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ADHM construction of (anti-)self-dual instantons in eight dimensions

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Abstract

We study the ADHM construction of (anti-)self-dual instantons in eight dimensions. We propose a general scheme to construct the (anti-)self-dual gauge field configurations $F \wedge F = \pm *_8 F \wedge F$ whose finite topological charges are given by the fourth Chern number. We show that our construction reproduces the known SO(8) one-instanton solution. We also construct multi-instanton solutions of the 't Hooft and the Jackiw–Nohl–Rebbi (JNR) types in the dilute instanton gas approximation. The well-separated configurations of multi-instantons reproduce the correct topological charges with high accuracy. We also show that our construction is generalized to (anti-)self-dual instantons in $4n$ $(n = 3, 4, \ldots)$ dimensions.

1. Introduction

It is well known that instantons in gauge theories play important roles in the study of non-perturbative effects [1,2]. Instantons in four-dimensional gauge theories with gauge group G are defined by configurations such that the gauge field strength 2-form F satisfies the (anti-)self-dual equation $F = \pm *_4 F$. Here $*_d$ is the Hodge dual operator in d-dimensional Euclid space. Due to the Bianchi identity, instanton solutions in four-dimensional Yang–Mills theory satisfy the equation of motion. The instanton solutions are classified by the second Chern number which is

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proportional to $\int \mathrm{Tr}[F \wedge F]$. They are characterized by the homotopy group $\pi_3(G)$. A salient feature of the (anti-)self-dual instantons in four dimensions is its systematic construction of solutions, known as the ADHM construction [3]. The ADHM construction reveals the Kähler quotient structure of the instanton moduli space and provides the scheme to calculate the non-perturbative corrections in the path integral.

It is natural to generalize the instantons in four dimensions to higher and lower dimensions. In the lower dimensions, the dimensional reduction of the (anti-)self-dual equation to three dimensions leads to the monopole equations. The ADHM construction is reduced to the Nahm construction of the monopoles [4]. In two dimensions, the (anti-)self-dual equations provide equations for the Hitchin system [5]. Further dimensional reductions of the (anti-)self-dual equation give equations in various integrable systems in one and two dimensions [6]. This is known as the Ward's conjecture [7].

On the other hand, instantons in dimensions higher than four have been studied in various contexts. It is known that there are several kinds of "instantons" in higher dimensions. A straightforward generalization of the (anti-)self-dual equation $F = \pm *_4 F$ to $d > 4$ dimensions is the linear equation $F_{\mu\nu} = \lambda T_{\mu\nu\rho\sigma} F^{\rho\sigma}, \lambda \neq 0$ $(\mu, \nu, \rho, \sigma = 1, \ldots, d)$ [8–10]. Here $T_{\mu\nu\rho\sigma}$ is an anti-symmetric constant tensor which respects subgroups of the $\mathrm{SO}(d)$ Lorentz group. This equation is called the secular type and solutions to this equation are sometimes called secular type instantons. Note that the secular type instantons satisfy the equation of motion for Yang–Mills theory but it is not always true that Chern numbers associated with the solutions are finite and quantized. An example of the secular type instanton is the Fubini–Nicolai instantons [11], also known as octonionic instantons, defined in eight dimensions. Other examples are BPS instantons that preserve fractions of supersymmetry in eight-dimensional super Yang–Mills theory [12]. An ADHM construction of secular type instantons in $4n$ $(n = 1, 2, 3, \ldots)$ dimensions has been studied [13].

Among other things, instantons in $4n$ $(n = 1, 2, 3, \ldots)$ dimensions provide special interests. This is because in these dimensions, the (anti-)self-dual equations of the field strengths are naturally defined. For example, in eight dimensions $(n = 2)$, we can define the (anti-)self-dual equation $F \wedge F = \pm *_8 F \wedge F$. We call solutions to this equation the (anti-)self-dual instantons in eight dimensions. We expect that configurations which satisfy the (anti-)self-dual equation have non-zero topological charges given by the fourth Chern number $k = \mathcal{N} \int \mathrm{Tr}[F \wedge F \wedge F \wedge F]$, where $\mathcal{N}$ is a normalization constant. Since the eight-dimensional (anti-)self-dual equation is highly non-linear and contains higher derivatives, only the one-instanton solution is known [14, 15]. This is called the SO(8) instanton. Note that the SO(8) instanton does not satisfy the secular equation in general.

In this paper, we study an ADHM construction of (anti-)self-dual instantons in eight dimensions. We will show that there is a general scheme to find the (anti-)self-dual instanton solutions. By introducing specific ADHM data which solve ADHM constraints, we will explicitly construct gauge field configurations whose fourth Chern numbers are integers. This implies that the solutions are characterized by the homotopy group $\pi_7(G)$. We will also discuss eight-dimensional higher derivative theories in which the (anti-)self-dual equation $F \wedge F = \pm *_8 F \wedge F$ becomes relevant.

The organization of this paper is as follows. In the next section, we study the ADHM construction of (anti-)self-dual instantons in eight dimensions. This is just an eight-dimensional analogue of the original ADHM construction of instantons in four dimensions. We find that there is an extra ADHM constraint in addition to the original one which is present in four dimensions. The gauge group and algebraic structures of the solutions are studied in detail. In section 3, we see that our construction precisely reproduces the well-known one-instanton profile of the solution

[14,15]. Furthermore we construct the so-called 't Hooft and the Jackiw–Nohl–Rebbi (JNR) type multi-instantons. The ADHM data associated with these solutions solve the ADHM constraints in the dilute instanton gas limit. We obtain the correct topological charges in a good accuracy. In section 4, we discuss eight-dimensional gauge field theories where the (anti-)self-dual instantons are analyzed. We observe that the multi-instantons of the 't Hooft type can be interpreted as D(−1)-branes embedded in the D7-branes in the small instanton limit. Section 5 is devoted to conclusion and discussions. The ADHM construction of instantons in four dimensions is briefly discussed in Appendix A. The Clifford algebras in $4n$ dimensions are shown in Appendix B. The explicit form of the eight-dimensional ADHM equations for the gauge group U(8) is found in Appendix C.

2. ADHM construction in eight dimensions

In this section, we study the ADHM construction of (anti-)self-dual instantons in eight-dimensional Euclid space with the flat metric. The (anti-)self-dual equation is defined by

$$F \wedge F = \pm *_8 (F \wedge F), \tag{1}$$

where the 2-form $F = \frac{1}{2!} F_{\mu\nu} dx^\mu \wedge dx^\nu$ is the gauge field strength whose component is defined by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{2}$$

The anti-Hermite gauge field A_μ takes value in $\mathcal{G}$. Here $\mathcal{G}$ is the Lie algebra associated with the non-Abelian gauge group G and $\mu, \nu, \ldots = 1, 2, \ldots, 8$ are the tensor indices in the eight-dimensional Euclid space. The (anti-)self-dual equation (1) in the component expression is given by

$$F_{[\mu\nu} F_{\rho\sigma]} = \pm \frac{1}{4!} \varepsilon_{\mu\nu\rho\sigma\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta}, \tag{3}$$

where $\varepsilon_{\mu\nu\rho\sigma\alpha\beta\gamma\delta}$ is the anti-symmetric epsilon symbol in eight dimensions and the bracket $[\mu_1\mu_2 \cdots \mu_n]$ stands for the anti-symmetrization of indices with the weight $1/n!$. In the following subsections, we look for a general scheme to find the solutions to the (anti-)self-dual equation (3). To this end, we follow the ADHM construction of instantons in four dimensions and generalize it to eight dimensions.

2.1. *(Anti-)self-dual basis in eight dimensions*

The first step toward the ADHM construction in eight dimensions is to find an appropriate basis which guarantees the (anti-)self-duality nature of the gauge field strength $F_{\mu\nu}$. The corresponding basis in four dimensions is the quaternions $\sigma_\mu = (-i\vec{\sigma}, \mathbf{1}_2)$ $(\mu = 1, \ldots, 4)$ where $\vec{\sigma}$ are the Pauli matrices. Using this basis, quantities $\eta^{(+)}_{\mu\nu} = \sigma^\dagger_\mu \sigma_\nu - \sigma^\dagger_\nu \sigma_\mu$, $\eta^{(-)}_{\mu\nu} = \sigma_\mu \sigma^\dagger_\nu - \sigma_\nu \sigma^\dagger_\mu$ that satisfy the (anti-)self-dual relations in four dimensions $\eta^{(\pm)}_{\mu\nu} = \pm \frac{1}{2!} \varepsilon_{\mu\nu\rho\sigma} \eta^{(\pm)}_{\rho\sigma}$ are defined. These $\eta^{(\pm)}_{\mu\nu}$ are just the 't Hooft symbol.

By the analogy of the quaternions in four dimensions, we define the following basis in eight dimensions:

$$e_\mu = \delta_{\mu 8} \mathbf{1}_8 + \delta_{\mu i} \Gamma^{(-)}_i, \qquad e^\dagger_\mu = \delta_{\mu 8} \mathbf{1}_8 + \delta_{\mu i} \Gamma^{(+)}_i \qquad (\mu = 1, \ldots, 8,\ i = 1, \ldots, 7), \tag{4}$$

where $\Gamma_i^{(\pm)}$ are 8×8 matrices that satisfy the relations $\{\Gamma_i^{(\pm)}, \Gamma_j^{(\pm)}\} = -2\delta_{ij}\mathbf{1}_8$. The matrices $\Gamma_i^{(\pm)}$ are defined by $\Gamma_i^{(\pm)} = \frac{1}{2}(1 \pm \omega)\Gamma_i$. We choose the matrices $\Gamma_i^{(\pm)}$ such that they satisfy the relation $\Gamma_i^{(+)} = -\Gamma_i^{(-)}$. Here Γ_i are given by the matrix representation of the seven-dimensional complex (real) Clifford algebra $\Gamma_i \in C\ell_7(\mathbb{C}(\mathbb{R}))$ and ω is a chirality matrix defined in Appendix B. Using this basis, we construct the eight-dimensional counterpart of the 't Hooft symbol. This is defined by

$$\Sigma_{\mu\nu}^{(+)} = e_\mu^\dagger e_\nu - e_\nu^\dagger e_\mu, \qquad \Sigma_{\mu\nu}^{(-)} = e_\mu e_\nu^\dagger - e_\nu e_\mu^\dagger. \tag{5}$$

We can confirm that $\Sigma_{\mu\nu}^{(\pm)}$ given above indeed satisfy the (anti-)self-dual relations in eight dimensions:

$$\Sigma_{[\mu\nu}^{(\pm)}\Sigma_{\rho\sigma]}^{(\pm)} = \pm\frac{1}{4!}\varepsilon_{\mu\nu\rho\sigma\alpha\beta\gamma\delta}\Sigma_{\alpha\beta}^{(\pm)}\Sigma_{\gamma\delta}^{(\pm)}, \tag{6}$$

where the upper script sign of $\Sigma_{\mu\nu}^{(\pm)}$ correspond the sign in the right-hand side of (6). We also observe that the basis e_μ satisfies the following useful relations:

$$e_\mu e_\nu^\dagger + e_\nu e_\mu^\dagger = e_\mu^\dagger e_\nu + e_\nu^\dagger e_\mu = 2\delta_{\mu\nu}\mathbf{1}_8, \tag{7a}$$

$$e_\mu e_\nu + e_\nu e_\mu = 2\delta_{\mu 8}e_\nu + 2\delta_{\nu 8}e_\mu - 2\delta_{\mu\nu}\mathbf{1}_8, \tag{7b}$$

$$e_\mu^\dagger e_\nu^\dagger + e_\nu^\dagger e_\mu^\dagger = 2\delta_{\mu 8}e_\nu^\dagger + 2\delta_{\nu 8}e_\mu^\dagger - 2\delta_{\mu\nu}\mathbf{1}_8. \tag{7c}$$

Furthermore the basis e_μ is normalized as $\mathrm{Tr}\left[e_\mu e_\nu^\dagger\right] = 8\delta_{\mu\nu}\mathbf{1}_8$. For later convenience we calculate the following quantities:

$$\begin{aligned}
&\mathrm{Tr}\Sigma_{12}^{(\pm)}\Sigma_{34}^{(\pm)}\Sigma_{56}^{(\pm)}\Sigma_{78}^{(\pm)} = \pm 16\mathrm{Tr}\mathbf{1}_8 = \pm 128,\\
&\Sigma_{\mu\nu}^{(\pm)}\Sigma_{\rho\sigma}^{(\pm)}\Sigma_{\alpha\beta}^{(\pm)}\Sigma_{\gamma\delta}^{(\pm)} = \varepsilon_{\mu\nu\rho\sigma\alpha\beta\gamma\delta}\Sigma_{12}^{(\pm)}\Sigma_{34}^{(\pm)}\Sigma_{56}^{(\pm)}\Sigma_{78}^{(\pm)} = \pm 16\varepsilon_{\mu\nu\rho\sigma\alpha\beta\gamma\delta}\mathbf{1}_8.
\end{aligned} \tag{8}$$

A comment is in order. One may consider that a natural candidate of the eight-dimensional counterpart of the quaternions is octonions. Indeed, an ADHM construction of (anti-)self-dual instantons with the octonion basis has been proposed and studied [16,17]. However, due to the well-known nature of octonions, the gauge field loses the associativity which would causes potential difficulties in field theories. We stress that the basis in (4) is defined by the complex (real) Clifford algebra $C\ell_7(\mathbb{C}(\mathbb{R}))$ which has the matrix representations and keeps the associativity.

