathbf{I}}_\mathbf{r}\mathbf{1_s}}+G_{\mathrm{l}}^{\bar{\mathbf{I}}_\mathbf{r}(-\mathbf{1_s})}\times G_{\mathrm{l}}^{\mathbf{I_r}(-\mathbf{1_s})})+G_{\mathrm{hh}}^{\mathbf{0_r1_s}}+G_{\mathrm{ll}}^{\mathbf{0_r}(-\mathbf{1_s})}$. By Proposition 4 (let $m=r$ in Eq. (21)), $r+q$ weighings suffice to identify the solution of $\mathscr{S}(r;0)$. For $\ell\geqslant 1$, applying the following forward recursion formula:

$$\{(\mathbf{I}'_\ell,\mathbf{J}'_\ell)\,|\,\mathbf{I}'_\ell\oplus\mathbf{J}'_\ell=\mathbf{1}_\ell\}\Leftrightarrow\{(i\mathbf{I}'_{\ell-1},j\mathbf{J}'_{\ell-1})\,|\,i\oplus j=1;\mathbf{I}'_{\ell-1}\oplus\mathbf{J}'_{\ell-1}=\mathbf{1}_{\ell-1}\}$$

$$\left\{(\mathbf{I}'_\ell,\mathbf{J}'_\ell)\;\middle|\;\begin{matrix}\mathbf{I}'_\ell\oplus\mathbf{J}'_\ell=\mathbf{1}_\ell\\ \mathbf{I}'_\ell>\mathbf{J}'_\ell\end{matrix}\right\}$$

$$=\left\{(i\mathbf{I}'_{\ell-1},j\mathbf{J}'_{\ell-1})\;\middle|\;\begin{matrix}i=1,j=0,\mathbf{I}'_{\ell-1}\oplus\mathbf{J}'_{\ell-1}=\mathbf{1}_{\ell-1}\text{ or}\\ i=j=1,\mathbf{I}'_{\ell-1}\oplus\mathbf{J}'_{\ell-1}=\mathbf{1}_{\ell-1},\mathbf{I}'_{\ell-1}>\mathbf{J}'_{\ell-1}\end{matrix}\right\}$$

$\mathscr{S}(r;\ell)$ can be rewritten as $\mathscr{S}(r;\ell)=\mathscr{S}^{-1}+\mathscr{S}^{0}+\mathscr{S}^{1}$ (see below). Let L be the collection of all sets which are marked by $(*L)$, R be the collection of all sets which are marked by $(*R)$. We note that $|L|\neq|R|$ and the difference $\beta(\ell)\triangleq|R|-|L|=|\{G^{\mathbf{0_r1_{s-\ell}}0\mathbf{J}'_{\ell-1}},\ G^{\mathbf{0_r}(-\mathbf{1_{s-\ell}})0\bar{\mathbf{J}}'_{\ell-1}}\,|\,\mathbf{J}'_{\ell-1}\in\{0,1\}^{\ell-1}\}|=n_q(2^\ell-1)$ if $\ell=s$, n_q2^ℓ if $1\leqslant\ell\leqslant s-1$. Since the number of the known good coins $g(s)=n_q\cdot 3^r\cdot(3^s-2^{s+1}+1)\geqslant\beta(\ell)$ do not hold only if $\ell=s=K=2$, $r=0$ (This exceptional case $\mathscr{S}(0;2)=S^{11}$ will be settled separately), these $\beta(\ell)$ coins can be used to balance the scale. After performing the weighing $L\cup\{\beta(\ell)\text{ good coins}\}:R$, it follows from Eq. (10) that

