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# The effects of learning on the optimal production lot size for deteriorating and partially backordered items with time varying demand and deterioration rates

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**Abstract**

In this paper the full transmission of learning for the general production lot size problem with an infinite planning horizon is studied under the following assumptions: (1) Items deteriorate while they are produced or stored. (2) Both demand and deterioration rates are (known) functions of time. (3) Shortages are allowed, but are partially backordered. (4) The production rate is defined as the number of units produced per unit time. A closed form for the total relevant cost as well as a rigorous mathematical method that leads to a minimum total cost of the underlying inventory system are introduced. An illustrative example which explains the applications of the theoretical results as well as its numerical verification are also given.


*Keywords:* Production lot size model; Inventory; Deterioration; Learning process

## 1. Introduction

It has been noted that the performance of a person, group of persons, or an organization, engaged in a repetitive task improves with time. Such a phenomenon is referred, in the literature, as the "Learning Phenomenon", which implies a reduction in the cost or the time required for producing each unit. For instance, the familiarity with operational tasks and their environments, and the effective use of tools and machines are usually increased with repetition. The simplest and most widely used model is due to Wright [18] who suggested the power function, known as the learning curve (LC), to express the relations of learning. The LC is represented as

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$$t_i = t_1 i^{-r} \tag{1}$$

where $t_i$ is the time elapsed to produce the $i$th unit, $t_1$ is the time required to produce the first unit, $i$ is the production count and $r$ is the slope of the (LC). Similarly, if interruption occurs in production the system may forget some of what it has learnt which means that the production at the recommencement is not as high as when the production ceased. This phenomenon leads to what so called the "forgetting curve" (FC) which is assumed to be

$$\hat{t}_x = \hat{t}_1 x^f \tag{2}$$

where $\hat{t}_x$, $\hat{t}_1$, $x$ and $f$ have equivalent meanings of forgetting as $t_i$, $t_1$, $i$ and $r$ in relation (1). Both learning and forgetting phenomenons have received considerably more attention of many researchers. Keachie and Fontana [7] have shown the importance of transmission of learning in optimal lot size models. Alder and Nanda [1] developed a general equation for the average production time per unit when some percentage of learning is not retained between lots. Muth and Spremann [14] introduced a transcendental cost function to determine the optimal lot size under learning effects. Elmaghraby [4] reviewed some previously proposed models and expand one of them to accommodate a finite horizon. He also suggested a model which gives more consistent relationship between learning and forgetting. Salameh et al. [16] described a production lot size model in which they incorporated the (LC) which has lead to a decrease in the optimum production quantity and the required production time for each unit when full transmission of learning (FTL) is assumed in successive cycles. In a subsequent paper Jaber and Salameh [8] generalized Salameh et al. model with the consideration of shortages. The effect of both learning and forgetting on the optimum production quantity and the minimum total inventory system cost where shortages are not allowed has been considered by Jaber and Bonney [10,13]. The effect of intracycle, within cycle, backorders on the optimal manufactured quantity and the total inventory system cost was studied by Jaber and Bonney [11] for both full and partial transmission of learning. A common theoretical drawback of (LC) formulae is that the results obtained are not plausible as the cumulative production approaches to infinity. To retain plausibility, a correction of Wrights model is made by adding a nonzero lower bound to give a new formulae known as DeJong learning curve which is given by

$$t_i = t_1 m + (1 - m) t_1 i^{-r} \tag{3}$$

where $t_i$, $t_1$, $i$ and $r$ have similar meanings as those of the (LC) and $m (0 \leqslant m \leqslant 1)$ is a new parameter, known as the incompressibility factor. The optimal lot sizing under the bounded learning case has been considered by so many researchers. The most recent articles on this aspect are of Jaber and Bonney [9] and Zhou and Lau [19]. For more details about the learning and forgetting phenomenons, including the bounded case, see the excellent review of Jaber and Bonney [12] and the references therein.

