Applied Mathematics Letters 23 (2010) 381–384

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

An inequality for continuous linear functionals

Ioan Gavrea, Mircea Ivan *

Department of Mathematics, Technical University Cluj-Napoca, Str. C. Daicoviciu 15, 400020 Cluj-Napoca, Romania

ARTICLE INFO

Article history:
Received 8 June 2009
Received in revised form 30 October 2009
Accepted 4 November 2009

Keywords:
Linear functional
Interpolation
Integral inequality
Remainder

ABSTRACT

Let $n > 1$ be an integer, $f \in C^n[a, b]$, and $A: C[a, b] \to \mathbb{R}$ a continuous linear functional which annihilates all polynomials of degree at most $n-1$. We give sharp inequalities of the form $|A(f)| \le M_k \|f^{(k)}\|_2$, $k = 2, \dots, n$.

1. Introduction

Let $n > 1$ be an integer and $f \in C^n[a, b]$. Denote by e_i the monomial functions $e_i(t) = t^i$, $i = 0, \dots, n-1$, $t \in [a, b]$. Let $A : C[a, b] \to \mathbb{R}$ be a continuous linear functional which annihilates all polynomials of degree at most $n-1$, i.e.,

$$A(e_i) = 0, \quad i = 0, \dots, n-1.$$

If $g: [a, b] \times [a, b] \to \mathbb{R}$ and, for $s \in [a, b]$, $h: [a, b] \to \mathbb{R}$, $h(t) = g(t, s)$, then we use the notation

$$A_t(g(t, s)) := A(h).$$

The divided difference of the function f on the distinct knots $x_0, \dots, x_n \in [a, b]$, is defined by

$$[x_0, \dots, x_n; f] = \sum_{i=0}^{n} \frac{f(x_i)}{(x_i - x_0)\dots(x_i - x_{i-1})(x_i - x_{i+1})\dots(x_i - x_n)}.$$

For $k \in \mathbb{N}$, $k \ge 1$, we recall the notation for the truncated power

$$(t-s)_+^k := \left(\frac{|t-s| + t - s}{2}\right)^k = \begin{cases} 0, & \text{if } t < s, \\ (t-a)^k, & \text{if } t \ge s, \end{cases} \quad t, s \in [a, b].$$

With $k = 0$, we obtain the Heaviside step function

$$(t-s)_+^0 := \begin{cases} 0, & \text{if } t < s, \\ 1, & \text{if } t \ge s. \end{cases}$$

As usual, we use the notation

$$\|f\|_2 := \sqrt{\int_a^b |f(x)|^2 \mathrm{d}x}.$$

Throughout the proof, we need the following simple result.

* Corresponding author.
E-mail addresses: Ioan.Gavrea@math.utcluj.ro (I. Gavrea), Mircea.Ivan@math.utcluj.ro (M. Ivan).

0893-9659/$ – see front matter
doi:10.1016/j.aml.2009.11.002

Proposition 1. *If* $g \in C^1[[a,b]\times[a,b]]$, *then we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}A_s(g(s,t)) = A_s\left(\frac{\partial}{\partial t}g(s,t)\right), \quad t,s\in[a,b].$$

2. Main result

The following is the main result of the paper.

Theorem 2. *The functional A satisfies the following inequalities:*

$$|A(f)| \le M_k \|f^{(k)}\|_2, \quad k = 2,\dots,n,$$

where

$$M_k = \sqrt{\frac{(-1)^k}{(2k-1)!}A_s\left(A_t\left((t-s)_+^{2k-1}\right)\right)}$$

are the best possible constants. The equality is attained if and only if f is of the form

$$f(s) = C\,\left(A_t(t-s)_+^{k-1}\right)^{(-k)}, \quad s\in[a,b],$$

where C is an arbitrary constant, and the symbol $(-k)$ *denotes a kth antiderivative of f.*

Proof. By applying A to both sides of the Taylor formula

$$f(t) = f(a) + \frac{f'(a)}{1!}(t-a) + \cdots + \frac{f^{(k-1)}(a)}{(k-1)!}(t-a)^{k-1} + \frac{1}{(k-1)!}\int_a^b (t-s)_+^{k-1} f^{(k)}(s)\,\mathrm{d}s, \quad t\in[a,b]$$

(see, e.g., [1, p. 122]), we get

$$A(f) = \frac{1}{(k-1)!}\int_a^b A_t\left((t-s)_+^{k-1}\right) f^{(k)}(s)\,\mathrm{d}s, \tag{1}$$

and using the Cauchy–Schwarz inequality yields

$$A^2(f) \le \left(\frac{1}{(k-1)!}\right)^2 \int_a^b \left(A_t\left((t-s)_+^{k-1}\right)\right)^2 \mathrm{d}s \int_a^b \left(f^{(k)}(s)\right)^2 \mathrm{d}s. \tag{2}$$

