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Uniqueness of the Measure-Valued Solution to a Conservation Law with Boundary Conditions

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Abstract—In this paper, we examine existence and uniqueness for the entropy measure-valued solution to a first-order hyperbolic equation in a bounded domain. © 1998 Elsevier Science Ltd. All rights reserved.

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Let us consider:

- Ω a bounded smooth domain of $\mathbb{R}^n$ with a boundary Γ and an outer external normal n;
- $T > 0$, $Q =]0,T[\times\Omega$ and $\Sigma =]0,T[\times\Gamma$;
- $p \in W^{2,\infty}(\Omega)$ is a solution of $-\Delta p = 0$ in Ω;
- f and g two lipschitz functions and v in $L^\infty(Q)$;
- u_0 and u^D, respectively, in $L^\infty(\Omega)$ and $L^\infty(\Sigma)$;
- the problem (P): Find $u \in L^\infty(Q)$, so that:

$$: \frac{\partial u}{\partial t} + \operatorname{div}(f(u)\nabla p) + g(u)v = 0 \qquad \text{in } \mathcal{D}'(Q), \tag{E}$$

$$: u(0,.) = u_0 \qquad \text{in } \Omega, \tag{CI}$$

$$: u = u^D \qquad \text{on } \Sigma. \tag{CB}$$

It is well known that for problems of that kind, if $u_0 \in L^\infty(Q) \cap \mathrm{BV}(Q)$, a unique solution exists in $L^\infty(Q) \cap \mathrm{BV}(Q)$ [1]. Thus, the initial and boundary conditions are understood in the sense of BV functions trace. Here, we only assume that $u_0 \in L^\infty(Q)$ and our solution is to be found in the set of Young measures. Using the construction in [2] of a trace for certain Young measures, through the entropy formulation described by Otto in [3] and [4], we are able to give a result of existence and uniqueness. At the end of the uniqueness method, we show that the unique measure-valued solution is, in fact, a Young measure associated with a function; so there exists a unique weak solution.

Moreover, a lot of sequences of approximate solutions, coming from numerical schemes, converge towards a measure-valued solution [5,6]. With this uniqueness result, it will be possible to

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conclude that the sequence tends (in all the $L^p(Q)$, $1 \le p < \infty$) to the weak solution without supposing that it is a BV solution.

1. RECALLS ON YOUNG MEASURES (EXTRACT FROM [7] AND [3])

If $\mathcal{L}$ denotes the Lebesgue measure, we state the following.

DEFINITION 1. *A positive measure ν on $Q\times\mathbb{R}$ is a Young measure on $Q\times\mathbb{R}$ if for every Borel set $A \subset Q$, $\nu(A\times\mathbb{R}) = \mathcal{L}(A)$. It is useful to describe a Young measure ν by its desintegration which is a unique family of probabilities on $\mathbb{R}$, $(dP_{(t,x)})$, $(t,x) \in Q$, characterized for every Caratheodory function $\psi(\lambda,t,x)$, $\lambda \in \mathbb{R}$, $(t,x) \in Q$, by:*

$$\int_{\mathbb{R}} \psi(\lambda,t,x)\,dP_{(t,x)}(\lambda) \text{ is } \mathcal{L}\text{-measurable on } Q \text{ and}$$
$$\int_{Q\times\mathbb{R}} \psi\,d\nu = \int_Q \int_{\mathbb{R}} \psi(\lambda,t,x)\,dP_{(t,x)}(\lambda)\,dt\,dx.$$

DEFINITION 2. *We call the Young measure associated with the measurable function u, the unique Young measure, denoted by ν_u, defined for every positive Caratheodory function ψ by:*

$$\int_{Q\times\mathbb{R}} \psi\,d\nu_u = \int_Q \psi(u(t,x),t,x)\,dt\,dx.$$

PROPOSITION 3. *If $u_n \rightharpoonup u$ in $L^\infty(Q) - *$, then there exists a subsequence u_{n_k} and a Young measure ν so that the sequence of Young measure ν_{n_k} associated with u_{n_k} converges narrowly towards ν (i.e., simply relatively to all bounded Caratheodory functions [7]). Moreover, if the limit ν is the Young measure associated with u, then u_{n_k} converges towards u in $L^p(Q)$, $1 \le p < \infty$.*

2. DEFINITION OF A MEASURE-VALUED SOLUTION FOR (P)

In order to define such a solution, we need to recall the following.