All the above mentioned authors assumed a constant demand rate and did not incorporate deterioration in their models. Also, with the exception of Jaber and Salameh [8] and Zhou and Lau [19] who assumed that shortages are allowed but are fully backordered, all other authors assumed that shortages are not permitted to occur.

In this paper we shall establish a general production lot size (GPLS) model under the following assumptions

(i) A single item is produced at an increasing rate, denoted by $P(t)$, and its production is subject to FTL (formulae (1)).
(ii) Items are subject to deterioration while in production or in storage. The deterioration and the demand rates are assumed to be a known functions of time say $\theta(t)$, $D(t)$, respectively.
(iii) Shortages are allowed but only a fraction $\beta(0 \leqslant \beta \leqslant 1)$ is backordered and the rest $(1-\beta)$ is lost. However, deterioration did not occur in the shortage periods.
(iv) To avoid mathematical complications, only FTL (i.e. formulae (1)) will be assumed, in which case the production rate $P(t)$ is defined in the natural sense as

$$P(t) = \frac{\text{Number of units produced up to time } t}{t} \tag{4}$$

and the initial production rate will be considered to be $1/t_1$ ($t_1$ is the time required to produce the first unit). However, the resulting mathematical model (though it is bit complicated one) can be considered as the starting point to build similar models in which one or more of the formulas (1), (2) or (3) can be incorporated.
(v) The cost parameters are as follows:
$c$ = unit production cost which includes material, labor and manufacturing costs.
$h$ = unit holding cost per unit time.
$b$ = unit shortage cost per unit time for backordered items.
$s$ = unit shortage cost per unit time for lost items.
$k$ = set up cost per set up.

Let $I(t)$ denote the inventory level at time $t$. The system starts operating at time $T_0$ (note that we can set $T_0 = 0$ without loss of generality) at a demand rate $D(t)$ which accumulates up to time $T_1$ leading to a maximum amount of $S_j$ units of backordered items. Then production starts to fulfill the demand and to clear the entire $S_j$ units where the inventory level reaches to zero by time $T_2$.

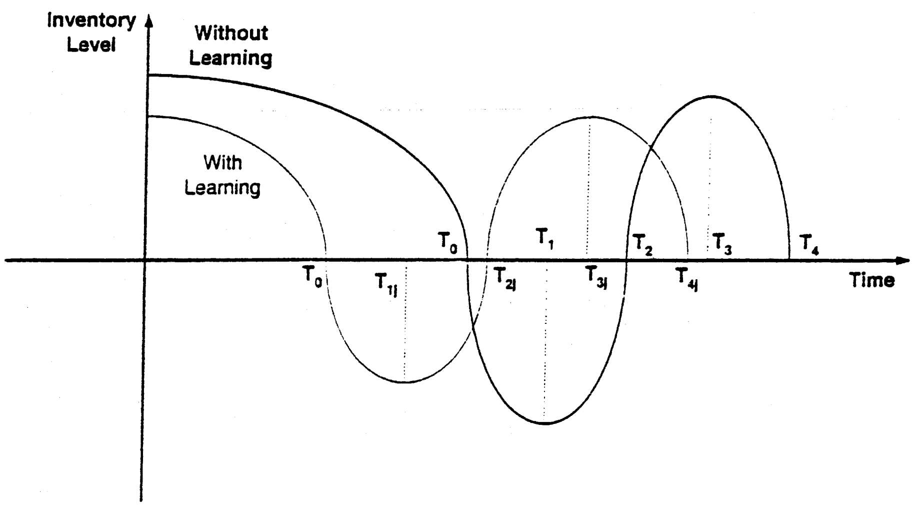


Fig. 1. The behaviour of the inventory system with and without learning.

Now, the inventory level starts to go up with the rate $P(t) - D(t) - \theta(t)I(t)$ till time $T_3$ where the inventory level reaches its maximum. Then the inventory declines continuously at a rate $D(t) - \theta(t)I(t)$ and it becomes zero by time $T_4$ (the end of the cycle). The process is repeated. For the learning case we shall use $T_{1j}$, $T_{2j}$, $T_{3j}$ and $T_{4j}$ for cycle $j$ with similar meanings as $T_1$, $T_2$, $T_3$ and $T_4$, respectively. The changes in the inventory levels for the with and without learning cases are depicted in Fig. 1.

