Remarks on a result about hypercyclic non-convolution operators

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Received 1 November 2004
Available online 26 February 2005
Submitted by R.M. Aron

Abstract

In this paper we correct a proof by Aron and Markose in [R. Aron, D. Markose, On universal functions, J. Korean Math. Soc. 41 (2004) 65–76] for the hypercyclicity of the operator $T : H(\mathbb{C}) \rightarrow H(\mathbb{C})$ given by $T(f)(z) = f'(\lambda z + b)$, $|\lambda| \geq 1$, in the case $b \neq 0$.

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Keywords: Hypercyclic operators; Non-convolution operators

A continuous linear operator $T$ on a Fréchet space $E$ is called hypercyclic if there is a vector $x \in E$ such that its orbit (under $T$) $O(x; T) = \{x, Tx, T^2x, \ldots\}$ is dense in $E$ (in this case the vector $x$ is called a hypercyclic vector for the operator $T$).

An important result by Godefroy and Shapiro in [2] states that if we consider the Fréchet space $H(\mathbb{C}^n)$ of analytic functions on $\mathbb{C}^n$ with the compact-open topology (that is, the topology of the uniform convergence on compact sets of $\mathbb{C}^n$), then every convolution oper-
ator (a continuous linear operator that commutes with all translations) that is not a multiple of the identity is hypercyclic. Therefore, it became natural and relevant to ask for explicit examples of hypercyclic non-convolution operators on these spaces \( H(\mathbb{C}^n) \), \( n \geq 1 \).

The linear operator \( T : H(\mathbb{C}) \to H(\mathbb{C}) \) given by \( T(f)(z) = f'(\lambda z + b) \), \( |\lambda| \geq 1 \) and \( b \) in \( \mathbb{C} \), was introduced by Aron and Markose in [1] as an example of a hypercyclic non-convolution (in the case \( \lambda \neq 1 \)) operator on \( H(\mathbb{C}) \).

The proof in [1] rests on a form of the Hypercyclicity Criterion (H.C.) known as Kitai’s Criterion. To apply the H.C., the authors use the space \( Z \) of analytic polynomials as a dense set in \( H(\mathbb{C}) \) and the idea is to provide a map \( S : Z \to Z \) so that

\begin{enumerate}
  \item \( T^n f \to 0 \) for all \( f \in Z \);
  \item \( S^n f \to 0 \) for all \( f \in Z \);
  \item \((T^n \circ S^n)f \to f \) for all \( f \in Z \).
\end{enumerate}

Although the statement \( (T \) is hypercyclic) is correct, the proof in [1] contains a gap, in that the expressions for \( S(z^k) \) and \( S^n(z^k) \) are not correct for the case \( b \neq 0 \). Here we address this problem by also proving the case \( b \neq 0 \).

First, let us recall that in the H.C. we do not need to consider the iterates of the same map \( S : Z \to Z \). If we have a dense set \( Z \subset H(\mathbb{C}) \) and a sequence of maps \( S_n : Z \to H(\mathbb{C}) \) so that (1) above is valid and also

\begin{enumerate}
  \item \( S_n f \to 0 \) for all \( f \in Z \);
  \item \((T^n \circ S_n)f \to f \) for all \( f \in Z \)
\end{enumerate}

are satisfied, then the Criterion works as well; that is, \( T \) is hypercyclic. Actually, in the general H.C. these conditions must hold only for a subsequence \((n_k)\) of indices [3].

Let us restate the result that will be proved.

**Theorem 1.** For \( \lambda, b \in \mathbb{C} \), define \( T : H(\mathbb{C}) \to H(\mathbb{C}) \) by \( T(f)(z) = f'(\lambda z + b) \). Then \( T \) is hypercyclic provided \( |\lambda| \geq 1 \).

**Proof.** We intend to apply the H.C. with \( Z \) being the set \( \mathcal{P}(\mathbb{C}) \) of all polynomials, which is dense in \( H(\mathbb{C}) \).

It is easy to see that condition (1) in the H.C. is satisfied.

To find a reasonable expression for each map \( S_n : Z \to H(\mathbb{C}) \) consider first that

\[ T = R_\lambda \circ \tau_b \circ D, \]

where \( R_\lambda, \tau_b \) and \( D \) are the operators on \( H(\mathbb{C}) \) given by

\[ R_\lambda(f)(z) = f(\lambda z), \]
\[ \tau_b(f)(z) = f(z + b) \quad \text{(translation by } b), \]
\[ D(f)(z) = f'(z) \quad \text{(differentiation).} \]

Observe that if \( \lambda \neq 1 \), then \( R_\lambda \) does not commute with \( \tau_a \) \((a \neq 0)\) or \( D \) and we have

\[ D \circ R_\lambda = \lambda(R_\lambda \circ D) \] and \( \tau_a \circ R_\lambda = R_\lambda \circ \tau_{\lambda a}. \) Hence \( T \) does not commute with all transla-
tions if $\lambda \neq 1$ ($\tau \circ T = T \circ \tau_{ab}$) and so $T$ is not a convolution operator in this case. With the above relations it is easy to compute

$$T^n = \lambda^{\frac{n(n-1)}{2}} R_{\lambda^n} \circ \tau_{-r_n} \circ D^n, \quad \text{where} \quad r_n = -b \sum_{l=0}^{n-1} \lambda^l.$$ 

