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Approximating fixed points of asymptotically nonexpansive mappings in Banach spaces by metric projections

Hossein Dehghan *

Department of Mathematics, Institute for Advanced Studies in Basic Sciences (IASBS), Gava Zang, Zanjan 45137-66731, Iran
Roozbeh Institute of Higher Education, Zanjan, Iran

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ABSTRACT

In this paper, a strong convergence theorem for asymptotically nonexpansive mappings in a uniformly convex and smooth Banach space is proved by using metric projections. This theorem extends and improves the recent strong convergence theorem due to Matsushita and Takahashi [S. Matsushita, W. Takahashi, Approximating fixed points of nonexpansive mappings in a Banach space by metric projections, Appl. Math. Comput. 196 (2008) 422–425] which was established for nonexpansive mappings.

1. Introduction

Let C be a closed convex subset of a real Banach space E. A mapping $T : C \to E$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. Also a mapping $T : C \to C$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|$$

for all $x, y \in C$ and each $n \geq 1$. The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [1] as an important generalization of nonexpansive mappings. It was proved in [1] that if C is a nonempty bounded closed convex subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive self-mapping on C, then $F(T)$ is nonempty closed convex subset of C, where $F(T)$ denotes the set of all fixed points of T. Strong convergence theorems for asymptotically nonexpansive mappings have been investigated with implicit and explicit iterative schemes (see [2–5] and references therein). On the other hand, using the metric projection, Nakajo and Takahashi [6] introduced the following iterative algorithm for the nonexpansive mapping T in the framework of Hilbert spaces: $x_0 = x \in C$ and

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n) T x_n, \\ C_n = \{z \in C : \|z - y_n\| \leq \|z - x_n\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \quad n = 0, 1, 2, \ldots, \end{cases} \tag{1.1}$$

where $\{\alpha_n\} \subset [0, \alpha]$, $\alpha \in [0, 1)$ and $P_{C_n \cap Q_n}$ is the metric projection from a Hilbert space H onto $C_n \cap Q_n$. They proved that $\{x_n\}$ generated by (1.1) converges strongly to a fixed point of T. Xu [7] extended Nakajo and Takahashi's theorem to Banach spaces by using the generalized projection.

* Corresponding address: Department of Mathematics, Institute for Advanced Studies in Basic Sciences (IASBS), Gava Zang, Zanjan 45137-66731, Iran.
E-mail addresses: h_dehghan@iasbs.ac.ir, hossein.dehgan@gmail.com.

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Matsushita and Takahashi [8] recently introduced the following iterative algorithm in the framework of Banach spaces: $x_0 = x \in C$ and

$$\begin{cases} C_n = \overline{\text{co}}\{z \in C : \|z - Tz\| \le t_n \|x_n - Tx_n\|\}, \\ D_n = \{z \in C : \langle x_n - z, J(x - x_n)\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap D_n} x, \quad n = 0, 1, 2, \ldots, \end{cases} \tag{1.2}$$

where $\overline{\text{co}}D$ denotes the convex closure of the set D, J is normalized duality mapping, $\{t_n\}$ is a sequence in $(0, 1)$ with $t_n \to 0$, and $P_{C_n \cap D_n}$ is the metric projection from E onto $C_n \cap D_n$. Then, they proved that $\{x_n\}$ generated by (1.2) converges strongly to a fixed point of nonexpansive mapping T.

In this paper, motivated by (1.1) and (1.2), we introduce the following iterative algorithm for finding fixed points of asymptotically nonexpansive mapping T in a uniformly convex and smooth Banach space: $x_1 = x \in C$, $C_0 = D_0 = C$ and

$$\begin{cases} C_n = \overline{\text{co}}\{z \in C_{n-1} : \|z - T^n z\| \le t_n \|x_n - T^n x_n\|\}, \\ D_n = \{z \in D_{n-1} : \langle x_n - z, J(x - x_n)\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap D_n} x, \quad n = 1, 2, \ldots, \end{cases} \tag{1.3}$$

where $\overline{\text{co}}D$ denotes the convex closure of the set D, J is normalized duality mapping, $\{t_n\}$ is a sequence in $(0, 1)$ with $t_n \to 0$, and $P_{C_n \cap D_n}$ is the metric projection from E onto $C_n \cap D_n$.