2.2. Solutions for gauge field

Now we have obtained the appropriate basis e_μ which supplants the quaternions in four dimensions. The next step is to find explicit solutions for the gauge field A_μ. In the following, we choose the minus sign in (3) and concentrate on the anti-self-dual equation. In order to find the anti-self-dual solution, we first introduce the eight-dimensional Weyl operator[1]

$$\Delta = C(x \otimes \mathbf{1}_k) + D, \tag{9}$$

where C and D are $(8 + 8k) \times 8k$ matrices, $x = x^\mu e_\mu$ and x^μ are the Cartesian coordinates of the eight-dimensional Euclid space. If we consider self-dual solutions, we choose the basis $e_\mu^\dagger$ instead of e_μ. The components of the matrices C and D are called the ADHM data. Note that

[1] Here the symbol $\otimes$ means the Kronecker product, *i.e.* the tensor product of matrices.

we can decompose the indices of an $8(1+k) \times 8k$ matrix into the instanton index that runs from 1 to k and the color indices that run from 1 to 8. As we will show, the integer k corresponds to the instanton number defined by the fourth Chern number $k = \mathcal{N} \int \mathrm{Tr}[F \wedge F \wedge F \wedge F]$. Now we introduce an $(8+8k) \times 8$ matrix $V(x)$ which satisfies the Weyl equation:

$$\Delta^\dagger V(x) = 0. \tag{10}$$

The matrix $V(x)$, which is called the zero-mode, is normalized as

$$V^\dagger V = \mathbf{1}_8. \tag{11}$$

The completeness condition of $V(x)$ implies the following relation[2]:

$$\mathbf{1}_{8+8k} - VV^\dagger = \Delta(\Delta^\dagger \Delta)^{-1}\Delta^\dagger. \tag{12}$$

Following the ADHM construction of instantons in four-dimensions [3], we employ the ansatz that the gauge field $A_\mu(x)$ is given by the pure gauge form:

$$A_\mu(x) = V^\dagger(x)\partial_\mu V(x). \tag{13}$$

Next we calculate the field strength $F_{\mu\nu}$ from the ansatz (13). Using the Weyl equation (10) and the completeness relation (12), the result is

$$F_{\mu\nu} = V^\dagger C(e_\mu \otimes \mathbf{1}_k)(\Delta^\dagger\Delta)^{-1}(e_\nu^\dagger \otimes \mathbf{1}_k)C^\dagger V - (\mu \leftrightarrow \nu). \tag{14}$$

We are now looking for conditions that the field strength (14) satisfies the anti-self-dual equation (3). One realizes that the basis e_μ should appear in the combination of $\Sigma^{(-)}_{\mu\nu}$ defined in (5). We then demand that the factor $(\Delta^\dagger\Delta)^{-1}$ in (14) commutes with the basis $e_\mu(\otimes \mathbf{1}_k)$:

$$e_\mu \otimes \mathbf{1}_k(\Delta^\dagger\Delta)^{-1} = (\Delta^\dagger\Delta)^{-1}e_\mu \otimes \mathbf{1}_k. \tag{15}$$

Then the product of the field strengths is calculated to be

$$F_{\mu\nu}F_{\rho\sigma} = \left(V^\dagger C(\Delta^\dagger\Delta)^{-1}\left(\Sigma^{(-)}_{\mu\nu} \otimes \mathbf{1}_k\right)C^\dagger V\right)\left(V^\dagger C\left(\Sigma^{(-)}_{\rho\sigma} \otimes \mathbf{1}_k\right)(\Delta^\dagger\Delta)^{-1}C^\dagger V\right). \tag{16}$$

In order that the field strength $F_{[\mu\nu}F_{\rho\sigma]}$ satisfies the anti-self-dual equation, $\Sigma^{(-)}_{\mu\nu} \otimes \mathbf{1}_k$ should commute with $(C^\dagger VV^\dagger C)$ in (16). Therefore we demand the following condition:

$$e_\mu \otimes \mathbf{1}_k\left(C^\dagger VV^\dagger C\right) = \left(C^\dagger VV^\dagger C\right)e_\mu \otimes \mathbf{1}_k. \tag{17}$$

Indeed, using the condition (17), the product of the field strengths becomes

$$F_{[\mu\nu}F_{\rho\sigma]} = V^\dagger C(\Delta^\dagger\Delta)^{-1}\left(\Sigma^{(-)}_{[\mu\nu}\Sigma^{(-)}_{\rho\sigma]} \otimes \mathbf{1}_k\right)C^\dagger VV^\dagger C(\Delta^\dagger\Delta)^{-1}C^\dagger V. \tag{18}$$

Since $\Sigma^{(-)}_{[\mu\nu}\Sigma^{(-)}_{\rho\sigma]}$ satisfies the anti-self-dual relation (6), we find that this is also true for $F_{[\mu\nu}F_{\rho\sigma]}$. Therefore the expression (13) with the constraints (15) and (17) gives the solution to the anti-self-dual equation (3) in eight dimensions.

It is desirable to find conditions on the ADHM data C and D corresponding to (15) and (17). The equation (15) is equivalent to the following constraint on the matrix Δ:

$$\Delta^\dagger\Delta = \mathbf{1}_8 \otimes E_k^{(1)}, \tag{19}$$

[2] The completeness relation (12) is derived by assuming an existence of $(\Delta^\dagger\Delta)^{-1}$.

where $E_k^{(1)}$ is an invertible $k \times k$ matrix. We call (19) the first ADHM constraint. The condition (19) is a natural generalization of the ADHM constraint in four dimensions. See Appendix A for the four-dimensional counterpart of the constraint.

On the other hand, the relation (12) allows us to rewrite the condition (17) as

$$
\begin{aligned}
& e_\mu \otimes \mathbf{1}_k \left(C^\dagger C \right) = \left(C^\dagger C \right) e_\mu \otimes \mathbf{1}_k, \\
& e_\mu \otimes \mathbf{1}_k \left(C^\dagger \Delta (\Delta^\dagger \Delta)^{-1} \Delta^\dagger C \right) = \left(C^\dagger \Delta (\Delta^\dagger \Delta)^{-1} \Delta^\dagger C \right) e_\mu \otimes \mathbf{1}_k .
\end{aligned} \tag{20}
$$

The first condition is automatically satisfied when the condition (15) holds. The second one in (20) is essentially the new condition for eight-dimensional anti-self-dual instantons. This is equivalent to the constraint

$$
C^\dagger \Delta (\Delta^\dagger \Delta)^{-1} \Delta^\dagger C = \mathbf{1}_8 \otimes E_k^{(2)}, \tag{21}
$$

where $E_k^{(2)}$ is an invertible $k \times k$ matrix. We call (21) the second ADHM constraint.

It is easy to find that the Weyl equation (10), the normalization condition (11), the first and the second ADHM constraints (19), (21) are invariant under the following transformations:

$$
C \mapsto C' = \mathcal{U} C \mathcal{R}, \quad D \mapsto D' = \mathcal{U} D \mathcal{R}, \quad V \mapsto V' = \mathcal{U} V, \tag{22}
$$

where $\mathcal{U} \in \mathrm{U}(8+8k)$ and $\mathcal{R} = \mathbf{1}_8 \otimes \mathcal{R}_k \in \mathbf{1}_8 \otimes \mathrm{GL}(k; \mathbb{C})$ for $\Gamma_i \in C\ell_7(\mathbb{C})$.[3] Using this $\mathrm{U}(8+8k) \times \mathrm{GL}(k, \mathbb{C})$ transformation, we can fix the ADHM data to the so-called canonical form. This is given by

$$
C = \begin{pmatrix} 0_{[8]\times[8k]} \\ \mathbf{1}_{8k} \end{pmatrix}_{[8+8k]\times[8k]}, \qquad D = \begin{pmatrix} S_{[8]\times[8k]} \\ T_{[8k]\times[8k]} \end{pmatrix}_{[8+8k]\times[8k]} = \begin{pmatrix} S_{[8]\times[8k]} \\ e_{\mu\ [8]} \otimes T^\mu_{[k]} \end{pmatrix}. \tag{23}
$$

Here the matrix subscript $[a] \times [b]$ means the matrix size. The symbol $S_{[8]\times[8k]}$ stands for $\begin{pmatrix} S_{1\ [8]\times[k]} & S_{2\ [8]\times[k]} & \ldots & S_{8\ [8]\times[k]} \end{pmatrix}$. We note that all the ADHM data are included in the $(8+8k) \times 8k$ matrix D in the canonical form. We find that there are residual symmetries which leave the canonical form (23) invariant. The transformations are given by

$$
S_\mu \mapsto S'_\mu = Q S_\mu R, \quad T^\mu \mapsto T'^\mu = R^\dagger T^\mu R, \tag{24}
$$

where $Q \in \mathrm{SU}(8)$ and $R \in \mathrm{U}(k)$ for $\Gamma_i \in C\ell_7(\mathbb{C})$.[4]

Now we have established the ADHM construction of (anti-)self-dual instantons in eight dimensions. Plugging the canonical form of C and D in (23) into the first and the second ADHM constraints (19), (21), we obtain the algebraic constraints on the matrices T and S. The explicit form of the constraints (that are called the ADHM equations) are found in Appendix C. Solutions S and T to these constraints lead to the profile functions of the gauge field A_μ corresponding to the anti-self-dual instantons. We will show the explicit solutions for S and T and its associated gauge field A_μ in Section 3. However, before going to the solutions, we discuss the gauge groups of the theory and the homotopy group which classify the solutions.

---

[3] When Γ_i take value in the real Clifford algebra $Cl_7(\mathbb{R})$ instead of the complex one and the ADHM data C are D are real valued, then the transformation groups are $\mathcal{U} \in \mathrm{O}(8+8k)$ and $\mathcal{R} \in \mathbf{1}_8 \otimes \mathrm{GL}(k; \mathbb{R})$.

[4] When $\Gamma_i \in C\ell_7(\mathbb{R})$ and the matrices S are T are real valued, the transformation groups are $Q \in \mathrm{SO}(8)$ and $R \in \mathrm{O}(k)$.

2.3. Gauge and homotopy groups

In this subsection, we discuss the gauge group of the theory and the homotopy class of the solutions.

The gauge transformation of the solution A_μ is induced by the transformation of the zero-mode $V(x)$ which preserves the normalization condition (11). Indeed, using the ansatz (13), the transformation of the zero-mode $V \mapsto Vg(x)$ induces the following gauge transformation:

$$A_\mu \mapsto g^{-1}(x)A_\mu g(x) + g^{-1}(x)\partial_\mu g(x). \tag{25}$$

We note that the transformation $V \mapsto Vg(x)$ is independent of the one in (22). The gauge group is determined as follows.

As we have mentioned, the group structure of the transformation matrix $g(x)$ is determined by the Clifford algebra which has been used to construct the basis e_μ. For example, when e_μ takes complex values, then Γ_i is the element of the complex Clifford algebra $C\ell_7(\mathbb{C})$. In this case, the Weyl operator Δ takes complex values and the solutions to the Weyl equation $\Delta^\dagger V = 0$ (that is the zero-mode V) is a complex $(8+8k) \times 8$ matrix. Therefore the gauge group associated with the transformation $V \mapsto Vg(x)$ is the unitary group $G = \mathrm{U}(8)$. On the other hand, when e_μ and the ADHM data take real values, then Γ_i belongs to the real Clifford algebra $C\ell_7(\mathbb{R})$. The Weyl operator Δ takes real values and the zero-mode V is a real $(8+8k) \times 8$ matrix. In this case, the gauge group associated with the transformation $V \mapsto Vg(x)$ is the orthogonal group $G = \mathrm{O}(8)$.

It is clear that the color size N of the gauge group depends on the matrix size of the basis e_μ. Here the matrix representations of the complex (real) Clifford algebra are given by the 8×8 complex (real) matrices. Therefore the basis e_μ are 8×8 matrices and the color size is eight, *i.e.* $N = 8$. Relations of gauge groups and Clifford algebras are discussed in detail in Appendix B. Note that the ADHM construction does not impose the specialty condition on the gauge group in general, namely, the gauge group G is not the special unitary group $\mathrm{SU}(N)$ nor the special orthogonal group $\mathrm{SO}(N)$ but they are $\mathrm{U}(N)$ or $\mathrm{O}(N)$. We can decompose the group $\mathrm{U}(N)$ (or $\mathrm{O}(N)$) into the special group $\mathrm{SU}(N)$ (or $\mathrm{SO}(N)$) part and $\mathrm{U}(1)$ (or S^0) part: $\mathrm{U}(N) = \mathrm{SU}(N) \ltimes \mathrm{U}(1)$ and $\mathrm{O}(N) = \mathrm{SO}(N) \ltimes S^0$. Usually, we have to fix the element of $\mathrm{U}(1)$ (or S^0) by hand when we consider $\mathrm{SU}(N)$ or $\mathrm{SO}(N)$ in the ADHM construction of instantons.