The paper is organized as follows. In the next section we formulate the (GPLS) model without the consideration of learning effects and we indicate in brief its solution in Section 3. The (GPLS) model is reformulated under (FTL) and a solution procedure for the resulting model are treated in Section 4. An illustrative example, which explains the application of the theoretical results of Section 3 together with its numerical verification as well as some concluding remarks are introduced in Section 5.

## 2. Model formulation in general (without learning)

The variations of inventory levels depicted in Fig. 1 are given by the following differential equations

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} = -\beta D(t) \quad T_0 \leqslant t < T_1 \tag{5}$$

with the initial condition $I(T_0) = 0$

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} = -[P(t) - D(t)] \quad T_1 \leqslant t < T_2 \tag{6}$$

with the ending condition $I(T_2) = 0$

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} = P(t) - D(t) - \theta(t)I(t) \quad T_2 \leqslant t < T_3 \tag{7}$$

with the initial condition $I(T_2) = 0$

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} = -D(t) - \theta(t)I(t) \quad T_3 \leqslant t < T_4 \tag{8}$$

with the final condition $I(T_2) = 0$.

Let

$$\alpha'(t) = \theta(t) \quad \text{and} \quad \gamma'(t) = \mathrm{e}^{-\alpha(t)} \tag{9}$$

Then the solutions of the above differential equations are

$$I(t) = -\beta \int_{T_0}^{t} D(u)\,\mathrm{d}u \quad T_0 \leqslant t < T_1 \tag{10}$$

$$I(t) = -\int_{t}^{T_2} [P(u) - D(u)]\,\mathrm{d}u \quad T_1 \leqslant t < T_2 \tag{11}$$

$$I(t) = e^{-\alpha(t)} \int_{T_2}^{t} [P(u) - D(u)] e^{\alpha(u)} \, du \quad T_2 \leqslant t < T_3 \tag{12}$$

$$I(t) = e^{-\alpha(t)} \int_{t}^{T_4} D(u) e^{\alpha(u)} \, du \quad T_3 \leqslant t < T_4 \tag{13}$$

respectively.

Let

$$I(t_1, t_2) = \int_{t_1}^{t_2} I(t) \, dt$$

then from (12) and (13) (using integration by parts) we can easily find

$$I(T_2, T_3) = \int_{T_2}^{T_3} [\gamma(T3) - \gamma(u)][P(u) - D(u)] e^{\alpha(u)} \, du \tag{14}$$

$$I(T_3, T_4) = \int_{T_3}^{T_4} [\gamma(u) - \gamma(T_3)] e^{\alpha(u)} \, du \tag{15}$$

Similarly (10) and (11) lead, respectively, to

$$I(T_0, T_1) = \beta \int_{T_0}^{T_1} (T_1 - u) D(u) \, du \tag{16}$$

$$I(T_1, T_2) = \int_{T_1}^{T_2} (u - T_1)[P(u) - D(u)] \, du \tag{17}$$

Let us assume, without loss of generality, that $T_0 = 0$. Then the relevant costs of the underlying inventory system consists of the following:

Item's production cost $= c \int_{T_1}^{T_3} P(u) \, du$.
Note that this cost includes the deterioration cost.
Holding cost $= h[I(T_2, T_3) + I(T_3, T_4)]$.
Shortage cost for backordered items $= b[I(0, T_1) + I(T_1, T_2)]$.
Shortage cost for lost items $= s(1 - \beta) \int_{0}^{T_1} D(u) \, du$.