The purpose of this paper is to establish a strong convergence theorem of the iterative algorithm (1.3) for asymptotically nonexpansive mappings in a uniformly convex and smooth Banach space. The results presented in this paper extend and improve the corresponding ones announced by Matsushita and Takahashi [8] and many others.

2. Preliminaries

In this section, we recall the well-known concepts and results which are needed to prove our main convergence theorem. Throughout this paper, we denote by $\mathbb{N}$ the set of all positive integers. Let E be a real Banach space and let E^* be the dual of E. We denote the value of $x^* \in E^*$ at $x \in E$ by $\langle x, x^*\rangle$. When $\{x_n\}$ is a sequence in E, we denote strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$. The normalized duality mapping J from E to 2^{E^*} is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$$

for all $x \in E$. Some properties of duality mapping have been given in [9].

A Banach space E is said to be *strictly convex* if $\|(x + y)/2\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \ne y$. A Banach space E is also said to be *uniformly convex* if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|x_n + y_n\| = 2$. We also know that if E is a uniformly convex Banach space, then $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ imply $x_n \to x$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be *smooth* if

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. It is known that a Banach space E is smooth if and only if the normalized duality mapping J is single-valued. Let C be a closed convex subset of a reflexive, strictly convex and smooth Banach space E. Then for any $x \in E$ there exists a unique point $x_0 \in C$ such that $\|x_0 - x\| = \min_{y\in C} \|y - x\|$. The mapping $P_C : E \to C$ defined by $P_C x = x_0$ is called the *metric projection* from E onto C. Let $x \in E$ and $u \in C$. Then, it is known that $u = P_C x$ if and only if

$$\langle u - y, J(x - u)\rangle \ge 0 \tag{2.1}$$

for all $y \in C$ (see [10,11]). The following proposition was proved by Bruck [12].

Proposition 2.1. *Let C be a bounded closed convex subset of a uniformly convex Banach space E. Then there exists a strictly increasing convex continuous function $\gamma : [0, \infty) \to [0, \infty)$ with $\gamma(0) = 0$ depending only on the diameter of C such that*

$$\gamma\left(\left\|T\left(\sum_{i=1}^n \lambda_i x_i\right) - \sum_{i=1}^n \lambda_i T x_i\right\|\right) \le \max_{1\le i<j\le n} (\|x_i - x_j\| - \|Tx_i - Tx_j\|)$$

holds for any nonexpansive mapping $T : C \to E$, any elements $x_1, \ldots, x_n$ in C and any numbers $\lambda_1, \ldots, \lambda_n \ge 0$ with $\lambda_1 + \cdots + \lambda_n = 1$. (Note that γ does not depend on T.)

Corollary 2.2. *Under the same suppositions as in Proposition 2.1, there exists a strictly increasing convex continuous function $\gamma : [0, \infty) \to [0, \infty)$ with $\gamma(0) = 0$ such that*

$$\gamma\left(\frac{1}{k_m}\left\|T^m\left(\sum_{i=1}^n \lambda_i x_i\right) - \sum_{i=1}^n \lambda_i T^m x_i\right\|\right) \le \max_{1\le i<j\le n} \left(\|x_i - x_j\| - \frac{1}{k_m}\|T^m x_i - T^m x_j\|\right)$$

for any asymptotically nonexpansive mapping $T : C \to C$ with $\{k_n\}$, any elements $x_1, \ldots, x_n$ in C, any numbers $\lambda_1, \ldots, \lambda_n \ge 0$ with $\lambda_1 + \cdots + \lambda_n = 1$ and each $m \ge 1$.

Proof. Define the mapping $S_m : C \to E$ as $S_m x = 1/k_m T^m x$, for all $x \in C$ and each $m \geq 1$. Then S_m is nonexpansive for all $m \geq 1$. From Proposition 2.1, there exists a strictly increasing convex continuous function $\gamma : [0, \infty) \to [0, \infty)$ with $\gamma(0) = 0$ such that

$$\gamma\left(\left\|S_m\left(\sum_{j=1}^{n}\lambda_j x_j\right) - \sum_{j=1}^{n}\lambda_j S_m x_j\right\|\right) \leq \max_{1\leq j<k\leq n}(\|x_j - x_k\| - \|S_m x_j - S_m x_k\|)$$

for all $m \geq 1$. Thus, by using the definition of S_m, we obtain the desired conclusion. □