DEFINITION 4. *The pair (η,q) is called an entropy pair if $\eta \in C^2(\mathbb{R})$ is positive and convex and q is a local-lipschitz function with $q'(x) = f'(x)\eta'(x)$.*

The pair (H,Q) is called a boundary entropy pair if H and Q are continuous functions on $\mathbb{R}^2$, if for all w in $\mathbb{R}$, $(H(.,w),Q(.,w))$ is an entropy pair, $H(w,w) = Q(w,w) = 0$ and $\partial_1 H(w,w) = 0$.

DEFINITION 5. *A Young measure ν on $\mathbb{R}\times Q$ is a measure-valued solution of (E) if using the desintegration form $d\nu(\lambda,t,x) = dP_{(t,x)}(\lambda)\,dt\,dx$,*

(i) $\exists M > 0$, $\mathrm{Supp}\,(dP_{(t,x)}) \subset [-M,M]$,

(ii) *for every positive function β in $H^1(Q)$ and all entropy pairs (η,q):*

$$\int_Q \int_{\mathbb{R}} \left\{\eta(\lambda)\frac{\partial\beta}{\partial t} + q(\lambda)\nabla p.\nabla\beta - \eta'(\lambda)g(\lambda)v\beta\right\} d\nu \ge 0.$$

REMARK 1. We mean implicitly that we are looking for entropy solutions.

In order to formulate the initial and the boundary conditions, let us give some properties of a measure-valued solution ν of (E).

PROPOSITION 6. *$\forall\alpha \in L^1(\Omega)$ positive, $\forall w \in L^\infty(\Omega)$,*

$$\lim_{t\to 0^+}\mathrm{ess} \int_Q \int_{\mathbb{R}} |\lambda - w(x)|\,\alpha(x)\,dP_{(t,x)}(\lambda)\,dx \text{ exists.}$$

DEFINITION 7. *Let us introduce the change of coordinates $x \to (\overline{x}, y)$ for x in the neighborhood of Γ: $x = \overline{x} - y.n(\overline{x})$ where $(\overline{x}, y) \in \Gamma x(0,\varepsilon)$ for some positive ε and denote by $J(\overline{x}, y)$ the Jacobian determinant associated with this change of coordinates. Then, there is a sequence y_i in $(0,\varepsilon)$, $y_i \to 0^+$, and a Young measure μ on $\Sigma \times \mathbb{R}$ described by $d\mu(\lambda, t, x) = dP^{\mu}_{(t,x)}(\lambda)\, dt\, d\mathcal{H}^{n-1}$, where $\mathcal{H}^{n-1}$ is the $n-1$-dimensional Hausdorff measure and $dP^{\mu}_{(t,x)}(\lambda)$ is a family of probabilities on $\mathbb{R}$, satisfying the point (i) of the above definition, so that: for every $L^1(\Sigma)$ function β and all continuous functions l,*

$$\lim_{j\to\infty} \int_\Sigma \int_{\mathbb{R}} l(\lambda)\beta(t,\overline{x})\, dP_{(t,x(\overline{x},y_j))}(\lambda)\, J(\overline{x}, y_j)\, dt\, d\mathcal{H}^{n-1} = \int_\Sigma \int_{\mathbb{R}} l(\lambda)\beta(t,\overline{x})\, d\mu.$$

Using this trace, we prove the following.

PROPOSITION 8. *For all boundary entropy pairs (H, Q), all positive α in $L^1(\Sigma)$, and all v^D in $L^\infty(\Sigma)$,*

$$\operatorname*{lim\,ess}_{y\to 0^+} \int_\Sigma \int_{\mathbb{R}} Q\left(\lambda, v^D\right) \nabla p(x).n(x)\, \alpha\, dP_{(t,x)}(\lambda)\, J\, dt\, d\mathcal{H}^{n-1}(\overline{x}) = \int_\Sigma \int_{\mathbb{R}} Q\left(\lambda, v^D\right) \nabla p.n\, \alpha\, d\mu.$$

DEFINITION 9. *We call the measure-valued solution for problem (P), every measure-valued solution of equation (E) if the following conditions are verified:*

(i) $\operatorname{lim\,ess}_{t\to 0^+} \int_Q \int_{\mathbb{R}} |\lambda - u_0(x)|\, dP_{(t,x)}(\lambda)\, dx = 0$,

(ii) *if μ is a Young measure trace, for all $L^1(\Sigma)$ positive α and all boundary entropy pairs (H, Q), $\int_\Sigma \int_{\mathbb{R}} Q(\lambda, u^D)\nabla p.n\alpha\, d\mu \geq 0$.*