$$\begin{aligned}\mathscr{S}^1=&\sum_{\mathbf{I_r}>\mathbf{0_r}}\sum_{\mathbf{I}'_{\ell-1}\oplus\mathbf{J}'_{\ell-1}=\mathbf{1}_{\ell-1}}(G_{\mathrm{h}}^{\mathbf{I_r1_{s-\ell}}1\mathbf{I}'_{\ell-1}}\times G_{\mathrm{h}}^{\bar{\mathbf{I}}_\mathbf{r}\mathbf{1_{s-\ell}}0\mathbf{J}'_{\ell-1}}(*R)\\&+G_{\mathrm{l}}^{\bar{\mathbf{I}}_\mathbf{r}(-\mathbf{1_{s-\ell}})(-1)\bar{\mathbf{I}}'_{\ell-1}}\times G_{\mathrm{l}}^{\mathbf{I_r}(-\mathbf{1_{s-\ell}})0\bar{\mathbf{J}}'_{\ell-1}}(*L))\\&+\sum_{\mathbf{I}'_{\ell-1}\oplus\mathbf{J}'_{\ell-1}=\mathbf{1}_{\ell-1}}G_{\mathrm{h}}^{\mathbf{0_r1_{s-\ell}}1\mathbf{I}'_{\ell-1}}\times G_{\mathrm{h}}^{\mathbf{0_r1_{s-\ell}}0\mathbf{J}'_{\ell-1}}(*R),\\ \mathscr{S}^{-1}=&\sum_{\mathbf{I_r}>\mathbf{0_r}}\sum_{\mathbf{I}'_{\ell-1}\oplus\mathbf{J}'_{\ell-1}=\mathbf{1}_{\ell-1}}(G_{\mathrm{h}}^{\mathbf{I_r1_{s-\ell}}0\mathbf{I}'_{\ell-1}}(*L)\times G_{\mathrm{h}}^{\bar{\mathbf{I}}_\mathbf{r}\mathbf{1_{s-\ell}}1\mathbf{J}'_{\ell-1}}\\&+G_{\mathrm{l}}^{\bar{\mathbf{I}}_\mathbf{r}(-\mathbf{1_{s-\ell}})0\bar{\mathbf{I}}'_{\ell-1}}(*R)\times G_{\mathrm{l}}^{\mathbf{I_r}(-\mathbf{1_{s-\ell}})(-1)\bar{\mathbf{J}}'_{\ell-1}})\\&+\sum_{\mathbf{I}'_{\ell-1}\oplus\mathbf{J}'_{\ell-1}=\mathbf{1}_{\ell-1}}G_{\mathrm{l}}^{\mathbf{0_r}(-\mathbf{1_{s-\ell}})(-1)\bar{\mathbf{I}}'_{\ell-1}}\times G_{\mathrm{l}}^{\mathbf{0_r}(-\mathbf{1_{s-\ell}})0\bar{\mathbf{J}}'_{\ell-1}}(*R),\\ \mathscr{S}^0=&\sum_{\mathbf{I_r}>\mathbf{0_r}}\sum_{\mathbf{I}'_{\ell-1}\oplus\mathbf{J}'_{\ell-1}=\mathbf{1}_{\ell-1}}(G_{\mathrm{h}}^{\mathbf{I_r1_{s-\ell}}1\mathbf{I}'_{\ell}-\mathbf{1}}\times G_{\mathrm{h}}^{\bar{\mathbf{I}}_\mathbf{r}\mathbf{1_{s-\ell}}1\mathbf{J}'_{\ell-1}}\\&+G_{\mathrm{l}}^{\bar{\mathbf{I}}_\mathbf{r}(-\mathbf{1_{s-\ell}})(-1)\bar{\mathbf{I}}'_{\ell-1}}\times G_{\mathrm{l}}^{\mathbf{I_r}(-\mathbf{1_{s-\ell}})(-1)\bar{\mathbf{J}}'_{\ell-1}})\end{aligned}$$

$$+ \sum_{\substack{\mathbf{I}'_{\ell-1}\oplus\mathbf{J}'_{\ell-1}=\mathbf{1}_{\ell-1}\\ \mathbf{I}'_{\ell-1}>\mathbf{J}'_{\ell-1}}} (G_{\mathrm{h}}^{\mathbf{0_r 1_{s-\ell}1 I'_{\ell-1}}} \times G_{\mathrm{h}}^{\mathbf{0_r 1_{s-\ell}1 J'_{\ell-1}}}$$

$$+ G_{\mathrm{l}}^{\mathbf{0_r(-1_{s-\ell})(-1)\overline{I'}_{\ell-1}}} \times G_{\mathrm{l}}^{\mathbf{0_r(-1_{s-\ell})(-1)\overline{J'}_{\ell-1}}}) + G_{\mathrm{hh}}^{\mathbf{0_r 1_s}} + G_{\mathrm{ll}}^{\mathbf{0_r(-1_s)}}.$$

We observe that $\mathscr{S}^{-1} \tilde{\cong} \mathscr{S}^1$ and $\mathscr{S}^1 \cong \mathscr{S}_b(r;\ell-1)$; $\mathscr{S}^0 = \mathscr{S}(r;\ell-1)$. Proposition 6 and Isomorphism Theorem imply that we can identify the solution of $\mathscr{S}^{-1}$, $\mathscr{S}^1$ by $(\ell-1)+r+q$ weighings; The hypothesis implies that we can identify the solution of $\mathscr{S}^0$ by $(\ell-1)+r+q$ weighings. Thus we can identify the solution of $\mathscr{S}(r;\ell)$ by $1+(\ell-1)+r+q=\ell+r+q$ weighings. By letting $\ell=s$ in $\mathscr{S}(r;\ell)$, we have $S^{\mathbf{0_r 1_s}} = \mathscr{S}(r;s)$ thus $s+r+q=K+q$ weighings suffice for $S^{\mathbf{0_r 1_s}}$.