Lemma 2.3 ([2, Lemma 1.6]). *Let E be a uniformly convex Banach space, C be a nonempty closed convex subset of E and $T : C \to C$ be an asymptotically nonexpansive mapping. Then $(I - T)$ is demiclosed at 0, i.e., if $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$, then $x \in F(T)$, where $F(T)$ is the set of all fixed points of T.*

3. Strong convergence theorem

In this section, we study the iterative algorithm (1.3) for finding fixed points of asymptotically nonexpansive mappings in a uniformly convex and smooth Banach space. We first prove that the sequence $\{x_n\}$ generated by (1.3) is well defined. Then, we prove that $\{x_n\}$ converges strongly to $P_{F(T)}x$, where $P_{F(T)}$ is the metric projection from E onto $F(T)$.

Lemma 3.1. *Let C be a nonempty closed convex subset of a reflexive, strictly convex and smooth Banach space E and let $T : C \to C$ be an asymptotically nonexpansive mapping. If $F(T) \neq \emptyset$, then the sequence $\{x_n\}$ generated by (1.3) is well defined.*

Proof. It is easy to check that $C_n \cap D_n$ is closed and convex and $F(T) \subset C_n$ for each $n \in \mathbb{N}$. Moreover $D_1 = C$ and so $F(T) \subset C_1 \cap D_1$. Suppose $F(T) \subset C_k \cap D_k$ for $k \in \mathbb{N}$. Then, there exists a unique element $x_{k+1} \in C_k \cap D_k$ such that $x_{k+1} = P_{C_k \cap D_k}x$. If $u \in F(T)$, then it follows from (2.1) that

$$\langle x_{k+1} - u, J(x - x_{k+1})\rangle \geq 0,$$

which implies $u \in D_{k+1}$. Therefore $F(T) \subset C_{k+1} \cap D_{k+1}$. By mathematical induction, we obtain that $F(T) \subset C_n \cap D_n$ for all $n \in \mathbb{N}$. Therefore, $\{x_n\}$ is well defined. □

In order to prove our main result, the following lemma is needed.

Lemma 3.2. *Let C be a nonempty bounded closed convex subset of a uniformly convex and smooth Banach space E and let $T : C \to C$ be an asymptotically nonexpansive mapping with $\{k_n\}$ and $\{x_n\}$ be the sequence generated by (1.3). Then for any $k \in \mathbb{N}$,*

$$\lim_{n\to\infty}\|x_n - T^{n-k}x_n\| = 0.$$

Proof. Fix $k \in \mathbb{N}$ and put $m = n - k$. Since $x_n = P_{C_{n-1}\cap D_{n-1}}x$, we have $x_n \in C_{n-1} \subseteq \cdots \subseteq C_m$. Since $t_m > 0$, there exist $y_1, \ldots, y_N \in C$ and $\lambda_1, \ldots, \lambda_N \geq 0$ with $\lambda_1 + \cdots + \lambda_N = 1$ such that

$$\left\|x_n - \sum_{i=1}^{N}\lambda_i y_i\right\| < t_m, \tag{3.1}$$

and $\|y_i - T^m y_i\| \leq t_m\|x_m - T^m x_m\|$ for all $i \in \{1, \ldots, N\}$. Put $M = \sup_{x\in C}\|x\|$, $u = P_{F(T)}x$ and $r_0 = \sup_{n\geq 1}(1 + k_n)\|x_n - u\|$. Since C and $\{k_n\}$ are bounded, (3.1) implies

$$\left\|x_n - \frac{1}{k_m}\sum_{i=1}^{N}\lambda_i y_i\right\| \leq \left(1 - \frac{1}{k_m}\right)\|x_n\| + \frac{1}{k_m}\left\|x_n - \sum_{i=1}^{N}\lambda_i y_i\right\| \leq \left(1 - \frac{1}{k_m}\right)M + t_m, \tag{3.2}$$

and $\|y_i - T^m y_i\| \leq t_m\|x_m - T^m x_m\| \leq t_m(1 + k_m)\|x_m - u\| \leq r_0 t_m$ for all $i \in \{1, \ldots, N\}$. Therefore

$$\left\|y_i - \frac{1}{k_m}T^m y_i\right\| \leq \left(1 - \frac{1}{k_m}\right)M + r_0 t_m \tag{3.3}$$

for all $i \in \{1, \ldots, N\}$. Moreover, asymptotically nonexpansiveness of T and (3.1) give that

$$\left\|\frac{1}{k_m}T^m\left(\sum_{i=1}^{N}\lambda_i y_i\right) - T^m x_n\right\| \leq \left(1 - \frac{1}{k_m}\right)M + t_m. \tag{3.4}$$