Thus, the total relevant cost per unit time denoted by $W$, is given by

$$W = \frac{1}{T_4} \Big\{ c \int_{T_1}^{T_3} P(u) \, du + h \Big[ \int_{T_2}^{T_3} \{\gamma(T_3) - \gamma(u)\}\{P(u) - D(u)\} \times e^{\alpha(u)} \, du + \int_{T_3}^{T_4} \{\gamma(u) - \gamma(T_3)\} D(u) e^{\alpha(u)} \, du \Big] + b \Big[ \beta \int_{0}^{T_1} (T_1 - u) D(u) \, du + \int_{T_1}^{T_2} (u - T_1)(P(u) - D(u)) \Big] \, du + s(1 - \beta) \int_{0}^{T_1} D(u) \, du + k \Big\} \tag{18}$$

Our objective is to find $T_1$, $T_2$, $T_3$ and $T_4$ that minimize $W$, where $W$ is given by (18). But, the variables $T_1$, $T_2$, $T_3$ and $T_4$ are related to each other through the following relations

$$0 < T_1 \leqslant T_2 \leqslant T_3 \leqslant T_4 \tag{19}$$

$$\beta \int_0^{T_1} D(u)\,\mathrm{d}u = \int_{T_1}^{T_2} [P(u) - D(u)]\,\mathrm{d}u \tag{20}$$

$$\mathrm{e}^{-\alpha(T_3)} \int_{T_2}^{T_3} [P(u) - D(u)]\mathrm{e}^{\alpha(u)}\,\mathrm{d}u = \mathrm{e}^{-\alpha(T_3)} \int_{T_3}^{T_4} D(u)\mathrm{e}^{\alpha(u)}\,\mathrm{d}u \tag{21}$$

Relation (19) express the natural monotonicity constraints, since otherwise, the given problem, would have no meaning. Relations (20) (and (21)) ensures that fact the the inventory levels must have equal values for $t = T_1$ (for $t = T_3$). Thus we need to solve the following optimization problem which we shall call problem $(P)$.

$$(P)\begin{cases} \text{Minimize } W & \text{subject to} \\ 19, & g_1 = 0 \text{ and } g_2 = 0 \end{cases}$$

where $W$ is a given by (18) and $g_1$, $g_2$ are given by

$$g_1 = \beta \int_0^{T_1} D(u)\,\mathrm{d}u - \int_{T_1}^{T_2} [P(u) - D(u)]\,\mathrm{d}u$$

$$g_2 = \int_{T_2}^{T_3} [P(u) - D(u)]\mathrm{e}^{\alpha(u)}\,\mathrm{d}u - \int_{T_3}^{T_4} D(u)\mathrm{e}^{\alpha(u)}\,\mathrm{d}u$$

## 3. Solution procedure for the general model

The solution procedure for problem $(P)$ goes as follows. First we ignore (19) and we solve the problem under the two equality constraints $g_1 = 0$ and $g_2 = 0$ (let us call this problem $(P_1)$).

Then we show that any solution to $(P_1)$ does satisfy the constraints (19) (hence it satisfies problem $(P)$).

To see this, let $Z(T_1, T_2, T_3, T_4, \lambda_1, \lambda_2)$ be the Lagrangean of $(P_1)$ where, $\lambda_1$, $\lambda_2$ are the Lagrange multipliers corresponding to $g_1 = 0$ and $g_2 = 0$, respectively, then

$$Z(T_1, T_2, T_3, T_4, \lambda_1, \lambda_2) = W(T_1, T_2, T_3, T_4) + \lambda_1 g_1 + \lambda_2 g_2$$

The necessary conditions to have a minimum are

$$\frac{\partial Z}{\partial T_1} = 0, \quad \frac{\partial Z}{\partial T_2} = 0, \quad \frac{\partial Z}{\partial T_3} = 0, \quad \frac{\partial Z}{\partial T_4} = 0, \quad \frac{\partial Z}{\partial \lambda_1} = 0, \quad \text{and,} \quad \frac{\partial Z}{\partial \lambda_2} = 0 \tag{22}$$