The problem now is to calculate the integral

$$\int_a^b \left(A_t\left((t-s)_+^{k-1}\right)\right)^2 \mathrm{d}s.$$

To do this, let

$$u(s) := A_t\left((t-s)_+^{2k-1}\right), \quad s\in[a,b].$$

Since A is linear and continuous, we obtain

$$u^{(k)}(s) = \frac{(-1)^k(2k-1)!}{(k-1)!}A_t\left((t-s)_+^{k-1}\right).$$

Eq. (1) gives

$$A(u) = \frac{(-1)^k(2k-1)!}{((k-1)!)^2}\int_a^b \left(A_t\left((t-s)_+^{k-1}\right)\right)^2 \mathrm{d}s$$

and hence

$$\int_a^b \left(A_t\left((t-s)_+^{k-1}\right)\right)^2 \mathrm{d}s = \frac{(-1)^k((k-1)!)^2}{(2k-1)!}A_s\left(A_t\left((t-s)_+^{2k-1}\right)\right).$$

Inserting this in (2) completes the proof of the theorem. □

3. Applications

Example 3 (*[2]*). For $A(f) = [x_0, \ldots, x_n; f]$, we obtain the following inequality:

$$[x_0, \ldots, x_n; f]^2 \le \frac{(-1)^k}{(2k-1)!}\left[x_0, \ldots, x_n; [x_0, \ldots, x_n; (t-s)_+^{2k-1}]_t\right]_s \|f^{(k)}\|_2^2$$

for all $f \in C^k[a, b]$, $k = 2, \ldots, n$.

Example 4 (*[3, p. 166, Ex. 130]*). Let $f' \in AC[a, b]$. Then for any $p > 1$ and $a < c < b$

$$\left|\frac{f(b)-f(c)}{b-c} - \frac{f(c)-f(a)}{c-a}\right| \le \left(\frac{p-1}{2p-1}(b-a)\right)^{\frac{p-1}{p}} \left(\int_a^b \left|f''(t)\right|^p \,\mathrm{d}t\right)^{\frac{1}{p}}.$$

For $p = 2$ and $c = (a+b)/2$, Example 4 yields

Example 5 (*[3, p. 166, Example 131], Zmorovich*). If $f' \in AC[a, b]$, then

$$\int_a^b \left(f''(t)\right)^2 \,\mathrm{d}t \ge \frac{12}{(b-a)^3}\left(f(a) - 2f\left(\frac{a+b}{2}\right) + f(b)\right)^2.$$

Example 6. With $A(f) = L(a, b; f)(x) - f(x)$, where $L(a, b; f)$ is the Lagrange interpolating polynomial

$$L(a, b; f)(x) = \frac{x-a}{b-a}f(b) + \frac{x-b}{a-b}f(a),$$

and $k = 2$, we get

$$A_t((t-s)_+^3) = \frac{a\ (b-s)^3}{a-b} + \frac{(b-s)^3\ x}{-a+b} - \frac{(-s+x+|s-x|)^3}{8},$$

$$M_2 = \frac{(x-a)(b-x)}{\sqrt{3}\sqrt{b-a}},$$

$$|L(a, b; f)(x) - f(x)| \le \frac{(x-a)(b-x)}{\sqrt{3}\sqrt{b-a}}\|f''\|_2, \quad x \in [a, b].$$

The equality is obtained for

$$f(s) = \frac{(b-a)|s-x|^3 + s^2(a(-6b+s+3x) + bs + 3bx - 2sx)}{12(b-a)}.$$

Example 7. With $A(f) = \int_a^b f(x)\,\mathrm{d}x - (b-a)\frac{f(a)+f(b)}{2}$, and $k = 2$, we get

$$A_t((t-s)_+^3) = -\frac{1}{4}(b-s)^3(-2a+b+s),$$

$$M_2 = \frac{(b-a)^{5/2}}{2\sqrt{30}},$$

$$\left|\int_a^b f(x)\,\mathrm{d}x - (b-a)\frac{f(a)+f(b)}{2}\right| \le \frac{(b-a)^{5/2}}{2\sqrt{30}}\|f''\|_2.$$

Example 8. With $A(f) = \int_a^b f(x)\,\mathrm{d}x - (b-a)f\left(\frac{a+b}{2}\right)$, and $k = 2$, we get

$$A_t((t-s)_+^3) = \frac{1}{8}\left(2\,(b-s)^4 + \frac{1}{8}(a-b)(a+b-2s+|a+b-2s|)^3\right),$$

$$M_2 = \frac{(b-a)^{5/2}}{8\sqrt{5}},$$

$$\left|\int_a^b f(x)\mathrm{d}x - (b-a)f\left(\frac{a+b}{2}\right)\right| \le \frac{(b-a)^{5/2}}{8\sqrt{5}}\|f''\|_2.$$