Finally, we give a brief discussion on the homotopy group. Instantons with gauge group[5] G in eight dimensions are classified by the homotopy group $\pi_7(G)$. We are interested in instantons that are characterized by an integer k. One observes that the gauge group G whose rank is small makes $\pi_7(G)$ be trivial. For example, the homotopy groups $\pi_7(G)$ for $G = \mathrm{SO}(N)$ $(N \le 4)$ and $G = \mathrm{SU}(N)$ $(N \le 3)$ become trivial. For larger rank groups, one obtains desired property $\pi_7(G) = \mathbb{Z}$ for $G = \mathrm{SU}(N)$ $(N \ge 4)$, $G = \mathrm{SO}(N)$ $(N \ge 5, N \ne 8)$, $G = \mathrm{Sp}(N)$ $(N \ge 2)$. The homotopy groups relevant to the eight-dimensional ADHM construction presented in this paper are $G = \mathrm{U}(8)$, $G = \mathrm{SU}(8)$ and $G = \mathrm{SO}(8)$. For the former two groups, we have

$$\pi_7(\mathrm{U}(8)) = \pi_7(\mathrm{SU}(8)) = \mathbb{Z}, \tag{26}$$

while for $G = \mathrm{SO}(8)$, we have

$$\pi_7(\mathrm{SO}(8)) = \mathbb{Z} \oplus \mathbb{Z}. \tag{27}$$

[5] Here we focus on the compact Lie group G.

We note that the SO(8) instanton solutions are embedded in solutions for the gauge groups SO(N) ($N \geq 8$). This is because the property of the homotopy class $\pi_7(\mathrm{SO}(N)) = \mathbb{Z}$ ($N > 8$). The same is true for SU(N) and U(N).

3. ADHM data and multi-instanton solutions

In this section, we introduce explicit ADHM data that satisfy the first and the second ADHM constraints (19), (21). We will show that the integer k in the construction is the topological charge of the eight-dimensional instantons. The topological charge Q for the eight-dimensional instantons is defined by the fourth Chern number:

$$Q = \mathcal{N} \int_{\mathbb{R}^8} \mathrm{Tr}\,(F \wedge F \wedge F \wedge F) = \mathcal{N} \int_{\mathbb{R}^8} d^8 x \ \mathrm{Tr} \left(\frac{1}{8!} \varepsilon_{\mu\nu\rho\sigma\alpha\beta\gamma\delta} F_{\mu\nu} F_{\rho\sigma} F_{\alpha\beta} F_{\gamma\delta} \right), \tag{28}$$

where $\mathcal{N}$ is the normalization constant which will be determined later. Using the expression (18) and the ADHM constraints (19), (21), the charge density $\mathcal{Q}$ is calculated to be

$$\mathcal{Q} = \pm 16 \mathrm{Tr} \left(V^\dagger C (\Delta^\dagger \Delta)^{-1} C^\dagger V \right)^4. \tag{29}$$

Here $\pm$ corresponds to the (anti-)self-dual solutions respectively.

In the next subsection, we introduce explicit ADHM data and calculate the topological charges associated with the solutions. We first introduce the eight-dimensional ADHM ansatz for the ADHM data on the analogy of the four-dimensional ones. Here the "ansatz" means that this ADHM data at least satisfy the first ADHM constraint (19).

For a technical reason, it is convenient to introduce the following form of the second ADHM constraint:

$$C^\dagger V V^\dagger C = \mathbf{1}_8 \otimes E_k^{(3)}, \tag{30}$$

where $E_k^{(3)}$ is an invertible $k \times k$ matrix. This is a stronger condition of the second ADHM constraint but more tractable than (21). The second ADHM constraint (21) is satisfied when (30) is satisfied. In the following, we will examine the second ADHM constraint (30) for given ansatz for ADHM data and determine the multi-instanton profiles.

3.1. BPST type one-instanton

We first reproduce the $k = 1$ instanton solution in eight dimensions. This is known as the SO(8) instanton [14]. In the case of $k = 1$, the ADHM ansatz in the canonical form is taken to be

$$C = \begin{pmatrix} 0 \\ \mathbf{1}_8 \end{pmatrix}, \qquad D = \begin{pmatrix} \lambda \mathbf{1}_8 \\ -a^\mu e_\mu \end{pmatrix}, \tag{31}$$

where $\lambda \in \mathbb{R}$ is the size modulus and $a^\mu \in \mathbb{R}$ is the position modulus of the instanton. It is easily shown that the ADHM ansatz (31) satisfies the first ADHM constraint (19).

The solution to the Weyl equation (10) associated with the ADHM ansatz (31) is found to be

$$V = \frac{1}{\sqrt{\rho}} \begin{pmatrix} \tilde{x}^\dagger \\ -\lambda \mathbf{1}_8 \end{pmatrix}, \tag{32}$$

where we have defined $\tilde{x}^\dagger = (x^\mu - a^\mu)e^\dagger_\mu$, $\|\tilde{x}\|^2 = \tilde{x}\tilde{x}^\dagger = \tilde{x}^\dagger\tilde{x} = (x^\mu - a^\mu)(x_\mu - a_\mu)$ and $\rho = \lambda^2 + \|\tilde{x}\|^2$. We next examine the constraint (30) for the ADHM ansatz (31). We find that the left-hand side of (30) associated with the ADHM ansatz (31) is proportional to the identity $\mathbf{1}_8$:

$$C^\dagger V V^\dagger C = \frac{\lambda^2}{\rho}\mathbf{1}_8. \tag{33}$$

Therefore, the second ADHM constraint (21) is trivially satisfied. Then the one-instanton solution to the anti-self-duality equation in eight dimensions is found to be

$$A_\mu = -\frac{1}{2}\frac{x^\nu - a^\nu}{\lambda^2 + \|\tilde{x}\|^2}\Sigma^{(-)}_{\mu\nu}. \tag{34}$$

This solution is nothing but the SO(8) instanton found in [14]. This is the eight-dimensional analogue of the Belavin–Polyakov–Schwarz–Tyupkin (BPST) instanton [1] in four dimensions. The associated field strength $F_{\mu\nu}$ is evaluated to be

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = \frac{\lambda^2}{(\lambda^2 + \|\tilde{x}\|^2)^2}\Sigma^{(-)}_{\mu\nu}. \tag{35}$$

Then the ADHM construction in eight dimensions have reproduced the known one-instanton solution. Next we calculate the topological charge and determine the normalization constant $\mathcal{N}$. The field strength for the SO(8) instanton (35) is very simple, so we are able to calculate the charge using (28). The result is

$$Q = \mathcal{N}\int_{\mathbb{R}^8} d^8x \left(\frac{\lambda^2}{(\lambda^2 + \tilde{x}^2)^2}\right)^4 \mathrm{Tr}\left(\Sigma^{(-)}_{[12}\Sigma^{(-)}_{34}\Sigma^{(-)}_{56}\Sigma^{(-)}_{78]}\right) = -\mathcal{N}\frac{16\pi^4}{105}. \tag{36}$$

Therefore the normalization constant $\mathcal{N}$ is determined to be

$$\mathcal{N} = \frac{105}{16\pi^4}. \tag{37}$$

This normalization is the same one employed in [14].

3.2. 't Hooft type solutions

We next study ADHM data for instantons with $k \geq 2$. A natural candidate for this is an eight-dimensional generalization of the 't Hooft type one [3]. The 't Hooft type ADHM ansatz are given by

$$\begin{aligned} T^\mu &= \mathrm{diag}^k_{p=1}\left(-a^\mu_p\right), \\ S &= \mathbf{1}_8 \otimes \begin{pmatrix}\lambda_1 & \lambda_2 & \ldots & \lambda_k\end{pmatrix}, \end{aligned} \tag{38}$$

where $a^\mu_p \in \mathbb{R}$ are position and $\lambda_p \in \mathbb{R}$ are size moduli respectively. The Weyl operator associated with the 't Hooft type ADHM ansatz is

$$\Delta^\dagger = \begin{pmatrix}S^\dagger & e^\dagger_\mu \otimes (x^\mu \mathbf{1}_k + T^\mu)\end{pmatrix}. \tag{39}$$

Then we find

$$\Delta^\dagger\Delta = \mathbf{1}_8 \otimes \begin{pmatrix} \lambda_1^2 + \|\tilde{x}_1\|^2 & \lambda_1\lambda_2 & \ldots & \lambda_1\lambda_k \\ \lambda_2\lambda_1 & \lambda_2^2 + \|\tilde{x}_2\|^2 & \ldots & \lambda_2\lambda_k \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_k\lambda_1 & \lambda_k\lambda_2 & \ldots & \lambda_k^2 + \|\tilde{x}_k\|^2 \end{pmatrix}. \tag{40}$$

Here $\tilde{x}_p$ is defined as $\tilde{x}_p = (x^\mu - a_p^\mu)e_\mu$. Therefore the ADHM ansatz (38) satisfies the first ADHM constraint (19).

The solution to the Weyl equation (10) is given by

$$V = \frac{1}{\sqrt{\phi}} \begin{pmatrix} -\mathbf{1}_8 \\ \left(e_\mu \otimes \mathrm{diag}_{p=1}^k \left(\frac{\tilde{x}_p^\mu}{\|\tilde{x}_p\|^2}\right)\right) S^\dagger \end{pmatrix}, \tag{41}$$

where $\phi = 1 + \sum_{p=1}^k \frac{\lambda_p^2}{\|\tilde{x}_p\|^2}$. We then examine the constraint (30). Plugging the zero-mode (41) into $C^\dagger V V^\dagger C$, we have

$$C^\dagger V V^\dagger C = \left(\delta_{\mu\nu}\mathbf{1}_8 + \Sigma_{\mu\nu}^{(-)}/2\right) \otimes E^{\mu\nu}_{(\text{'t Hooft})}, \tag{42}$$

where

$$E^{\mu\nu}_{(\text{'t Hooft})} = \begin{pmatrix} \lambda_1^2 X_1^\mu X_1^\nu & \lambda_1\lambda_2 X_1^\mu X_2^\mu & \dots & \lambda_1\lambda_k X_1^\mu X_k^\nu \\ \lambda_2\lambda_1 X_2^\mu X_1^\nu & \lambda_2^2 X_2^\mu X_2^\nu & \dots & \lambda_2\lambda_k X_2^\mu X_k^\nu \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_k\lambda_1 X_k^\mu X_1^\nu & \lambda_k\lambda_2 X_k^\mu X_2^\nu & \dots & \lambda_k^2 X_k^\mu X_k^\nu \end{pmatrix}. \tag{43}$$

Here we have used the relation $e_\mu e_\nu^\dagger = \delta_{\mu\nu}\mathbf{1}_8 + \Sigma_{\mu\nu}^{(-)}/2$ and defined $X_m^\mu = \tilde{x}_m^\mu / \|\tilde{x}_m\|^2$. Since the constraint (30) requires that the right-hand side of (42) is proportional to $\mathbf{1}_8$, we have the following conditions on the moduli λ_a and a_m^μ:

$$\lambda_m\lambda_n(x^\mu - a_m^\mu)(x^\nu - a_n^\nu)\Sigma_{\mu\nu}^{(-)} = 0. \tag{44}$$

Here the indices m, n run from 1 to k and not summed. We find that the conditions (44) are satisfied in the well-separated limit of each instanton:

$$\|a_m^\mu - a_n^\mu\|^2 \gg \lambda_m\lambda_n, \tag{45}$$

for all m and n. In the well-separated limit (45), we can neglect all the off-diagonal components in the matrix in (40):

$$\begin{pmatrix} \lambda_1^2 + \|\tilde{x}_1\|^2 & \dots & \lambda_1\lambda_k \\ \vdots & \ddots & \vdots \\ \lambda_k\lambda_1 & \dots & \lambda_k^2 + \|\tilde{x}_k\|^2 \end{pmatrix} \simeq \begin{pmatrix} \lambda_1^2 + \|\tilde{x}_1\|^2 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_k^2 + \|\tilde{x}_k\|^2 \end{pmatrix}. \tag{46}$$

Indeed, in this limit we have

$$C^\dagger \Delta(\Delta^\dagger\Delta)^{-1}\Delta^\dagger C \simeq \mathbf{1}_8 \otimes \mathrm{diag}_{p=1}^k \left[\frac{\|\tilde{x}_p\|^2}{\lambda_p^2 + \|\tilde{x}_p\|^2}\right]. \tag{47}$$

Therefore for the ’t Hooft type ansatz (38), the second ADHM constraint is satisfied in the well-separated limit. Since the number of instanton density becomes dilute in this limit, this is called the dilute instanton gas approximation [18].