It follows from Corollary 2.2, (3.2)–(3.4) that

$$\begin{aligned}\|x_n - T^m x_n\| &\le \left\| x_n - \frac{1}{k_m}\sum_{i=1}^{N}\lambda_i y_i \right\| + \frac{1}{k_m}\left\| \sum_{i=1}^{N}\lambda_i \left(y_i - T^m y_i\right)\right\| \\ &\quad + \frac{1}{k_m}\left\| \sum_{i=1}^{N}\lambda_i T^m y_i - T^m\left(\sum_{i=1}^{N}\lambda_i y_i\right)\right\| + \left\| \frac{1}{k_m}T^m\left(\sum_{i=1}^{N}\lambda_i y_i\right) - T^m x_n\right\| \\ &\le 2\left(1-\frac{1}{k_m}\right)M + 2t_m + \frac{r_0 t_m}{k_m} + \gamma^{-1}\left(\max_{1\le i<j\le N}\left(\|y_i - y_j\| - \frac{1}{k_m}\left\|T^m y_i - T^m y_j\right\|\right)\right) \\ &\le 2\left(1-\frac{1}{k_m}\right)M + 2t_m + \frac{r_0 t_m}{k_m} + \gamma^{-1}\left(\max_{1\le i<j\le N}\left(\left\|y_i - \frac{1}{k_m}T^m y_i\right\| + \left\|y_j - \frac{1}{k_m}T^m y_j\right\|\right)\right) \\ &\le 2\left(1-\frac{1}{k_m}\right)M + 2t_m + \frac{r_0 t_m}{k_m} + \gamma^{-1}\left(2\left(1-\frac{1}{k_m}\right)M + 2r_0 t_m\right).\end{aligned}$$

Since $\lim_{n\to\infty} k_n = 1$ and $\lim_{n\to\infty} t_n = 0$, it follows from the last inequality that $\lim_{n\to\infty}\|x_n - T^m x_n\| = 0$. This completes the proof. □

Theorem 3.3. *Let C be a nonempty bounded closed convex subset of a uniformly convex and smooth Banach space E and let $T : C \to C$ be an asymptotically nonexpansive mapping and let $\{x_n\}$ be the sequence generated by (1.3). Then $\{x_n\}$ converges strongly to the element $P_{F(T)}x$ of $F(T)$, where $P_{F(T)}$ is the metric projection from E onto $F(T)$.*

Proof. Put $u = P_{F(T)}x$. Since $F(T) \subset C_n \cap D_n$ and $x_{n+1} = P_{C_n\cap D_n}x$, we have that

$$\|x - x_{n+1}\| \le \|x - u\| \tag{3.5}$$

for all $n \in \mathbb{N}$. By Lemma 3.2, we have

$$\begin{aligned}\|x_n - Tx_n\| &\le \|x_n - T^{n-1}x_n\| + \|T^{n-1}x_n - Tx_n\| \\ &\le \|x_n - T^{n-1}x_n\| + k_1\|T^{n-2}x_n - x_n\| \to 0 \quad \text{as } n\to\infty.\end{aligned}$$

Since $\{x_n\}$ is bounded, there exists $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup v$. It follows from Lemma 2.3 that $v \in F(T)$. From the weakly lower semicontinuity of norm and (3.5), we obtain

$$\|x - u\| \le \|x - v\| \le \liminf_{i\to\infty}\|x - x_{n_i}\| \le \limsup_{i\to\infty}\|x - x_{n_i}\| \le \|x - u\|.$$

This together with the uniqueness of $P_{F(T)}x$, implies $u = v$, and hence $x_{n_i} \rightharpoonup u$. Therefore, we obtain $x_n \rightharpoonup u$. Furthermore, we have that

$$\lim_{n\to\infty}\|x - x_n\| = \|x - u\|.$$

Since E is uniformly convex, we have that $x - x_n \to x - u$. It follows that $x_n \to u$. This completes the proof. □

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