The exceptional case: $\ell=s=K=2$ and $r=0$. Let $\mathbf{I}_2=i_1i_2$, $\mathbf{J}_2=j_1j_2\in\{-1,0,1\}^2$ then $\{(\mathbf{I}_2,\mathbf{J}_2)\,|\,\mathbf{I}_2\oplus\mathbf{J}_2=11;\mathbf{I}_2>\mathbf{J}_2\}=\{(i_1i_2,j_1j_2)\,|\,(11,10),(11,01),(11,00),(10,01)\}$. By Structure Theorem,

$$S^{11} = G_{\mathrm{h}}^{11}\times\{G^{10},G^{01},G^{00}\}_{\mathrm{h}} + G_{\mathrm{h}}^{10}\times G_{\mathrm{h}}^{01} + G_{\mathrm{hh}}^{11}$$

$$+ G_{\mathrm{l}}^{-1-1}\times\{G^{-10},G^{0-1},G^{00}\}_{\mathrm{l}} + G_{\mathrm{l}}^{-10}\times G_{\mathrm{l}}^{0-1} + G_{\mathrm{ll}}^{-1-1}$$

and $|G^{ij}|=n_q$ for all $G^{ij}\in S^{11}$. Now the number of the known good coins $g(s)=n_q\cdot 3^r\cdot(3^s-2^{s+1}+1)=2n_q$ ($G^{1-1}\cup G^{-11}$ is the set of $2n_q$ good coins). Let $L(S^{11})=\{G^{10},G^{-10},G^{00}\}$, $R(S^{11})=\{G^{01},G^{0-1},n_q$ good coins$\}$, it follows from Eq. (10) that

$$S^{110} = G_{\mathrm{h}}^{10}\times G_{\mathrm{h}}^{01} + G_{\mathrm{l}}^{-10}\times G_{\mathrm{l}}^{0-1} + G_{\mathrm{hh}}^{11} + G_{\mathrm{ll}}^{-1-1},$$

$$S^{111} = G_{\mathrm{h}}^{11}\times G_{\mathrm{h}}^{01} + G_{\mathrm{l}}^{-1-1}\times G_{\mathrm{l}}^{-10} + G_{\mathrm{l}}^{-1-1}\times G_{\mathrm{l}}^{00},$$

$$S^{11-1} = G_{\mathrm{l}}^{-1-1}\times G_{\mathrm{l}}^{0-1} + G_{\mathrm{h}}^{11}\times G_{\mathrm{h}}^{10} + G_{\mathrm{h}}^{11}\times G_{\mathrm{h}}^{00}.$$

We observe that $S^{11-1}\tilde{\cong}S^{111}$. The test-sets of the second weighing of S^{110} and S^{111} are given by $\{2n_q$ good coins$\}:\{G^{01},G^{-10}\}$ then $S^{1100}=G_{\mathrm{hh}}^{11}+G_{\mathrm{ll}}^{-1-1}$, $S^{1101}=G_{\mathrm{h}}^{10}\times G_{\mathrm{h}}^{01}$, $S^{110-1}=G_{\mathrm{l}}^{-10}\times G_{\mathrm{l}}^{0-1}$ and $S^{1110}=G_{\mathrm{l}}^{-1-1}\times G_{\mathrm{l}}^{00}$, $S^{1111}=G_{\mathrm{h}}^{11}\times G_{\mathrm{h}}^{01}$, $S^{111-1}=G_{\mathrm{l}}^{-1-1}\times G_{\mathrm{l}}^{-10}$. We observe that $S^{1100}\cong\sigma_f(n_q)$, other five search domains are isomorphic (or symmetrically isomorphic) to $\sigma_e(n_q)$. The assumptions on $\sigma_e(n_q)$ and $\sigma_f(n_q)$ imply that q weighings can identify the solution of above six search domains. Therefore $K+q=q+2$ weighings can identify the solution of S^{11}. □

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