By way of motivation, observe that condition $(3')$ in the H.C. gives us a hint to find the maps $S_n$. The maps $S_n : Z \to H(\mathbb{C})$ shall behave (asymptotically, when $n \to \infty$) as right inverses on $Z$ for the operators $T^n$. So, if we denote by $AD$ (anti-differentiation) the “formal operator” defined only on the set $A = \{1\} \cup \{d(z + c)^k; \ k \in \mathbb{N}, c, d \in \mathbb{C}\}$ by

$$AD[d(z + c)^k] = \frac{d(z + c)^{k+1}}{k+1}, \quad k \in \mathbb{N}, c, d \in \mathbb{C},$$

$$AD[1] = (z - b)$$

and since $D \circ AD = \text{Id}_A$, the third condition of the H.C. makes it natural to try (on $1, z, z^2, \ldots, z^k, \ldots$)

$$S = AD \circ \tau_{-b} \circ R_{1/\lambda} \quad \text{and} \quad S_n = S^n \quad \text{for all} \quad n \in \mathbb{N}.$$ 

Note that $AD$ is just an “operational tool” to lead us to explicit, well-defined expressions for the maps $S_n : Z \to H(\mathbb{C})$ on the basis $\{1, z, z^2, \ldots, z^k, \ldots\}$ of $Z = P(\mathbb{C})$.

So we get

$$S_n(z^k) = \frac{k!}{(k+n)! \lambda^{kn+1}} (z + r_n)^{k+n} \quad (k \in \mathbb{N}_0, n \in \mathbb{N})$$

and extend linearly to $Z$.

As expected, this works well for condition $(3')$, but when we try to check $(2')$ we run into problems with high powers of $\lambda$ in $(z + r_n)^{k+n}$. However, if we consider

$$S_n(z^k) = \frac{k!}{(k+n)! \lambda^{kn+1}} [(z + r_n)^{k+n} + \Delta_{n,k}(z)], \quad n \in \mathbb{N}, k \in \mathbb{N}_0,$$

where $\Delta_{n,k}(z)$ is any polynomial in $z$ with degree less than or equal to $n - 1$, it is easy to see that we will still have $(3')$ satisfied (look at the expression for $T^n$).

So the idea now is to use an appropriate $\Delta_{n,k}(z)$ to “kill” the high powers of $\lambda$ in the binomial $(z + r_n)^{k+n}$ (and we can do this because high powers of $\lambda$ are paired with small powers of $z$ in the binomial expansion).

Now fix

$$\Delta_{n,k}(z) = \sum_{j=k+1}^{k+n} \binom{k+n}{j} z^{k+n-j} r_n^j, \quad n \in \mathbb{N}, k \in \mathbb{N}_0,$$

and we will finally have
\[ S_n(z^k) = \frac{k!}{(k+n)!\lambda^{kn}\lambda^{\frac{n(n-1)}{2}}} \left[ (z + r_n)^{k+n} + \Delta_{n,k}(z) \right] \]
\[ = \frac{k!}{(k+n)!\lambda^{kn}\lambda^{\frac{n(n-1)}{2}}} \left[ \sum_{j=0}^{k} \binom{k+n}{j} z^{k+n-j} r_n^j \right]. \]

which we extend linearly to \( Z \).

Since \( \text{deg}(\Delta_{n,k}(z)) = n - 1 \), (3') is satisfied and now, if we consider \( |z| \leq R \) and fix \( k \in \mathbb{N}_0 \), we have, for all \( j = 0, 1, \ldots, k \),
\[ \left| \frac{k!}{(k+n)!\lambda^{kn}\lambda^{\frac{n(n-1)}{2}}} \binom{k+n}{j} z^{k+n-j} r_n^j \right| \leq \frac{k!(k+n)! R^{k+n-j} |r_n|^j}{(k+n)!|\lambda|^{kn}|\lambda|^{\frac{n(n-1)}{2}} (k+n-j)! j!} \]
\[ \leq \frac{k! R^{k+n-j} |b|^{j} |n|^j |\lambda|^j n}{|\lambda|^{kn}|\lambda|^{\frac{n(n-1)}{2}} (k+n-j)! j!} \xrightarrow{n \to 0} 0 \quad (|\lambda| \geq 1). \]

So \( S_n(z^k) \xrightarrow{n} 0 \), (2') is now satisfied and we can conclude (by the H.C.) that the operator \( T \) is hypercyclic. \( \square \)

Acknowledgments

The authors produced this paper during their visit to Kent State University and are specially thankful to Prof. Richard Aron. We also thank the referee for some useful comments on the paper.

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