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Note

Cycle packings in graphs and digraphs

Jennifer J. Quinn [1]

Department of Mathematics, Occidental College, Los Angeles, CA 90041, USA

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Abstract

A cycle packing in a (directed) multigraph is a vertex disjoint collection of (directed) elementary cycles. If D is a demiregular multidigraph we show that the arcs of D can be partitioned into Δ_{in} cycle packings — where Δ_{in} is the maximum indegree of a vertex in D. We then show that the maximum length cycle packings in any digraph contain a common vertex.

1. Introduction

A *cycle packing* $\mathscr{C}$ in a (directed) graph is defined in [6] as the vertex disjoint union of elementary (directed) cycles. The size of $\mathscr{C}$ equals the number $|\mathscr{C}|$ of vertices covered by the packing. The packing $\mathscr{C}$ is a *maximal cycle packing* provided no cycle is disjoint from $\mathscr{C}$ and $\mathscr{C}$ is a *maximum cycle packing* provided $|\mathscr{C}|$ is maximum. For a digraph D of order n let $\varrho(D)$ denote the size of a maximum cycle packing in D. If $\varrho(D) = n$ then D has a *perfect* cycle packing (or 2–factor).

In a graph G, the maximum degree of a vertex in G is denoted $\Delta(G)$. In a digraph D, the maximum in- and out-degrees occurring in D are denoted $\Delta_{\text{in}}(D)$ and $\Delta_{\text{out}}(D)$, respectively. A digraph D is *regular* if the $\text{indeg}_D(v) = \text{outdeg}_D(w) = \Delta_{\text{in}}(D) = \Delta_{\text{out}}(D)$ for all vertices v and w in D. A digraph D is *demiregular* if the $\text{indeg}_D(v) = \text{outdeg}_D(v)$ for all vertices v in D. We show that the arcs of a demiregular multidigraph D can always be partitioned into the minimum possible number of cycle packings; that is, the arcs of D can always be partitioned into $\Delta_{\text{in}}(D)$ cycle packings.

In [6], it was conjectured that the maximum cycle packings in graphs and digraphs contain a common vertex. This conjecture was then verified for outerplanar graphs and outerplanar digraphs. Here, we prove that the maximum cycle packings in a directed

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multigraph contain a common vertex by utilizing a cycle packing decomposition of the arcs in a related demiregular multidigraph. Unfortunately, a similar argument is not valid for undirected multigraphs.

2. Results

The connected components of a demiregular multidigraph D contain directed Eulerian cycles (see e.g. [1,2]), so the arcs of D can be greedily decomposed into cycle packings. Since $\Delta_{\text{in}}(D)$ is the maximum indegree of a vertex in D, it cannot be decomposed into fewer than $\Delta_{\text{in}}(D)$ cycle packings. Kotzig [3] showed that $\Delta_{\text{in}}(D)$ packings suffice for regular multidigraphs by proving that its arcs can be partitioned into perfect cycle packings. We shall show that $\Delta_{\text{in}}(D)$ cycle packings suffice for any demiregular digraph.

Lemma 2.1. *Assume D is a demiregular multidigraph of order n. Then D can be partitioned into $\Delta_{\text{in}}(D)$ cycle packings.*

Proof. Let D' be the multidigraph obtained by adding $\Delta_{\text{in}}(D) - \text{indeg}(v)$ loops at vertex v for each vertex v in D. The resulting multidigraph D' is regular of indegree $\Delta_{\text{in}}(D)$. By Kotzig's work, the arcs of D' can be decomposed into $\Delta_{\text{in}}(D)$ perfect cycle packings. So

$$D' = \mathscr{C}'_1 \oplus \mathscr{C}'_2 \oplus \cdots \oplus \mathscr{C}'_{\Delta_{\text{in}}(D)},$$

where each $\mathscr{C}'_i$ is a perfect cycle packing of D'. By removing the loops in D' which do not occur in D, each $\mathscr{C}'_i$ restricts to a cycle packing $\mathscr{C}_i$ of D. Thus

$$D = \mathscr{C}_1 \oplus \mathscr{C}_2 \oplus \cdots \oplus \mathscr{C}_{\Delta_{\text{in}}(D)},$$

where each $\mathscr{C}_i$ is a cycle packing of D. So a demiregular multidigraph can be partitioned into $\Delta_{\text{in}}(D)$ cycle packings. $\square$

Lemma 2.1 can be used to prove that the maximum cycle packings in a multidigraph have a common vertex.