PROPOSITION 10. *From then on, such a solution is characterized by the following entropy formulation.*

If μ is a Young measure trace, $k \in \mathbb{R}$ and β a $H^1(Q)$ positive function, then

$$-\int_Q \int_{\mathbb{R}} |\lambda - k|\frac{\partial \beta}{\partial t} + F(\lambda, k)\, \nabla p.\nabla\beta\, - \operatorname{Sgn}_0(\lambda - k) g(\lambda) v\, d\nu \leq \int_\Omega |u_0 - k|\, \beta(0)\, dx$$
$$+ \int_\Sigma \int_{\mathbb{R}} F\left(\lambda, u^D\right)\, \beta\, \nabla p.n\, d\mu - \int_\Sigma F\left(k, u^D\right)\, \beta\, \nabla p.n\, d\mathcal{H}^{n-1}\, dt,$$

where $F(x, k) = \operatorname{Sgn}(x - k)[f(x) - f(k)]$.

3. ABOUT EXISTENCE AND UNIQUENESS

- In order to prove the existence of a measure-valued solution, we use an artificial viscosity technique, meaning that we consider the solution u_n of the parabolic problem:

$$\frac{\partial u_n}{\partial t} + \operatorname{div}(f(u_n)\nabla p) + g(u_n)v = \frac{1}{n}\Delta u_n \qquad \text{in } \mathcal{D}'(Q),$$

  where the problem data have to be regularized.

As (u_n) is a bounded sequence in $L^\infty(Q)$, it is possible to extract a subsequence that converges narrowly (i.e., in the sense of the Young measures [7]). Then, fitting the Otto demonstrations (developed in [3]) to the context of the measure-valued solutions, we are able to state the following.

PROPOSITION 11. *There exists a measure-valued solution to problem (P), so that, $\forall\ \beta \in H^1(Q)$ positive, $\forall\ k \in \mathbb{R}$, and $\forall\,(H, Q)$ boundary entropy pair,*

$$-\int_Q \int_{\mathbb{R}} H(\lambda, k)\frac{\partial \beta}{\partial t} + Q(\lambda, k)\, \nabla p.\nabla\beta - \partial_1 H(\lambda, k) g(\lambda) v\beta\, d\nu$$
$$\leq \int_\Omega H(u_0, k)\, \beta(0)\, dx + cte(\nabla p, f) \int_\Sigma H(u^D, k)\, \beta\, d\mathcal{H}^{n-1}\, dt.$$

- In order to show the uniqueness of such a solution, we use the technique of [8] in the domain Q. Then, we have to process the boundary integrations like [3], to obtain the following.

PROPOSITION 12. *If ν and $\hat{\nu}$ are two measure-valued solutions of (P), with $d\nu = dP_{(t,x)}(\lambda)\,dt\,dx$ and $d\hat{\nu} = d\widehat{P}_{(t,x)}(\lambda)\,dt\,dx$, for the initial and boundary conditions u_0, $\hat{u}_0$ and $u^D, \hat{u}^D$, then*

$$\int_Q \int_{\mathbb{R}\times\mathbb{R}} |\lambda - k|\, dP_{(t,x)}(\lambda)\, d\widehat{P}_{(t,x)}(k)$$
$$\leq e^{cte(g,v).t}\left[\int_\Omega |u_0 - \hat{u}_0|\, dx + cte(\nabla p, f)\int_0^t \int_\Sigma |u^D - \hat{u}^D|\, d\mathcal{H}^{n-1}\, ds\right].$$

According to [8], the above inequality allows us to say that if $u_0 = \hat{u}_0$ and $u^D = \hat{u}^D$, then u exists in $L^\infty(Q)$ with $\hat{\nu} = \nu = \nu_u$, where ν_u is the Young measure associated with the function u.

4. CONCLUSION

There exists a unique measure-valued solution to problem (P) and a unique measurable function u, weak solution to problem (P); moreover, if u and $\hat{u}$ are two solutions for the initial conditions u_0, $\hat{u}_0$ and boundary conditions u^D, $\hat{u}^D$, one gets the following stability result:

$$\|u - \hat{u}\|_{L^1(\Omega)} \leq e^{cte(g,v).t}\left[\|u_0 - \hat{u}_0\|_{L^1(\Omega)} + cte(\nabla p, f)\int_0^t \left\|u^D - \hat{u}^D\right\|_{L^1(\Gamma)}\, ds\right].$$

One can find details of the demonstrations in [9].

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