We proceed to evaluate the instanton charge for the ’t Hooft type ADHM data. In the well-separated limit (45), by using (29), the charge density for general k instantons is calculated as

$$\mathcal{Q} \simeq \frac{-16}{\phi^4}\mathrm{Tr}\left\{f^{\mu\nu} e_\mu^\dagger e_\nu\right\}^4, \tag{48}$$

where $f^{\mu\nu}$ is given by

$$f^{\mu\nu} = \begin{pmatrix} \frac{\lambda_1 \tilde{x}_1^\mu}{\|\tilde{x}_1\|^2} & \frac{\lambda_2 \tilde{x}_2^\mu}{\|\tilde{x}_2\|^2} & \cdots & \frac{\lambda_k \tilde{x}_k^\mu}{\|\tilde{x}_k\|^2} \end{pmatrix} \begin{pmatrix} \lambda_1^2 + \|\tilde{x}_1\|^2 & \lambda_1\lambda_2 & \cdots & \lambda_1\lambda_k \\ \lambda_1\lambda_2 & \lambda_2^2 + \|\tilde{x}_2\|^2 & \cdots & \lambda_2\lambda_k \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1\lambda_k & \lambda_2\lambda_k & \cdots & \lambda_k^2 + \|\tilde{x}_k\|^2 \end{pmatrix}^{-1} \times \begin{pmatrix} \frac{\lambda_1 \tilde{x}_1^\nu}{\|\tilde{x}_1\|^2} \\ \frac{\lambda_2 \tilde{x}_2^\nu}{\|\tilde{x}_2\|^2} \\ \vdots \\ \frac{\lambda_k \tilde{x}_k^\nu}{\|\tilde{x}_k\|^2} \end{pmatrix}. \tag{49}$$

In order to illustrate multi-instanton solutions, we write down the charge densities for $k = 1, 2, 3$ explicitly.

For $k = 1$, the charge density is given by

$$\mathcal{Q}^{(k=1)}_{\text{'t Hooft}} = -128 \left(\frac{\lambda^2}{\left(\lambda^2 + \|\tilde{x}\|^2\right)^2} \right)^4. \tag{50}$$

This is nothing but the one calculated in (36). We note that the one-instanton solution in the 't Hooft ADHM data is singular at the instanton position:

$$A_\mu^{\text{singular}} = \frac{1}{4} \Sigma^{(+)}_{\mu\nu} \partial_\nu \ln \left(1 + \frac{\lambda^2}{\|\tilde{x}\|^2} \right), \tag{51}$$

while the BPST type solution (34) discussed in the previous subsection is non-singular. These solutions are connected by the following singular gauge transformation:

$$A_\mu^{\text{non-singular}} = g_1 A_\mu^{\text{singular}} g_1^{-1} + g_1 \partial_\mu g_1^{-1}, \qquad g_1 = \frac{\tilde{x}}{\sqrt{\|\tilde{x}\|^2}}. \tag{52}$$

For $k = 2$ and $k = 3$, the charge densities in the dilute gas approximation are evaluated as

$$\mathcal{Q}^{(k=2)}_{\text{'t Hooft}} \simeq -128 \left(\frac{\lambda_1^2 \|\tilde{x}_2\|^4 + \lambda_2^2 \|\tilde{x}_1\|^4 + \lambda_1^2\lambda_2^2 \left(\|\tilde{x}_1\|^2 + \|\tilde{x}_2\|^2 - 2\tilde{x}_1^\mu \tilde{x}_2^\mu \right)}{\left(\lambda_1^2 \|\tilde{x}_2\|^2 + \lambda_2^2 \|\tilde{x}_1\|^2 + \|\tilde{x}_1\|^2 \|\tilde{x}_2\|^2 \right)^2} \right)^4, \tag{53}$$

$$\begin{aligned} \mathcal{Q}^{(k=3)}_{\text{'t Hooft}} \simeq -128 \Bigg[\gamma \Bigg(& \lambda_1^2 \|\tilde{x}_2\|^4 \|\tilde{x}_3\|^4 + \lambda_2^2\lambda_3^2 \|\tilde{x}_1\|^4 \left(\|\tilde{x}_2\|^2 + \|\tilde{x}_3\|^2 - 2\tilde{x}_2^\mu \tilde{x}_3^\mu \right) \\ & + \lambda_2^2 \|\tilde{x}_1\|^4 \|\tilde{x}_3\|^4 + \lambda_1^2\lambda_3^2 \|\tilde{x}_2\|^4 \left(\|\tilde{x}_1\|^2 + \|\tilde{x}_3\|^2 - 2\tilde{x}_1^\mu \tilde{x}_3^\mu \right) \\ & + \lambda_3^2 \|\tilde{x}_1\|^4 \|\tilde{x}_2\|^4 + \lambda_1^2\lambda_2^2 \|\tilde{x}_3\|^4 \left(\|\tilde{x}_1\|^2 + \|\tilde{x}_2\|^2 - 2\tilde{x}_1^\mu \tilde{x}_2^\mu \right) \Bigg) \Bigg]^4. \end{aligned} \tag{54}$$

Here we have defined

$$\gamma = \frac{1}{\left(\lambda_1^2 \|\tilde{x}_2\|^2 \|\tilde{x}_3\|^2 + \lambda_2^2 \|\tilde{x}_1\|^2 \|\tilde{x}_3\|^2 + \lambda_3^2 \|\tilde{x}_1\|^2 \|\tilde{x}_2\|^2 + \|\tilde{x}_1\|^2 \|\tilde{x}_2\|^2 \|\tilde{x}_3\|^2 \right)^2}. \tag{55}$$

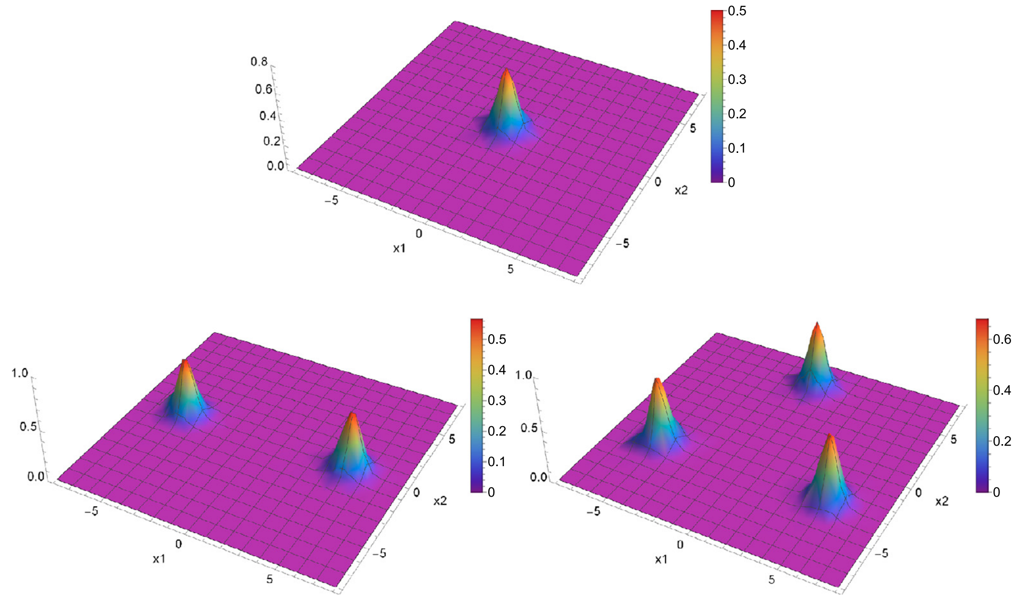

Fig. 1. The charge density plots of the 't Hooft type solutions. The upper figure corresponds to $k = 1$, left and right ones in the lower figure correspond to $k = 2, 3$ respectively. All the plots are projected to a two-dimensional subspace in the eight-dimensional space.

The numerical profiles for the $k = 1, 2, 3$ charge densities are found in Fig. 1. Here the parameters that satisfy the well-separated limit (45) are chosen such that $a^{\mu} = 0$, $\lambda = 2$ for $k = 1$, $a_1^1 = -5$, $a_2^1 = 5$, $a_1^{\mu} = a_2^{\mu} = 0$ $(\mu > 1)$, $\lambda_1 = \lambda_2 = 2$, for $k = 2$ and $a_m^1 = 10/\sqrt{3} \times \sin(2\pi(m-1)/3)$, $a_m^2 = 10/\sqrt{3} \times \cos(2\pi(m-1)/3)$, $a_m^{\mu} = 0$ $(\mu > 2)$, $\lambda_m = 2$ $(m = 1, 2, 3)$ for $k = 3$. For these parameters, the numerical results of instanton charges are evaluated as $Q \simeq 2 \times 1.02$ $(k = 2)$, $Q \simeq 3 \times 1.03$ $(k = 3)$. Therefore we find that the dilute instanton gas approximation, which is needed to solve the second ADHM constraint, works well.

We also observe that the topological charge defined by the fourth Chern number is quantized in the well-separated limit. Indeed, using the property of the basis e_{μ}, the charge density formula for general anti-self-dual instantons (29) is rewritten as

$$\mathcal{Q} = -128 \mathrm{Tr}_k \left(\left(E_k^{(1)} \right)^{-1} \left(\mathbf{1}_k - E_k^{(2)} \right) \right)^4, \tag{56}$$

where ADHM data have been fixed to the canonical form. For the 't Hooft type ADHM data in the dilute gas approximation, we have

$$\mathcal{Q}_{\text{'t Hooft}} \simeq -128 \sum_{p=1}^{k} \left(\frac{\lambda_p^2}{(\lambda_p^2 + \|\tilde{x}_p\|^2)^2} \right)^4. \tag{57}$$

This is just the summation of the one-instanton charge density (50) and gives $Q = k$.

A few comments are in order. First, we find the special solutions to the condition (44). The condition is exactly solved by $a_m = a_n$ $(m \neq n)$ which implies that all the instantons are localized at the same point. However, we find that the corresponding solution is equivalent to the one-instanton (51). On the other hand, another exact solution $\lambda_n = 0$ (for all n) make the solution be trivial.[6] Namely, it is a vacuum configuration.

Second, there is a principal difference between the four- and the eight-dimensional (anti-)self-dual equations. In four dimensions, the 't Hooft type ADHM data provides the *exact* solutions to the (anti-)self-dual equation $F = \pm *_4 F$ [3]. However, in eight dimensions, this provides only the *approximate* solutions. The reason is that the (anti-)self-dual equation is linear in F only in four dimensions. The (anti-)self-dual equations in dimensions greater than four contain multiple F. For example in $4n$ $(n \geq 2)$ dimensions, the equation is given by

$$F^{\wedge n} = \pm *_{4n} F^{\wedge n}, \tag{58}$$

where $F^{\wedge n}$ is the wedge products of n field strengths F. The equations (58) are non-linear in F when $n \geq 2$. The intrinsic origin of the second ADHM constraint (21) comes from this non-linearity of the (anti-)self-dual equations. Therefore the situation in $4n$ $(n \geq 2)$ dimensions is quite different from the four-dimensional case.

3.3. Jackiw–Nohl–Rebbi type solutions

We then study a generalization of the 't Hooft solutions which is so-called Jackiw–Nohl–Rebbi (JNR) type solutions.

The JNR type ansatz [19] is given by

$$\Delta = \begin{pmatrix} \mathbf{1}_8 \otimes \Lambda \\ \mathbf{1}_8 \otimes \mathbf{1}_k \end{pmatrix} \cdot x \otimes \mathbf{1}_k + \begin{pmatrix} -a_0 \otimes \Lambda \\ \mathrm{diag}_{p=1}^k(-a_p) \end{pmatrix} = \begin{pmatrix} \tilde{x}_0 \otimes \Lambda \\ \tilde{X}_{[8k]\times[8k]} \end{pmatrix} = e_\mu \otimes \begin{pmatrix} \tilde{x}_0^\mu \Lambda \\ \mathrm{diag}_{p=1}^k(\tilde{x}_p^\mu) \end{pmatrix}, \tag{59}$$

where $\Lambda = \begin{pmatrix} \lambda_1/\lambda_0 & \ldots & \lambda_k/\lambda_0 \end{pmatrix}$, $\tilde{x}_i = (x^\mu - a_i^\mu)e_\mu$, $a_i = a_i^\mu e_\mu$ and $\tilde{X} = \mathrm{diag}(\tilde{x}_1, \ldots, \tilde{x}_k)$. Here $\lambda_i \in \mathbb{R}$ and $a_i^\mu \in \mathbb{R}$ $(i = 0, \ldots, k)$ are moduli parameters. We note that the JNR ansatz (59) is not in the canonical form and contain more moduli parameters than the 't Hooft one. The latter is obtained from the former by the limit $a_0 \to \infty$, $\lambda_0 \to \infty$ with fixed $a_0/\lambda_0 = 1$.