Note that the last two equations of (22) repeat (20) and (21), respectively. From the expression of $W$, $g_1$ and $g_2$, we can easily obtain the following results

$$T_4 \cdot \frac{\partial Z}{\partial T_1} = -cP(T_1) + b\left\{\beta \int_0^{T_1} D(u)\,\mathrm{d}u - \int_{T_1}^{T_2} [P(u) - D(u)]\,\mathrm{d}u\right\} + s(1-\beta)D(T_1) + \lambda_1[\beta D(T_1) + P(T_1) - D(T_1)] = 0$$

Recalling (19) we get:

$$\lambda_1 = \frac{cP(T_1) - s(1-\beta)D(T_1)}{P(T_1) - (1-\beta)D(T_1)} \tag{23}$$

Note that $\lambda_1 \geqslant 0$ if $c \geqslant s$ or if $\beta = 1$ (no lost items)

$$T_4 \cdot \frac{\partial Z}{\partial T_2} = h[-\gamma(T_3) + \gamma(T_2)][P(T_2) - D(T_2)]\mathrm{e}^{\alpha(T_2)} + b(T_2 - T_1)[P(T_2) - D(T_2)] - \lambda_1[P(T_2) - D(T_2)] - \lambda_2[P(T_2) - D(T_2)]\mathrm{e}^{\alpha(T_2)} = 0$$

From which and (23) we have

$$\lambda_2 = \mathrm{e}^{-\alpha(T_2)}\left\{h[\gamma(T_2) - \gamma(T_3)]\mathrm{e}^{\alpha(T_2)} + b(T_2 - T_1) - \frac{s(1-\beta)D(T_1) - cP(T_1)}{P(T_1) - (1-\beta)D(T_1)}\right\} \tag{24}$$

$$T_4 \cdot \frac{\partial Z}{\partial T_3} = cP(T_3) + h\left[\gamma'(T_3)\int_{T_2}^{T_3} [P(u) - D(u)]\mathrm{e}^{\alpha(u)}\,\mathrm{d}u - \gamma'(T_3)\int_{T_3}^{T_4} D(u)\mathrm{e}^{\alpha(u)}\,\mathrm{d}u\right] + \lambda_2[P(T_3) - D(T_3)]\mathrm{e}^{\alpha(T_3)} + D(T_3)\mathrm{e}^{\alpha(T_3)}] = 0$$

recalling (21) we obtain

$$\lambda_2 = -c\mathrm{e}^{-\alpha(T_3)} \tag{25}$$

From (24) and (25) we have

$$c\mathrm{e}^{-\alpha(T_3)} = \mathrm{e}^{-\alpha(T_2)}\left\{h[\gamma(T_3) - \gamma(T_2)] - b(T_2 - T_1) - \frac{cP(T_1) - s(1-\beta)D(T_1)}{P(T_1) - (1-\beta)D(T_1)}\right\} \tag{26}$$

Finally, let $W = w/T_4 \Longleftrightarrow w = T_4 \cdot W$, $w' = \partial W/\partial T_4$, then

$$\frac{\partial Z}{\partial T_4} = \frac{w'T_4 - w}{T_4^2} + \lambda_2\frac{\partial g_2}{\partial T_4} = \frac{\{h[\gamma(T_4) - \gamma(T_3)]D(T_4)\mathrm{e}^{\alpha(T_4)}\}T_4 - w}{T_4^2} + c\mathrm{e}^{-\alpha(T_3)}D(T_4)\mathrm{e}^{\alpha(T_4)}$$

$$\frac{\partial Z}{\partial T_4} = 0 \Longleftrightarrow W = \{h[\gamma(T_4) - \gamma(T_3)] + cT_4\mathrm{e}^{-\alpha(T_3)}\}\mathrm{e}^{\alpha(T_4)}D(T_4) \tag{27}$$