Theorem 2.2. *The maximum cycle packings of a directed multigraph D have a common vertex.*

Proof. Assume D is a multidigraph and assume to the contrary that for each vertex v in D there is a maximum cycle packing $\mathscr{C}_v$ which does not contain the vertex v. Define a directed multigraph D' on the vertices of D as

$$D' = \sum_{v \in V} \mathscr{C}_v,$$

where an arc (x, y) occurs with multiplicity $|\{v | (x, y) \in \mathscr{C}_v\}|$. Then D' is a demiregular multidigraph and $\Delta_{\text{in}}(D') < |V(D)|$. By Lemma 2.1 the arcs of D' can be partitioned in $\Delta_{\rm in}(D')$ cycle packings. Since D' has exactly $|V(D)|\varrho(D)$ arcs, the average size of a cycle packing in the partition is

$$\frac{|V(D)|\varrho(D)}{\Delta_{\rm in}(D')} > \varrho(D).$$

So there is at least one cycle packing of D' of size strictly greater than $\varrho(D)$. But this contradicts the fact that any cycle packing of D' is also a cycle packing of D and so the size of any cycle packing in D' cannot exceed $\varrho(D)$. $\square$

A digraph is *hypohamiltonian* provided the digraph minus any vertex contains a directed Hamilton cycle (see [8]). A nontrivial example of a directed hypohamiltonian graph is $\vec{C}_p \times \vec{C}_q$ where $\vec{C}_i$ represents the directed cycle on i vertices and p and q are relatively prime. We now show that every hypohamiltonian digraph contains a perfect cycle packing.

Corollary 2.3. *If D is a directed hypohamiltonian graph, then D contains a perfect cycle packing.*

Proof. Assume D is a hypohamiltonian digraph of order n. For any vertex $v \in V(D)$, the digraph $D\setminus\{v\}$ contains a Hamilton cycle, so $\varrho(D) \geqslant n-1$ and the intersection of all cycles of length $n-1$ is empty. Hence the intersection of all cycle packings of size $n-1$ is empty. Theorem 2.2 implies that $\varrho(D)$ cannot equal $n-1$, forcing $\varrho(D)=n$. Hence D contains a perfect cycle packing. $\square$

An undirected version of Lemma 2.1 clearly holds. We shall state it here without proof.

Lemma 2.4. *If G is a multigraph with every vertex having even degree, then G can be partitioned into $\Delta(G)/2$ cycle packings.*

The next logical step is to try and prove the corresponding undirected version of Theorem 2.2. Mimicking the proof of Theorem 2.2, assume G is a multigraph and for each vertex v in G assume there exists a maximum cycle packing $\mathscr{C}_v$ which does not contain v. The multigraph

$$G' = \sum_{v\in V(G)} \mathscr{C}_v,$$

where the multiplicity of the edge $\{x,y\}$ is $|\{v \mid \{x,y\} \in \mathscr{C}_v\}|$, can be partitioned into $\Delta(G')/2$ cycle packings. The average size of a cycle packing in this decomposition is strictly greater than $\varrho(G)$. However, a cycle packing in G' may not be a cycle packing of G since G' can contain multiple edges (forming 2-cycles) that do not occur in G. Hence the proof of Theorem 2.2 will not generalize to undirected graphs. We can only continue to *conjecture* that the maximum cycle packings of a multigraph have a common vertex.

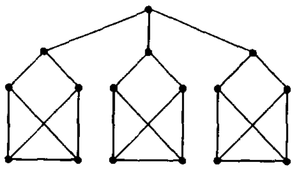

Fig. 1. A cubic graph on 16 vertices whose maximum cycle packings contain only 15 vertices.

Notice, any graph that contains a perfect cycle packing satisfies this conjecture since the maximum cycle packings must have every vertex of the graph as a common vertex. Thus regular multigraphs of even degree and vertex transitive multigraphs satisfy the conjecture since they always contain perfect cycle packings (see [4]). Regular multigraphs of odd degree need not contain perfect cycle packings. The regular graph of degree 3 (a *cubic* graph) in Fig. 1 has 16 vertices and maximum cycle packings of size 15. But, cubic multigraphs will also satisfy the conjecture that the maximum cycle packings share a common vertex.

A classical result of Peterson, states sufficient conditions for a cubic multigraph to contain a perfect matching:

> Every connected cubic multigraph with no more than two bridges
> has a perfect matching.

In a cubic multigraph, a pendant block is incident to exactly one bridge. Using Peterson's result, we see that any pendant block in a cubic multigraph G must contain a perfect cycle packing. Thus G either contains a perfect cycle packing or every maximum cycle packing of G contains all the vertices of the pendant blocks.

Peterson's result can be used further to find a lower bound on the size of a maximum cycle packing in cubic graphs. Finding and characterizing such bounds is the focus of [7].

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