We can confirm that the JNR ansatz satisfies the first ADHM constraint (19):

$$\Delta^\dagger \Delta = \mathbf{1}_8 \otimes \left(\|\tilde{x}_0\|^2 {}^t\Lambda \Lambda + \mathrm{diag}_{p=1}^k(\|\tilde{x}_p\|^2) \right) = \mathbf{1}_8 \otimes E_k^{(\mathrm{JNR})}, \tag{60}$$

where the symbol ${}^tM$ means the transposed matrix of M, so ${}^t\Lambda$ is k-column vector and ${}^t\Lambda\Lambda$ is $k \times k$ matrix. The solution to the Weyl equation (10) is given by

$$V = \frac{1}{\sqrt{\phi}} \begin{pmatrix} -\mathbf{1}_8 \\ \mathrm{diag}_{p=1}^k \left(\frac{\tilde{x}_p}{\|\tilde{x}_p\|^2} \right) \cdot \tilde{x}_0^\dagger \otimes {}^t\Lambda \end{pmatrix}, \tag{61}$$

where $\phi = 1 + \frac{\|\tilde{x}_0\|^2}{\lambda_0^2} \sum_{p=1}^k \left(\frac{\lambda_p^2}{\|\tilde{x}_p\|^2} \right)$.

Now we examine the second ADHM constraint. The left-hand side of (30) is evaluated to be:

[6] Strictly speaking, the solution becomes singular at the instanton positions $\tilde{x}_m^\mu = 0$ in the limit $\lambda_m \to 0$. We will discuss the physical meaning of this limit in Section 4.

$$C^\dagger V V^\dagger C = \frac{1}{\phi\lambda_0^2}\left(e_\mu e_\nu^\dagger e_\rho e_\sigma^\dagger \otimes E^{\mu\nu\rho\sigma}_{(\mathrm{JNR})}\right),$$

$$E^{\mu\nu\rho\sigma}_{(\mathrm{JNR})} = \begin{pmatrix} \lambda_1^2 Y_1^{\mu\nu} Y_1^{\sigma\rho} & \lambda_1\lambda_2 Y_1^{\mu\nu} Y_2^{\sigma\rho} & \dots & \lambda_1\lambda_k Y_1^{\mu\nu} Y_k^{\sigma\rho} \\ \lambda_2\lambda_1 Y_2^{\mu\nu} Y_1^{\sigma\rho} & \lambda_2^2 Y_2^{\mu\nu} Y_2^{\sigma\rho} & \dots & \lambda_2\lambda_k Y_2^{\mu\nu} Y_k^{\sigma\rho} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_k\lambda_1 Y_k^{\mu\nu} Y_1^{\sigma\rho} & \lambda_k\lambda_2 Y_k^{\mu\nu} Y_2^{\sigma\rho} & \dots & \lambda_k^2 Y_k^{\mu\nu} Y_k^{\sigma\rho} \end{pmatrix}, \tag{62}$$

where $Y_m^{\mu\nu} = \tilde{x}_m^\mu \tilde{x}_0^\nu / \|\tilde{x}_m\|^2 - \delta^{\mu 8}\delta^{\nu 8}$ and $m = 1, \dots, k$ is not summed. In each component in the matrix in (62), we have

$$Y_m^{\mu\nu} Y_n^{\sigma\rho} e_\mu e_\nu^\dagger e_\rho e_\sigma^\dagger = \frac{\|\tilde{x}_0\|^2}{\|\tilde{x}_m\|^2 \|\tilde{x}_n\|^2} \tilde{x}_m \tilde{x}_n^\dagger - \frac{1}{\|\tilde{x}_n\|^2} \tilde{x}_0 \tilde{x}_n^\dagger - \frac{1}{\|\tilde{x}_m\|^2} \tilde{x}_m \tilde{x}_0^\dagger + \mathbf{1}_8. \tag{63}$$

For $k = 1$, since we have the relation $\tilde{x}_a \tilde{x}_b^\dagger + \tilde{x}_b \tilde{x}_a^\dagger = 2\tilde{x}_a^\mu \tilde{x}_b^\mu \mathbf{1}_8$, the right-hand side of (63) is proportional to $\mathbf{1}_8$ and the second ADHM constraint is satisfied. The charge density of the $k = 1$ JNR solution is given by

$$\mathcal{Q}^{(k=1)}_{\mathrm{JNR}} = -128 \left(\frac{\bar{\lambda}_1^2 \left(\|\tilde{x}_1\|^2 + \|\tilde{x}_0\|^2 - 2\tilde{x}_0^\mu \tilde{x}_1^\mu\right)}{\left(\|\tilde{x}_0\|^2 \bar{\lambda}_1^2 + \|\tilde{x}_1\|^2\right)^2} \right)^4, \tag{64}$$

where $\bar{\lambda}_m = \lambda_m/\lambda_0$. The moduli parameters are $\lambda_1/\lambda_0 = \lambda$ and $a_1^\mu - a_0^\mu = a^\mu$, so the $k = 1$ JNR solution has total nine parameters. Therefore the $k = 1$ JNR data is essentially equal to the $k = 1$ 't Hooft data, and we find that the numerical results of the $k = 1$ instanton charge (64) is $Q = 1$.

For $k \geq 2$ case, it is not straightforward to solve the constraint (30) in a general fashion. However, a solution is found in the well-separated limit (45). In this limit, we can neglect all the off-diagonal components in $E_k^{(\mathrm{JNR})}$:

$$E_k^{(\mathrm{JNR})} \simeq \mathrm{diag}_{p=1}^k \left(\|\tilde{x}_0\|^2 \bar{\lambda}_p^2 + \|\tilde{x}_p\|^2\right). \tag{65}$$

Then, the second ADHM constraint is satisfied:

$$C^\dagger \Delta (\Delta^\dagger \Delta)^{-1} \Delta^\dagger C \simeq \mathbf{1}_8 \otimes \mathrm{diag}_{p=1}^k \left(\frac{\|\tilde{x}_0\|^2 \bar{\lambda}_p^4 + 2\bar{\lambda}_p^2 \tilde{x}_0^\mu \tilde{x}_p^\mu + \|\tilde{x}_p\|^2}{\|\tilde{x}_0\|^2 \bar{\lambda}_p^2 + \|\tilde{x}_p\|^2} \right). \tag{66}$$

We also observe that the instanton charge is quantized in this limit by using the same formula of the 't Hooft ones. We note that the JNR data is not in the canonical form. In this case, the charge density formula (29) is rewritten as

$$\mathcal{Q} = -128 \mathrm{Tr}_k \left(\left(E_k^{(1)}\right)^{-1} (C^{(2)} - E_k^{(2)}) \right)^4, \tag{67}$$

where $C^{(2)}$ is defined by $C^\dagger C = \mathbf{1}_8 \otimes C^{(2)}$. In the limit (45), we have $C^\dagger C \simeq \mathbf{1}_8 \otimes (\mathrm{diag}_{p=1}^k \bar{\lambda}_p^2 + \mathbf{1}_k)$. Therefore we obtain

$$\mathcal{Q}_{\mathrm{JNR}} \simeq -128 \sum_{p=1}^k \left(\frac{\bar{\lambda}_p^2 \left(\|\tilde{x}_p\|^2 + \|\tilde{x}_0\|^2 - 2\tilde{x}_0^\mu \tilde{x}_p^\mu\right)}{\left(\|\tilde{x}_0\|^2 \bar{\lambda}_p^2 + \|\tilde{x}_p\|^2\right)^2} \right)^4. \tag{68}$$

This is just the summation of the JNR type one-instanton charge density and the charge associated with (68) is $Q = k$.

We note that these three type ADHM data (BPST type, 't Hooft type and JNR type) take real values. Therefore we can choose the gauge group G by using the Clifford algebras: $C\ell_7(\mathbb{C})$ or $C\ell_7(\mathbb{R})$. If we choose the complex Clifford algebra $C\ell_7(\mathbb{C})$ then the gauge group is the unitary group $G = \mathrm{U}(8)$. On the other hand, we choose the real Clifford algebra $C\ell_7(\mathbb{R})$ for the orthogonal group $G = \mathrm{O}(8)$. We find explicit form of the complex (real) basis in Appendix B.

4. Higher derivative field theories in eight dimensions

In this section we discuss eight-dimensional gauge field theories where the (anti-)self-dual instantons are relevant. Since the (anti-)self-dual equations in dimensions greater than four contain multi-field strengths, the theories inevitably contain higher derivative terms. In the following, we consider a gauge field A_μ and a non-Abelian gauge group G whose Lie algebra is $\mathcal{G}$ in eight-dimensional Euclid space. The generators of the gauge group T^a $(a = 1, \ldots, \dim \mathcal{G})$ are normalized as $\mathrm{Tr} T^a T^b = \kappa \delta^{ab}$ where κ is a constant. We also introduce the gauge coupling constant g whose mass dimension is -2 in eight dimensions. The constant α' is the string Regge slope parameter.

Quartic Yang–Mills model The first example is the so called quartic Yang–Mills model whose Lagrangian is given by the 4th products of the gauge field strengths F and no Yang–Mills kinetic term. The action of the quartic Yang–Mills model is given by

$$S = \frac{\alpha}{\kappa g^2} \int \mathrm{Tr} \left[\frac{1}{2} *_8 (F \wedge F) \wedge (F \wedge F) \right], \tag{69}$$

where α is a constant whose mass dimension is $[\alpha] = -4$. The action (69) is classically conformal and the Derrick's theorem implies that the theory admits stable static solitons. It is straightforward to show that the Bogomol'nyi completion of the action is

$$S = \frac{\alpha}{\kappa g^2} \int \mathrm{Tr} \left[(F \wedge F \pm *_8(F \wedge F))^2 \mp F \wedge F \wedge F \wedge F \right], \tag{70}$$

where we have defined

$$(F \wedge F \pm *_8 F \wedge F)^2 = (F \wedge F \pm *_8 F \wedge F) \wedge *_8 (F \wedge F \pm *_8 F \wedge F). \tag{71}$$

Then the action is bounded from below by the fourth Chern number $S \geq \pm \frac{\alpha}{\kappa g^2} \int \mathrm{Tr}[F \wedge F \wedge F \wedge F]$. The Bogomol'nyi bound is saturated when (1) is satisfied. It is easy to show that the (anti-)self-dual solution satisfies the full equation of motion for the quartic model (69).

This kind of quartic Yang–Mills theory is not the standard gauge field theory but appears in some physically appropriate situations. For example, the quartic model has been introduced to provide the solution for the fundamental strings in SO(32) heterotic string theory [20,21]. Tree-level heterotic five-brane is expected to induce a quartic Yang–Mills theory whose (anti-)self-dual instantons in eight dimensions precisely reproduce the energy–momentum tensor for fundamental strings. On the other hand, the quadratic Yang–Mills part is expected to appear at the one-loop level in perturbative heterotic five-brane theory [22]. This is in contrast to the heterotic fundamental string theory where the quartic Yang–Mills term appears in the one-loop level.

There are other applications of the quartic model. For example, various topological solitons specific for the quartic model have been studied in [23–25].

D7-brane effective action and D-instantons We next consider more physically relevant models. Higher dimensional gauge theories are naturally realized as low-energy effective field theories on D-branes. The $(p+1)$-dimensional quadratic Yang–Mills theory appears in the zero-slope limit $\alpha' \to 0$ of the open string sector on Dp-branes. The four-dimensional instantons are interpreted as D-instantons (or D(-1)-branes) embedded in D3-branes [26]. The ADHM moduli are interpreted as the zero-dimensional fields on the D-instanton world-volume. The ADHM constraint comes from the supersymmetric D-term condition of the D3–D(-1), D(-1)–D(-1) open string sectors [27]. This interpretation is generalized to Dp–D$(p-4)$ brane systems.