Now, let $T_1 > 0$, then it is evident from (20) that $T_2 \geqslant T_1$ since the RHS of (20) is >0 and since $P(u) \geqslant D(u)$. On the other hand, (26) implies that $h[\gamma(T_3) - \gamma(T_2)] > b(T_2 - T_1) + \lambda_1 \geqslant 0$ (for $c \geqslant s$) which implies that $\gamma(T_3) \geqslant \gamma(T_2) \Longleftrightarrow T_3 \geqslant T_2$ (recall $\gamma(t)$ is an increasing function since $\gamma'(t) = \mathrm{e}^{-\alpha(t)} > 0$). This and (21) implies that $T_4 \geqslant T_3$. Thus (19) can be replaced by $T_1 > 0$. But the constraint $T_1 > 0$ need not to be considered since its corresponding Lagrange multiplier is then equal to zero as an implication of Kuhn–Tucker necessary conditions.

Sufficient conditions for the global optimality of the solution(s) to problem $(P_1)$ can be established by quite similar methods to the methods that have been used in Balkhi [2,3].

## 4. Model motivation and formulation under (FTL)

### 4.1. Model motivation

A very simple example for learning phenomena, which expresses the time reduction as increasing practice is gained, is that clerks become quicker in accomplishing their daily routines within time upon reaching to some kind of steady state situation. Such learning phenomena, is also applicable to many kinds of productions especially those which their productivity are influenced by human labour needed for final assembly (Examples: cars, aircrafts, ships, machines, electronics, ...). The theory can be applicable to single operator or trainee learning a new task just as much as a complete organization doing a new type of production where the organizational learning is, conveniently, viewed as the sum of individual learning. Different learning curve models for certain individual, organizational learning or a combination of both have been introduced in the literature and will not be examined here again, because they have had adequate description in their papers. The learning curves in those papers have been imposed and their parameters have been estimated on the basis of aggregated empirical and/or simulated data usually taken from assembly lines (see for instance [5–7,15,17,18]). To the author's knowledge and with the exception of the papers mentioned in Section 1, non of those learning curves has been incorporated in an inventory model in order to study the impact of the learning phenomena on the production outputs in terms of time, cost or cumulative work. However, the incorporation of learning phenomena in the (GPLS) model introduced here, has the features that it covers many of production lot size inventory models. To avoid mathematical complications, the learning curve (1) (which was proposed by Wright [18] to show the variation of labour cost with the quantity produced airplanes) is going to be incorporated in our (GPLS) model. This curve is, in fact, the most common curve still in use since it is simple to explain and as accurate as other learning curves. Though our model is mainly a theoretical one but it can be applied to many real life situations. A scenario of practical example can still be given. The example consists of assembling a number (say 1000) of electronic machines (say washing machines). This requires a staff of some men and women (say 50) who are newly trained for such task but are not completely skilled in this work. There are neither imposed production rates nor bonuses given to the workers, hence their actual production rates are recorded. Thus, the actual times required to assemble successive units are recorded for each worker. Also, the cumulative outputs versus the elapsed time for the whole work shop are also recorded, then the average times required by the whole work shop to produce successive units are calculated. The resulting data can then be used to estimate the parameters of the proposed learning curve (1), namely $t_1$ and $r$, by using suitable statistical method. For instance, the logarithmic transformation of LC (1) is

$$\log(t_i) = \log(t_1) - r\log(i)$$

which is an equation of a straight line with slope $(-r)$.

A simple linear regression can then be used to compute $r$ and $t_1$ from the empirical data obtained (as in the manner explained above). The resulting learning curve, can now be incorporated in the underlying production lot size model to study its impact on the levels of productivity.

Further, from (1) we have $t_2 = t_1 2^{-r}$ and for the value $r = 0.075$, (which is being used below in Section 5.1) then $2^{-r} = 94.9\%$. This result means that the time needed to produce the second unit is 94.9% of the time needed to produce the first unit. The value $r = 0.075$ has, infact, been taken from an empirical study of an assembly line case investigated in [5] to estimate the parameters of LC (1).