Example 9. With $A(f)=\int_a^b f(x)\,\mathrm{d}x-\frac{b-a}{6}\left(f(a)+f\left(\frac{a+b}{2}\right)+f(b)\right)$, and $k=4$, we get

$$A_t((t-s)_+^7)=\frac{1}{8}(b-s)^8-\frac{1}{768}(b-a)\left(128(b-s)^7+\frac{1}{128}(a+b-2s+|a+b-2s|)^7\right)$$

$$M_4=\frac{(b-a)^{9/2}}{1152\sqrt{14}},$$

$$\left|\int_a^b f(x)\,\mathrm{d}x-\frac{b-a}{6}\left(f(a)+f\left(\frac{a+b}{2}\right)+f(b)\right)\right|\le\frac{(b-a)^{9/2}}{1152\sqrt{14}}\|f^{(4)}\|_2.$$

Example 10. With $A(f)=L(0,0,1,1;f)(x)-f(x)$, and $x\in[0,1]$, $k=4$, where $L(0,0,1,1;f)$ is the Lagrange–Hermite interpolating polynomial attached to f on the double knots 0 and 1, we get

$$M_4=\frac{x^2(1-x)^2\sqrt{3+x-x^2}}{6\sqrt{35}}$$

$$\|L(0,0,1,1;f)(x)-f(x)\|\le\frac{x^2(1-x)^2\sqrt{3+x-x^2}}{6\sqrt{35}}\ \|f^{(4)}\|_2.$$

Example 11. With $A(f)=\int_a^b f(x)\,P_n(x)\,w(x)\,\mathrm{d}x$, where $P_0,\ldots,P_n$ is a system of orthonormal polynomials with respect to the weight function w on $[a,b]$,

$$\int_a^b P_i(x)P_j(x)w(x)\,\mathrm{d}x=\delta_{i,j},\quad i,j=0,\ldots,n,$$

and $D=[a,b]\times[a,b]$, we obtain

$$\int_a^b\left(\int_a^b(t-s)_+^{k-1}P_n(t)\,w(t)\,\mathrm{d}t\right)^2\mathrm{d}s=\frac{(-1)^k((k-1)!)^2}{(2k-1)!}\int_a^b\int_a^b(t-s)_+^{2k-1}P_n(t)\,w(t)\,P_n(s)\,w(s)\,\mathrm{d}t\mathrm{d}s,$$

and

$$A^2(f)\le\frac{(-1)^k}{(2k-1)!}\int\int_D(t-s)_+^{2k-1}\,P_n(t)P_n(s)\,w(t)w(s)\,\mathrm{d}t\mathrm{d}s\ \|f^{(k)}\|_2^2,$$

$k=2,\ldots,n.$

For some generalizations of Zmorovich's inequality see [4,5]. See also [6–9].

Acknowledgement

We gratefully acknowledge Professor Xinlong Zhou for valuable suggestions regarding the references. We wish to express gratitude to the referee for valuable remarks.

References

[1] R.A. DeVore, G.G. Lorentz, Constructive Approximation, Springer-Verlag, Berlin, Heidelberg, New York, 1993.
[2] I. Gavrea, M. Ivan, Sur une suite des opérateurs d'interpolation et approximation, in: Proceedings of the Itinerant Seminar on Functional Equations, Approximation and Convexity, Iaşi 1986, 1986, pp. 36–39.
[3] L.C. Hsu, X. Wang, Methods and Examples in Analysis, High Education Publishing Company, Beijing, 1955 (in Chinese).
[4] R.Ž Đorđević, G.V. Milovanović, On some generalizations of Zmorovič's inequality, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. 544–576 (1976) 25–30.
[5] R.Ž Đorđević, G.V. Milovanović, J.E. Pečarić, Some estimates of L^r norm on the set of continuously-differentiable functions, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. 634–677 (1979) 57–61.
[6] A. Lupaş, Inequalities for divided differences, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. 678–715 (1980) 24–28. (1981).
[7] V.A. Zmorovich, N.I. Cherneĭ, Some integral inequalities, Dokl. Akad. Nauk Ukrain. SSR Ser. A (6) (1983) 13–16 (in Russian).
[8] I. Gavrea, M. Ivan, An inequality for two-dimensional divided differences, Automat. Comput. Appl. Math. 4 (2) (1995) 100–103. (1996).
[9] V.A. Zmorovich, On some inequalities, Izv. Polytehn. Inst. Kiev 19 (1956) 92–107 (in Russian).