For the eight-dimensional gauge theory, we consider Euclidean D7-branes in type IIB string theory. In order to see the (anti-)self-dual instanton effects, we consider the α' corrections to the eight-dimensional Yang–Mills theory. This is obtained from the dimensional reduction of the α' corrected super Yang–Mills theory in ten-dimensions [28]. The gauge field part of the D7-brane world-volume Lagrangian is given by

$$\mathcal{L}_F = \frac{1}{\kappa g^2}\mathrm{Tr}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right] + \frac{(2\pi\alpha')^2}{12\kappa g^2}t_8\mathrm{Tr}F^4 + \mathcal{O}\left(\frac{\alpha'^4}{g^2}\right). \tag{72}$$

Here the first term is the eight-dimensional quadratic Yang–Mills part. The second part is the first α' correction given by

$$\begin{aligned} t_8\mathrm{Tr}F^4 = -4\mathrm{Tr}\Bigg[&\frac{1}{32}F_{\mu\nu}F_{\rho\sigma}F^{\mu\nu}F^{\rho\sigma} + \frac{1}{16}(F_{\mu\nu}F^{\mu\nu})^2 \\ &- \frac{1}{8}F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu} - \frac{1}{4}F^{\mu\rho}F_{\rho\nu}F_{\mu\sigma}F^{\sigma\nu}\Bigg]. \end{aligned} \tag{73}$$

We now interpret the eight-dimensional instantons as the D-instantons embedded in the D7-branes. The D-instantons are the sources of the R–R 0-form $C^{(0)}$. The $C^{(0)}$ coupling to the D7-branes is given by the Wess–Zumino term of the effective action:

$$\begin{aligned} \mathcal{L}_{\mathrm{WZ}} &= \frac{\mu_7}{\kappa g^2}\mathrm{Tr}\left[\frac{(2\pi\alpha')^2}{4!}C^{(0)}F\wedge F\wedge F\wedge F\right] \\ &= \frac{\mu_7}{\kappa g^2}\mathrm{Tr}\left[\frac{(2\pi\alpha')^2}{4!\cdot 2^4}C^{(0)}\varepsilon^{\mu_1\cdots\mu_8}F_{\mu_1\mu_2}\cdots F_{\mu_7\mu_8}\right]d^8x. \end{aligned} \tag{74}$$

Here μ_7 is the R–R charge of the D7-brane. In order that the eight-dimensional instantons $F\wedge F = \pm *_8 F\wedge F$ whose instanton number $k = \frac{1}{4!(2\pi)^4}\mathrm{Tr}\int F\wedge F\wedge F\wedge F$ become the source of the R–R 0-form, the quartic term in $\mathcal{L}_F$ evaluated on the instantons should coincide with $\mathcal{L}_{\mathrm{WZ}}$. We also need the condition that the quadratic term in $\mathcal{L}_F$ on the instantons vanish. Therefore the eight-dimensional (anti-)self-dual instantons become the D-instantons when the following conditions are satisfied:

$$\frac{1}{g^2}\int d^8x\ \mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}] = 0, \qquad \frac{1}{12}t_8F^4 = \pm\frac{1}{4!\cdot 2^4}\varepsilon^{\mu_1\cdots\mu_8}F_{\mu_1\mu_2}\cdots F_{\mu_7\mu_8}, \tag{75}$$

and all the $\mathcal{O}(\alpha'^4/g^2)$ terms vanish. We call (75) the D-instanton conditions. When the D-instanton conditions holds on the instanton solution, then, the quartic term in $\mathcal{L}_F$ agrees with the effective action of k D-instantons:

$$S_{D(-1)} = \mu_{-1}C^{(0)}\frac{1}{\kappa}\mathrm{Tr}\mathbf{1}_k = \frac{k\mu_{-1}}{\kappa}C^{(0)}. \tag{76}$$

Here μ_{-1} is the R–R charge of the D-instanton and we have used the relation $\frac{\lambda^2}{g_7^2} = \frac{1}{(2\pi)^3 g_s}$, $\mu_{-1} = \frac{2\pi}{g_s}$ and g_s is the string coupling.

We consider the zero-slope limit $\alpha' \to 0$ with fixed α'^2/g^2 to obtain the effective action (72). In this limit, the $\mathcal{O}(\alpha'^4/g^2)$ terms vanish and the F quartic term remains finite while the Yang–Mills part diverges in general. The situation where the conditions (75) are satisfied has been analyzed in [30] where the instanton partition function for the D7/D(−1) system is studied. In there, it is shown that the one-instanton solution [14,15] satisfies the D-instanton condition (75) when the size modulus becomes zero.

We find that our general solution (13) actually satisfies the second condition in (75). This is due to the property of the basis e_μ defined by the Clifford algebra. For the first condition in (75), we can evaluate the quadratic Yang–Mills term for the 't Hooft type instantons in the well-separated limit (45). The Yang–Mills quadratic term becomes

$$\frac{1}{g^2}\int d^8x\ \mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}] = \frac{1}{g^2}\mathrm{Tr}[\Sigma^{(-)}_{\mu\nu}\Sigma^{(-)\mu\nu}]\sum_{p=1}^{k}\int d^8x \left(\frac{\lambda_p^2}{(\lambda_p^2+\|\tilde{x}_p\|^2)^2}\right)^2 . \tag{77}$$

This is just the k times the one-instanton contribution. The radial part of the space–time integral in (77) has been calculated to be [29,30]

$$\int_0^\Lambda r^7 dr \left(\frac{\lambda_p^2}{(\lambda_p^2+r^2)^2}\right)^2 = \frac{\lambda_p^2}{2}\log\left(1+\frac{\Lambda^2}{\lambda_p^2}\right) - \frac{11}{12}\lambda_p^2, \tag{78}$$

where we have introduced the cutoff Λ in the space–time integral and neglected sub-leading terms of $1/\Lambda$. Then, when the instantons shrink to zero-size $\lambda_p \to 0$, the first condition in (75) is satisfied.[7] This result is a multi-instanton generalization of the one-instanton calculations in [29,30]. Therefore we conclude that the (anti-)self-dual instantons in the small instanton limit correspond to the D-instantons embedded in the D7-branes. We emphasize that the small instanton $\lambda_p \to 0$ is the strict limit of the dilute instanton gas approximation (45). In this limit, all the instantons show singular behavior and they satisfy the equation of motion. Note that the fourth Chern number is kept finite in this limit. The string origin of the zero-size limit of instantons is also discussed in [31].

A few comments are in order. First, the SO(8) instantons in eight dimensions are studied in the context of heterotic/type I string duality [32]. Consider the (Euclidean) D7-branes in type IIB orientifold theory compactified on two torus T^2. The D7-branes are placed on top of the O7-planes. There are four SO(8) sectors in the theory. Let us concentrate on the one sector among them. The world-volume theory of the D7-branes is given by the eight-dimensional $\mathcal{N}=2$ super Yang–Mills theory with the gauge group SO(8). The self-dual instantons give non-perturbative effects in eight-dimensional gauge theories [33,34]. This is a non-perturbative test of the string duality.

Second, the famous anomaly cancellation term $B\mathrm{Tr}F^4$, where B is the NS–NS B-field, in heterotic string theory indicates that configurations with the finite fourth Chern number become sources of fundamental strings [35]. This configuration is nothing but the (anti-)self-dual instanton in eight dimensions.

[7] This zero-size limit should be taken so that $\frac{\lambda_p^2}{g^2}\log(\frac{\Lambda_p^2}{\lambda_p^2}) \to 0$ in the limit $\alpha' \to 0$ with fixed α'^2/g^2.

5. Conclusion and discussions

In this paper we have studied ADHM construction of (anti-)self-dual instantons in eight dimensions. The instantons satisfy the (anti-)self-dual equations $F \wedge F = \pm *_8 F \wedge F$. The gauge field is given by the pure gauge form (13) which is a natural generalization of the four-dimensional (anti-)self-dual instantons. The ADHM construction is based on the basis e_μ which is constructed from the Clifford algebra in seven dimensions. Due to the property of the basis e_μ, the eight-dimensional anti-self-dual equation reduces to a set of algebraic constraints on matrices (the ADHM data). Compared with the (anti-)self-dual equation in four dimensions, the equation in eight dimensions is non-linear in F and contains terms with space–time derivative of second order. We have found that there are the first and the second ADHM constraints on the ADHM data. The former is the same form of the four-dimensional one while the latter comes from the non-linearity of the equation and essentially a new ingredient. We have also pointed out that the gauge group of the theory is determined by the structure of the seven-dimensional Clifford algebra.

We have shown that our construction precisely reproduces the known one-instanton profile, namely, the SO(8) instanton [14,15]. We have also found the $k = 2, 3$ multi-instanton solutions based on the 't Hooft and JNR ansatz. The JNR type solution contain more moduli parameters compared with the 't Hooft type. We have shown that the first and the second ADHM constraints are explicitly solved in the dilute instanton gas approximation. The topological charges are evaluated numerically and we have shown that the consistent results are found in a good accuracy. It is obvious that any higher charge solutions can be systematically constructed. Although they are approximate solutions, as far as we know, they are the first explicit examples of higher charge solutions that do not show spherical symmetry in eight dimensions.

We have discussed the eight-dimensional gauge theories where the (anti-)self-dual equation is relevant. The instanton configurations extremize the action of the quadratic field strength. Therefore the theory inevitably contain higher derivative terms. As in the four-dimensional case, the eight-dimensional ADHM construction enjoys the space–time gauge symmetry and the gauge symmetry in the instanton space (dual space). This fact strongly suggests that the ADHM construction presented in this paper has string theory origin in D-brane configurations [27,36]. Indeed, in [29,30], the authors studied D7/D(-1)-brane configurations in type IIB orientifold. The open string scattering amplitudes including zero-modes associated with strings that end on these branes reveal that the moduli action for eight-dimensional $k = 1$ self-dual instanton is given by the D-instanton effective action. We have exhibited a strong evidence that this is true even for the multi-instantons in the small instanton limit. In this limit, the 't Hooft type multi-instantons become exact solutions of the (anti-)self-dual equation. They also satisfy the D-instanton conditions in this limit and identified with the D-instantons embedded in the D7-brane world-volume.

In four dimensions, the ADHM construction of instantons in noncommutative space has been studied where the ADHM constraint is modified by the noncommutativity parameter. It is interesting to study the ADHM construction of instantons in noncommutative space–time in eight dimensions. It is also interesting to study monopoles in seven dimensions [37] and its Nahm construction. Using the ADHM data we can also construct calorons in seven dimensions. In the high temperature limit, we expect that the seven-dimensional monopoles are realized.

It is interesting to study (anti-)self-dual equations in dimensions greater than eight. In $4n$ dimensions, we can consider the (anti-)self-dual equation $F \wedge F \wedge \cdots \wedge F = \pm *_{4n} F \wedge F \wedge \cdots \wedge F$, where $F \wedge F \wedge \cdots \wedge F$ in both sides are $2n$-forms. Solutions to the (anti-)self-dual equation are expected to have finite $2n$-th Chern number $k = \frac{1}{n!(2\pi)^n} \int \mathrm{Tr}[F \wedge F \wedge \cdots \wedge F]$, for appropriate basis e_μ. We find that the ADHM construction of instantons in eight dimensions presented in this paper is generalized to $4n$ dimensions. Supersymmetric generalization including higher derivative interactions [38] is also important to study the relation to string theories. We will come back to these issues in future studies.

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Appendix A. ADHM construction with U(2) gauge in four dimensions

In this section, we give a brief review on the ADHM construction of instantons in four dimensions. We consider the gauge group U(2).

The four-dimensional Weyl operator $\Delta_{(4)}$ is defined by

$$\Delta_{(4)} = C(x \otimes \mathbf{1}_k) + D, \tag{79}$$

where C and D are quaternionic $(k+1) \times k$ matrices, k is the instanton charge and $x = x^\mu e_\mu$. Here x^μ $(\mu = 1, \ldots, 4)$ is the Cartesian coordinate of the four-dimensional Euclid space, $e_\mu = (-i\sigma_i, \mathbf{1}_2)$ is the basis of the quaternion and σ_i are the Pauli matrices. The Weyl operator $\Delta_{(4)}$ is assumed to satisfy the ADHM constraint:

$$\Delta^\dagger_{(4)} \Delta_{(4)} = \mathbf{1}_2 \otimes E_k, \tag{80}$$

where $\Delta^\dagger_{(4)}$ is the quaternionic conjugate of $\Delta_{(4)}$ and E_k is an invertible $k \times k$ matrix.

In order to construct the instanton solution for the gauge field $A_\mu(x)$, it is necessary to find a quaternionic $(k+1)$ column vector $V(x)$ obeying the Weyl equation:

$$\Delta^\dagger_{(4)} V(x) = 0, \tag{81}$$

where $V(x)$ is the zero-mode normalized as $V^\dagger(x)V(x) = \mathbf{1}_2$. The gauge field $A_\mu(x)$ of instantons is given by

$$A_\mu(x) = V^\dagger(x)\partial_\mu V(x). \tag{82}$$

Using the expression (82), the field strength is calculated as

$$F_{\mu\nu} = \partial_\mu V^\dagger \left(\mathbf{1}_{2+2k} - VV^\dagger\right) \partial_\nu V - (\mu \leftrightarrow \nu). \tag{83}$$

Here we use the completeness relation:

$$\mathbf{1}_{2+2k} - VV^\dagger = \Delta_{(4)}(\Delta^\dagger_{(4)}\Delta_{(4)})^{-1}\Delta^\dagger_{(4)}. \tag{84}$$

Then (83) is rewritten as

$$\begin{aligned} F_{\mu\nu} &= V^\dagger C(e_\mu \otimes \mathbf{1}_k)(\Delta^\dagger_{(4)}\Delta_{(4)})^{-1}(e^\dagger_\nu \otimes \mathbf{1}_k)C^\dagger V - (\mu \leftrightarrow \nu) \\ &= V^\dagger C(\Delta^\dagger_{(4)}\Delta_{(4)})^{-1}\left(\eta^{(-)}_{\mu\nu} \otimes \mathbf{1}_k\right) C^\dagger V, \end{aligned} \tag{85}$$

where we have used the ADHM constraint (80). Here $\eta^{(\pm)}_{\mu\nu}$ is the 't Hooft symbol defined by

$$\eta^{(+)}_{\mu\nu} = e^\dagger_\mu e_\nu - e^\dagger_\nu e_\mu, \quad \eta^{(-)}_{\mu\nu} = e_\mu e^\dagger_\nu - e_\nu e^\dagger_\mu. \tag{86}$$

The 't Hooft symbol satisfies the four-dimensional (anti-)self-dual relation:

$$\eta^{(\pm)}_{\mu\nu} = \pm\frac{1}{2!}\varepsilon_{\mu\nu\rho\sigma}\eta^{(\pm)}_{\rho\sigma}. \tag{87}$$

Therefore the field strength $F_{\mu\nu}$ associated with the solution (82) automatically satisfies the (anti-)self-dual equation $F = \pm *_4 F$.