### 4.2. Model formulation

Since FTL implies that learning takes its effects in successive cycles and units we shall rewrite the (LC) given by (1) as

$$t_{ij} = t_{1j} i^{-r} \tag{28}$$

where $t_{ij}$ is the elapsed time to produce the $i$th unit in production cycle $j$, $t_{1j}$ is the time required to produce the first unit in cycle $j$, and $r$ is the slop reflecting the decrease in the production time required per unit. To see the effects of learning in the production lot size models, we shall include the above learning curve (viz (28)) in our (GPLS) model where it is necessary, namely in production times and rates. (Note that the LC (28) need not to be included in the factors that are not related to the learning phenomena, namely, $D(t)$ and $\theta(t)$.) To do this, we first give another (approximated) form for (28). Let $t_j$ be the time required to produce $i$ units in the $j$th cycle then

$$t_j = \sum_{k=1}^{i} t_{1j} k^{-r} = t_{1j} \sum_{k=1}^{i} k^{-r} \approx t_{1j} \int_0^i k^{-r}\,\mathrm{d}k, \quad \text{or} \quad t_j = t_{1j} \frac{i^{1-r}}{1-r} \tag{29}$$

Also, from the definition of $P(t)$ we have

$$P(t_j) = \frac{i}{t_j} = \frac{1-r}{t_{1j} i^{-r}} \tag{30}$$

Now, let $Q_j$ be the amount produced between $T_{2j}$ and $T_{3j}$ and $S_j$ as defined above.

Then, from (29) we have

$$T_{3j} - T_{2j} = t_{1j} \frac{Q_j^{1-r}}{1-r} \tag{31}$$

Also,

$$S_j = \beta \int_0^{T_{1j}} D(t)\,\mathrm{d}t \tag{32}$$

Note that the RHS of (32) is an increasing function of $T_{1j}$, so $T_{1j}$ can be uniquely determined as a function of $S_j$, say

$$T_{1j} = f_1(S_j) \tag{33}$$

From (33) and (20), $T_{2j}$ can be uniquely determined as a function of $S_j$, say

$$T_{2j} = f_2(S_j) \tag{34}$$

Substituting (34) in (31) we find that $T_{3j}$ can be uniquely determined as a function of $S_j$ and $Q_j$, say

$$T_{3j} = f_3(S_j, Q_j) \tag{35}$$

from which and (21), $T_{4j}$ can be uniquely determined as a function of $S_j$ and $Q_j$, say

$$T_{4j} = f_4(S_j, Q_j) \tag{36}$$

Note that (34)–(36) have resulted from (direct) substitutions of the constraints $g_1 = 0$ and $g_2 = 0$. Now substituting these results in (18) we then obtain the following unconstrained optimization problem for the (FTL) case in which $S_j$ and $Q_j$ are our decision variables which we shall call it problem $(P_2)$

$$\begin{aligned}\text{Minimize } & L(S_j, Q_j)\\ &= \frac{1}{f_4}\left\{c\int_{f_1}^{f_3} P(u)\,\mathrm{d}u + h\left[-\int_{f_2}^{f_3}\gamma(u)(P(u)-D(u))\mathrm{e}^{\alpha(u)}\,\mathrm{d}u + \int_{f_3}^{f_4}\gamma(u)D(u)\mathrm{e}^{\alpha(u)}\,\mathrm{d}u\right]\right.\\ &\quad \left. + b\left[-\beta\int_0^{f_1} uD(u)\,\mathrm{d}u + \int_{f_1}^{f_2} u(P(u)-D(u))\,\mathrm{d}u\right] + s(1-\beta)\int_0^{f_1} D(u)\,\mathrm{d}u + k\right\}\end{aligned} \tag{37}$$

Letting

$$l(S_j, Q_j) = \frac{L(S_j, Q_j)}{f_4} \iff L(S_j, Q_j) = f_4 \cdot l(S_j, Q_j)$$