From the above discussion, we find that a key point of the ADHM construction is that the 't Hooft symbol $\eta^{(\pm)}_{\mu\nu}$ constructed from the basis e_μ satisfies the (anti-)self-dual relation. Therefore if we formulate the ADHM construction of instantons in higher dimensions then we need to find the basis that satisfies the (anti-)self-dual relation in higher dimensions.

Appendix B. Clifford algebra and 4*n*-dimensional (anti-)self-dual tensor

In this section, we construct the $4n$-dimensional generalization of the 't Hooft symbol which satisfies the (anti-)self-dual relation. We first introduce ($m = 4n - 1$)-dimensional Clifford algebra $C\ell_m(K)$ on the (number) field K. Elements of the Clifford algebra $\Gamma_i \in C\ell_m(K)$ satisfy the relation:

$$\Gamma_i\Gamma_j + \Gamma_j\Gamma_i = -2\delta_{ij}, \tag{88}$$

where the indices i, j run from 1 to $4n - 1$. For $K = \mathbb{R}$, $C\ell_m(\mathbb{R})$ is called "the real Clifford algebra". On the other hand, for $K = \mathbb{C}$, $C\ell_m(\mathbb{C})$ is called "the complex Clifford algebra". In $4n - 1$ dimensions, the chirality element ω is defined by

$$\omega = (-1)^{\lfloor (m+5)/4\rfloor}\Gamma_1\Gamma_2\ldots\Gamma_m, \quad \Gamma_i \in C\ell_m(\mathbb{R}), \tag{89a}$$

$$\omega = i^{\lfloor (m+5)/2\rfloor}\Gamma_1\Gamma_2\ldots\Gamma_m, \quad \Gamma_i \in C\ell_m(\mathbb{C}), \tag{89b}$$

where the symbol $\lfloor x\rfloor$ is the floor function (for example: $\lfloor 2.8\rfloor = 2$, $\lfloor 3\rfloor = 3$). Here we define the overall factor of the chirality element ω for later convenience. It is well known that we can decompose the $(4n - 1)$-dimensional Clifford algebra by using the chirality element. The projection operator is defined by

$$P_\pm = \frac{1}{2}(1 \pm \omega). \tag{90}$$

Using $P_\pm$, we can decompose the Clifford algebra as

$$C\ell_m(K) = C\ell^{(+)}_m(K) \oplus C\ell^{(-)}_m(K), \tag{91}$$

where $C\ell^{(\pm)}_m(K)$ are defined by elements in $C\ell_m(K)$ projected by $P_\pm$. We call $C\ell^{(\pm)}_m(K)$ "the decomposed Clifford algebra". Now we choose the elements of the decomposed Clifford algebra $\Gamma^{(\pm)}_i \in C\ell^{(\pm)}_m(K)$ that satisfy the relation $\Gamma^{(+)}_i = -\Gamma^{(-)}_i$.

Note that the elements of the decomposed Clifford algebra $\Gamma^{(\pm)}_i \in C\ell^{(\pm)}_m(K)$ satisfy the relation $\{\Gamma^{(\pm)}_i, \Gamma^{(\pm)}_j\} = -2\delta_{ij}$, but $\Gamma^{(\pm)}_i$ are not elements of the Clifford algebra. Because the elements of the decomposed Clifford algebra are not the algebraic generators. The algebraic generators have the property that each element of the algebra is not produced by a product of other elements, that is $e_ie_j\cdots \neq e_t$ where $e_i, e_j, \ldots, e_t \in Q(K)$ and $Q(K)$ is an algebra on the field K. The elements of the Clifford algebra Γ_i are algebraic generators, therefore Γ_i satisfies the relation $\Gamma_i\Gamma_j\cdots \neq \Gamma_t$, where $\Gamma_i, \Gamma_j, \ldots, \Gamma_t \in C\ell_m(K)$. On the other hand, the element of

Table 1
The matrix rings $\mathrm{GL}(N;K)$ which are isomorphic to the $(4n-1)$-dimensional complex (real) Clifford algebra $C\ell_{4n-1}(\mathbb{C}(\mathbb{R}))$. Here N is the matrix size and the symbol $\mathbb{H}$ means the quaternion.

$4n$-dim. mod 8	$C\ell_{4n-1}(\mathbb{C})$	$C\ell_{4n-1}(\mathbb{R})$
4	$\mathrm{GL}(2^{2n-1};\mathbb{C})\oplus\mathrm{GL}(2^{2n-1};\mathbb{C})$	$\mathrm{GL}(2^{2n-2};\mathbb{H})\oplus\mathrm{GL}(2^{2n-2};\mathbb{H})$
8	$\mathrm{GL}(2^{2n-1};\mathbb{C})\oplus\mathrm{GL}(2^{2n-1};\mathbb{C})$	$\mathrm{GL}(2^{2n-1};\mathbb{R})\oplus\mathrm{GL}(2^{2n-1};\mathbb{R})$

the decomposed Clifford algebra $\Gamma_i^{(\pm)}$ does not satisfy the relation $\Gamma_i^{(\pm)}\Gamma_j^{(\pm)}\cdots\neq\Gamma_t^{(\pm)}$, where $\Gamma_i^{(\pm)},\Gamma_j^{(\pm)},\ldots,\Gamma_t^{(\pm)}\in C\ell_m^{(\pm)}(K)$.

We can construct the $4n$-dimensional (anti-)self-dual tensor $\Sigma_{\mu\nu}^{(\pm)}$ form the $(4n-1)$-dimensional Clifford algebra $C\ell_{4n-1}(K)$. Here the $4n$-dimensional "(anti-)self-dual tensor" means that the tensor satisfies the (anti-)self-dual relation in $4n$ dimensions. We define the $4n$-dimensional basis e_μ by

$$e_\mu=\delta_{\mu 4n}1+\delta_{\mu i}\Gamma_i^{(-)},\quad e_\mu^\dagger=\delta_{\mu 4n}1+\delta_{\mu i}\Gamma_i^{(+)}, \tag{92}$$

where 1 is an identity element (such that $1\Gamma_i^{(\pm)}=\Gamma_i^{(\pm)}1$) and the indices $\mu,\nu,\ldots$ run from 1 to $4n$. Using this basis, we define the $4n$-dimensional (anti-)self-dual tensor by

$$\Sigma_{\mu\nu}^{(+)}=e_\mu^\dagger e_\nu-e_\nu^\dagger e_\mu,\quad \Sigma_{\mu\nu}^{(-)}=e_\mu e_\nu^\dagger-e_\nu e_\mu^\dagger. \tag{93}$$

We can confirm that $\Sigma_{\mu\nu}^{(\pm)}$ satisfies the $4n$-dimensional (anti-)self-dual relation:

$$\Sigma_{[a_1a_2}^{(\pm)}\ldots\Sigma_{a_{2n-1}a_{2n}]}^{(\pm)}=\pm\frac{1}{2n!}\varepsilon_{a_1a_2\ldots a_{2n}b_1b_2\ldots b_{2n}}\Sigma_{b_1b_2}^{(\pm)}\ldots\Sigma_{b_{2n-1}b_{2n}}^{(\pm)} \tag{94}$$

where $\Sigma_{\mu\nu}^{(+)}$ satisfies the self-dual equation and $\Sigma_{\mu\nu}^{(-)}$ satisfies the anti-self-dual equation respectively.

In a $4n$-dimensional ADHM construction, we have to represent the $(4n-1)$-dimensional Clifford algebra $C\ell_{4n-1}(K)$ by matrices. It is well known that the complex (real) Clifford algebra has an isomorphism with a matrix ring. Furthermore the complex (real) Clifford algebra has the period with two (eight) from the Bott periodicity theorem [39]. Therefore we can naturally obtain the matrix representations of the complex (real) Clifford algebra (Table 1).

Note that the gauge group of the ADHM construction based on the (anti-)self-dual tensor (93) is determined by the (number) field of the Clifford algebra. Therefore the size of the gauge group (color size) N is dependent on a matrix size of the matrix representation of the Clifford algebra $\mathrm{GL}(N;K)$.

Now we have obtained the $4n$-dimensional (anti-)self-dual tensor. We construct the four- and eight-dimensional (anti-)self-dual basis explicitly. Note that the representation of the basis is not unique. We use the tensor product of the following 2×2 matrices. The complex Clifford algebra $C\ell_m(\mathbb{C})$ is constructed by the Pauli matrices:

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad \sigma_0=\mathbf{1}_2. \tag{95}$$

On the other hand, the real Clifford algebras $C\ell_m(\mathbb{R})$ are constructed by the following matrices [40]:

$$\tau_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \tau_2=\begin{pmatrix}0&-1\\1&0\end{pmatrix},\quad \tau_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad \tau_0=\mathbf{1}_2. \tag{96}$$

For simplicity, we omit the tensor (Kronecker) product symbol $\otimes$ in the following discussions. For example, σ_{ij} means $\sigma_i \otimes \sigma_j$.

The complex basis in four dimensions We construct the four-dimensional (anti-)self-dual tensor from the three-dimensional Clifford algebra. The matrix representation of the three-dimensional complex Clifford algebra $C\ell_3(\mathbb{C})$ is given by

$$\Gamma_1 = \begin{pmatrix} i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}, \quad \Gamma_3 = \begin{pmatrix} i\sigma_3 & 0 \\ 0 & -i\sigma_3 \end{pmatrix}. \tag{97}$$

The chiral matrix ω and the projection operators $P_\pm$ are

$$\omega = \Gamma_1\Gamma_2\Gamma_3 = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix}, \qquad P_+ = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_- = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1}_2 \end{pmatrix}. \tag{98}$$

Using these matrices, we obtain

$$\Gamma_i^{(\pm)} = \pm i\sigma_i, \tag{99}$$

where $i = 1, 2, 3$. Therefore we obtain the four-dimensional (anti-)self-dual complex basis:

$$e_\mu = \delta_{\mu 4}\mathbf{1}_2 - i\delta_{\mu i}\sigma_i, \quad e_\mu^\dagger = \delta_{\mu 4}\mathbf{1}_2 + i\delta_{\mu i}\sigma_i. \tag{100}$$

This basis is nothing but the quaternion basis which is used in the four-dimensional ADHM construction. In the previous discussion in subsection 2.3, the gauge group is U(2) for this basis.

The real basis in four dimensions For Table 1, the three-dimensional real Clifford algebra $C\ell_3(\mathbb{R})$ is isomorphic to $\mathbb{H} \oplus \mathbb{H}$. However we use real matrix representation to implement gauge group O(4). The real matrix representation of $C\ell_3(\mathbb{R})$ is given by

$$\Gamma_1 = \begin{pmatrix} \tau_{12} & 0 \\ 0 & -\tau_{12} \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} \tau_{20} & 0 \\ 0 & -\tau_{20} \end{pmatrix}, \quad \Gamma_3 = \begin{pmatrix} \tau_{32} & 0 \\ 0 & -\tau_{32} \end{pmatrix}. \tag{101}$$

The chiral matrix ω and the projection operators $P_\pm$ are

$$\omega = \Gamma_1\Gamma_2\Gamma_3 = \begin{pmatrix} -\mathbf{1}_4 & 0 \\ 0 & \mathbf{1}_4 \end{pmatrix}, \qquad P_+ = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1}_4 \end{pmatrix}, \quad P_- = \begin{pmatrix} \mathbf{1}_4 & 0 \\ 0 & 0 \end{pmatrix}. \tag{102}$$

Therefore $\Gamma_i^{(\pm)}$ are

$$\Gamma_1^{(\pm)} = \mp\tau_{12}, \quad \Gamma_2^{(\pm)} = \mp\tau_{20}, \quad \Gamma_3^{(\pm)} = \mp\tau_{32}, \tag{103}$$

and we obtain the four-dimensional (anti-)self-dual tensor by using (92) and (93). If this real basis is used in the four-dimensional ADHM construction, the gauge group becomes $G = \mathrm{O}(4)$.

The complex basis in eight dimensions The matrix representation of $C\ell_7(\mathbb{C})$ is given by

$$\begin{aligned}
&\Gamma_1 = \begin{pmatrix} i\sigma_{133} & 0 \\ 0 & -i\sigma_{133} \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} i\sigma_{233} & 0 \\ 0 & -i\sigma_{233} \end{pmatrix}, \quad \Gamma_3 = \begin{pmatrix} i\sigma_{013} & 0 \\ 0 & -i\sigma_{013} \end{pmatrix}, \\
&\Gamma_4 = \begin{pmatrix} i\sigma_{023} & 0 \\ 0 & -i\sigma_{023} \end{pmatrix}, \quad \Gamma_5 = \begin{pmatrix} i\sigma_{001} & 0 \\ 0 & -i\sigma_{001} \end{pmatrix}, \quad \Gamma_6 = \begin{pmatrix} i\sigma_{002} & 0 \\ 0 & -i\sigma_{002} \end{pmatrix}, \\
&\Gamma_7 = \begin{pmatrix} i\sigma_{333} & 0 \\ 0 & -i\sigma_{333} \end{pmatrix}.
\end{aligned} \tag{104}$$

Using (89b), the chiral matrix ω is given by

$$\omega = (-1)\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5\Gamma_6\Gamma_7 = \begin{pmatrix} \mathbf{1}_8 & 0 \\ 0 & -\mathbf{1}_8 \end{pmatrix}. \tag{105}$$

The projection operators $P_\pm$ are

$$P_+ = \begin{pmatrix} \mathbf{1}_8 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_- = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1}_8 \end{pmatrix}. \tag{106}$$

Therefore we obtain

$$\begin{aligned} &\Gamma_1^{(\pm)} = \pm i\sigma_{133}, \quad \Gamma_2^{(\pm)} = \pm i\sigma_{233}, \quad \Gamma_3^{(\pm)} = \pm i\sigma_{013}, \\ &\Gamma_4^{(\pm)} = \pm i\sigma_{023}, \quad \Gamma_5^{(\pm)} = \pm i\sigma_{001}, \quad \Gamma_6^{(\pm)} = \pm i\sigma_{002}, \quad \Gamma_7^{(\pm)} = \pm i\sigma_{333}. \end{aligned} \tag{107}$$

Of course, we can take another matrix representation:

$$\begin{aligned} &\Gamma_1 = \begin{pmatrix} i\sigma_{112} & 0 \\ 0 & -i\sigma_{112} \end{pmatrix}, \; \Gamma_2 = \begin{pmatrix} i\sigma_{120} & 0 \\ 0 & -i\sigma_{120} \end{pmatrix}, \; \Gamma_3 = \begin{pmatrix} -i\sigma_{132} & 0 \\ 0 & i\sigma_{132} \end{pmatrix}, \\ &\Gamma_4 = \begin{pmatrix} -i\sigma_{221} & 0 \\ 0 & i\sigma_{221} \end{pmatrix}, \; \Gamma_5 = \begin{pmatrix} i\sigma_{223} & 0 \\ 0 & -i\sigma_{223} \end{pmatrix}, \; \Gamma_6 = \begin{pmatrix} -i\sigma_{202} & 0 \\ 0 & i\sigma_{202} \end{pmatrix}, \\ &\Gamma_7 = \begin{pmatrix} i\sigma_{300} & 0 \\ 0 & -i\sigma_{300} \end{pmatrix}. \end{aligned} \tag{108}$$

In this case, $\Gamma_i^{(\pm)}$ are

$$\begin{aligned} &\Gamma_1^{(\pm)} = \pm i\sigma_{112}, \quad \Gamma_2^{(\pm)} = \pm i\sigma_{120}, \quad \Gamma_3^{(\pm)} = \mp i\sigma_{132}, \\ &\Gamma_4^{(\pm)} = \mp i\sigma_{221}, \quad \Gamma_5^{(\pm)} = \pm i\sigma_{223}, \quad \Gamma_6^{(\pm)} = \mp i\sigma_{202}, \quad \Gamma_7^{(\pm)} = \pm i\sigma_{300}. \end{aligned} \tag{109}$$

The basis (109) is used to construct the Grossman's one-instantons [14]. These bases take complex values and the matrix size of $\Gamma_i^{(\pm)}$ is eight. Therefore the gauge group becomes U(8) for this basis.

The real basis in eight dimensions The matrix representation of $C\ell_7(\mathbb{R})$ is given by

$$\begin{aligned} &\Gamma_1 = \begin{pmatrix} \tau_{222} & 0 \\ 0 & -\tau_{222} \end{pmatrix}, \; \Gamma_2 = \begin{pmatrix} \tau_{012} & 0 \\ 0 & -\tau_{012} \end{pmatrix}, \; \Gamma_3 = \begin{pmatrix} \tau_{201} & 0 \\ 0 & -\tau_{201} \end{pmatrix}, \\ &\Gamma_4 = \begin{pmatrix} \tau_{032} & 0 \\ 0 & -\tau_{032} \end{pmatrix}, \; \Gamma_5 = \begin{pmatrix} \tau_{120} & 0 \\ 0 & -\tau_{120} \end{pmatrix}, \; \Gamma_6 = \begin{pmatrix} \tau_{320} & 0 \\ 0 & -\tau_{320} \end{pmatrix}, \\ &\Gamma_7 = \begin{pmatrix} \tau_{203} & 0 \\ 0 & -\tau_{203} \end{pmatrix}. \end{aligned} \tag{110}$$

Using (89a), the chiral matrix ω is given by

$$\omega = (-1)\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5\Gamma_6\Gamma_7 = \begin{pmatrix} \mathbf{1}_8 & 0 \\ 0 & -\mathbf{1}_8 \end{pmatrix}. \tag{111}$$

The projection operators $P_\pm$ are

$$P_+ = \begin{pmatrix} \mathbf{1}_8 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_- = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1}_8 \end{pmatrix}. \tag{112}$$

Therefore we obtain

$$\Gamma_1^{(\pm)} = \pm\tau_{222}, \quad \Gamma_2^{(\pm)} = \pm\tau_{012}, \quad \Gamma_3^{(\pm)} = \pm\tau_{201},$$
$$\Gamma_4^{(\pm)} = \pm\tau_{032}, \quad \Gamma_5^{(\pm)} = \pm\tau_{120}, \quad \Gamma_6^{(\pm)} = \pm\tau_{320}, \quad \Gamma_7^{(\pm)} = \pm\tau_{203}. \tag{113}$$

This basis is real valued, therefore the gauge group becomes O(8).

Appendix C. Eight-dimensional U(8) ADHM equations

In this section, we explicitly write down the eight-dimensional ADHM equations for U(8) gauge group. Here we use the complex basis (109).[8] If we use the real basis (113) then we obtain the eight-dimensional ADHM equations for O(8) gauge group.

We assume that ADHM data S is expanded by the basis e_μ, that is $S = e_\mu \otimes \tilde{S}^\mu$. The first ADHM equations are given by the following equations (114), (115) and (116).

$$[T^2, T^5] - [T^3, T^6] + \frac{i}{2}\left(S_4^\dagger S_4 - S_1^\dagger S_1\right) = 0,$$
$$[T^3, T^6] - [T^1, T^4] + \frac{i}{2}\left(S_4^\dagger S_4 - S_2^\dagger S_2\right) = 0,$$
$$[T^1, T^4] - [T^2, T^5] + \frac{i}{2}\left(S_4^\dagger S_4 - S_3^\dagger S_3\right) = 0. \tag{114}$$

$$[T^1, T^2] + [T^4, T^5] + \frac{1}{2}\left(S_1^\dagger S_2 - S_2^\dagger S_1\right) = 0,$$
$$[T^1, T^5] - [T^4, T^2] + \frac{i}{2}\left(S_1^\dagger S_2 + S_2^\dagger S_1\right) = 0,$$
$$[T^1, T^3] + [T^4, T^6] + \frac{1}{2}\left(S_1^\dagger S_3 - S_3^\dagger S_1\right) = 0,$$
$$[T^1, T^6] - [T^4, T^3] + \frac{i}{2}\left(S_1^\dagger S_3 + S_3^\dagger S_1\right) = 0,$$
$$[T^2, T^3] + [T^5, T^6] + \frac{1}{2}\left(S_2^\dagger S_3 - S_3^\dagger S_2\right) = 0,$$
$$[T^2, T^6] - [T^5, T^3] + \frac{i}{2}\left(S_2^\dagger S_3 + S_3^\dagger S_2\right) = 0,$$
$$[T^1, T^2] - [T^4, T^5] + \frac{1}{2}\left(S_4^\dagger S_3 - S_3^\dagger S_4\right) = 0,$$
$$[T^1, T^5] + [T^4, T^2] - \frac{i}{2}\left(S_4^\dagger S_3 + S_3^\dagger S_4\right) = 0,$$
$$[T^2, T^3] - [T^5, T^6] + \frac{1}{2}\left(S_4^\dagger S_1 - S_1^\dagger S_4\right) = 0,$$
$$[T^2, T^6] + [T^5, T^3] - \frac{i}{2}\left(S_4^\dagger S_1 + S_1^\dagger S_4\right) = 0,$$
$$[T^3, T^1] - [T^6, T^4] + \frac{1}{2}\left(S_4^\dagger S_2 - S_2^\dagger S_4\right) = 0,$$

[8] Note that we can use other basis (107).

$$
\begin{aligned}
&[T^3, T^4] + [T^6, T^1] - \frac{i}{2}\left(S_4^\dagger S_2 + S_2^\dagger S_4\right) = 0, \\
&[T^8, T^1] + [T^7, T^4] + \frac{1}{2}\left(S_2^\dagger S_7 - S_7^\dagger S_2\right) = 0, \\
&[T^8, T^4] - [T^7, T^1] + \frac{i}{2}\left(S_2^\dagger S_7 + S_7^\dagger S_2\right) = 0, \\
&[T^8, T^2] + [T^7, T^5] + \frac{1}{2}\left(S_3^\dagger S_5 - S_5^\dagger S_3\right) = 0, \\
&[T^8, T^5] - [T^7, T^2] + \frac{i}{2}\left(S_3^\dagger S_5 + S_5^\dagger S_3\right) = 0, \\
&[T^8, T^3] + [T^7, T^6] + \frac{1}{2}\left(S_1^\dagger S_6 - S_6^\dagger S_1\right) = 0, \\
&[T^8, T^6] - [T^7, T^3] + \frac{i}{2}\left(S_1^\dagger S_6 + S_6^\dagger S_1\right) = 0, \\
&[T^8, T^1] - [T^7, T^4] + \frac{1}{2}\left(S_4^\dagger S_5 - S_5^\dagger S_4\right) = 0, \\
&[T^8, T^4] + [T^7, T^1] - \frac{i}{2}\left(S_4^\dagger S_5 + S_5^\dagger S_4\right) = 0, \\
&[T^8, T^2] - [T^7, T^5] + \frac{1}{2}\left(S_4^\dagger S_6 - S_6^\dagger S_4\right) = 0, \\
&[T^8, T^5] + [T^7, T^2] - \frac{i}{2}\left(S_4^\dagger S_6 + S_6^\dagger S_4\right) = 0, \\
&[T^8, T^3] - [T^7, T^6] + \frac{1}{2}\left(S_4^\dagger S_7 - S_7^\dagger S_4\right) = 0, \\
&[T^8, T^6] + [T^7, T^3] - \frac{i}{2}\left(S_4^\dagger S_7 + S_7^\dagger S_4\right) = 0.
\end{aligned} \tag{115}
$$

$$
-[T^1, T^4] - [T^2, T^5] - [T^3, T^6] - [T^8, T^7] - \frac{i}{2}\left(S_4^\dagger S_4 - S_8^\dagger S_8\right) = 0. \tag{116}
$$

Now an invertible $k \times k$ matrix f is defined by the first ADHM constraint $\Delta^\dagger \Delta = \mathbf{1}_8 \otimes f^{-1}$ (that is $f = (E_k^{(1)})^{-1}$). The second ADHM equations are given by the following equations (117), (118) and (119).

$$
T^\mu f = f T^\mu \tag{117}
$$

$$
\begin{aligned}
&[T^1, T^2] = 0, \qquad [T^2, T^3] = 0, \qquad [T^3, T^1] = 0, \\
&[T^4, T^5] = 0, \qquad [T^5, T^6] = 0, \qquad [T^6, T^4] = 0, \\
&[T^1, T^8] = 0, \qquad [T^2, T^8] = 0, \qquad [T^3, T^8] = 0, \\
&[T^4, T^7] = 0, \qquad [T^5, T^7] = 0, \qquad [T^6, T^7] = 0.
\end{aligned} \tag{118}
$$

$$
\begin{aligned}
&[T^1, T^5] + [T^4, T^2] = 0, \qquad [T^2, T^6] + [T^5, T^3] = 0, \qquad [T^3, T^4] + [T^6, T^1] = 0, \\
&[T^1, T^7] + [T^4, T^8] = 0, \qquad [T^2, T^7] + [T^5, T^8] = 0, \qquad [T^3, T^7] + [T^6, T^8] = 0.
\end{aligned} \tag{119}
$$

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