Then the necessary conditions for having a minimum are

$$\frac{\partial L}{\partial S_j} = 0 \quad \text{and} \quad \frac{\partial L}{\partial Q_j} = 0 \tag{38}$$

which are, respectively, equivalent to

$$l'S_j \cdot f_4 - f'_{4,S_j} \cdot l = 0 \tag{39}$$

and

$$l'_{Q_j} \cdot f_4 - f'_{4,Q_j} \cdot l = 0 \tag{40}$$

where $l'_x$ and $f'_{i,x}$ are the derivatives of $l$ and $f_i$ with respect to (w.r.t) $x$, respectively. But (39) and (40) lead to

$$l'Q_j \cdot f'_{4,S_j} = l'_{S_j} \cdot f'_{4,Q_j} \tag{41}$$

Taking the derivative of both sides of (21) w.r.t. $Q_j$ we have

$$f'_{3,Q_j}P(f_3)\mathrm{e}^{\alpha(f_3)} = f'_{4,Q_j}D(f_4)\mathrm{e}^{\alpha(f_4)}$$

from which and (33)–(37) we obtain

$$l'_{Q_j} = f'_{3,Q_j}P(f_3)\{c + h\mathrm{e}^{\alpha(f_3)}[\gamma(f_4) - \gamma(f_3)]\} \tag{42}$$

Also (33)–(37) lead to

$$\begin{aligned}l'S_j = {} & c[f'_{2,S_j}P(f_2) - f'_{1,S_j}P(f_1)] + h\{-f'_{3,S_j}\gamma(f_3)[P(f_3) - D(f_3)]\mathrm{e}^{\alpha(f_3)} + f'_{2,S_j}\gamma(f_2)[P(f_2) - D(f_2)]\mathrm{e}^{\alpha(f_2)}\\ & + f'_{4,S_j}\gamma(f_4)D(f_4)\mathrm{e}^{\alpha(f_4)} - f'_{3,S_j}\gamma(f_3)D(f_3)\mathrm{e}^{\alpha(f_3)}\} + b\{-\beta f'_{1,S_j}f_1D(f_1) + f'_{2,S_j}f_2[P(f_2) - D(f_2)]\\ & - f'_{1,S_j}f_1[P(f_1) - D(f_1)]\} + s(1-\beta)f'_{1,S_j}D(f_1)\end{aligned}$$

From (20), (33) and (21) we, respectively, have

$$\beta f'_{1,S_j} D(f_1) = f'_{2,S_j}[P(f_2) - D(f_2)] - f'_{1,S_j}[P(f_1) - D(f_1)], \quad f'_{1,S_j} D(f_1) = \frac{1}{\beta}$$

and

$$f'_{3,S_j} P(f_3) e^{\alpha(f_3)} - f'_{2,S_j}[P(f_2) - D(f_2)] e^{\alpha(f_2)} = f'_{4,S_j} D(f_4) e^{\alpha(f_4)}$$

Substituting in $l'_{S_j}$ we obtain (recall that $f'_{3,S_j} = f'_{2,S_j}$ by (31))

$$\begin{aligned} l'_{S_j} &= c[f'_{2,S_j} P(f_2) - f'_{1,S_j} P(f_1)] + h f'_{2,S_j} P(f_3) e^{\alpha(f_3)}[\gamma(f_4) - \gamma(f_3)] \\ &\quad + f'_{2,S_j}[P(f_2) - D(f_2)]\{b(f_2 - f_1) - h e^{\alpha(f_2)}[\gamma(f_4) - \gamma(f_2)]\} + \frac{s(1-\beta)}{\beta} \end{aligned} \tag{43}$$

From (41)–(43) we have

$$\begin{aligned} & f'_{4,S_j} f'_{3,Q_j} P(f_3)\{c + h e^{\alpha(f_3)}[\gamma(f_4) - \gamma(f_3)]\} \\ &\quad = f'_{4,Q_j}\Bigg( c[f'_{2,S_j} P(f_2) - f'_{1,S_j} P(f_1)] + h f'_{2,S_j} P(f_3) e^{\alpha(f_3)}[\gamma(f_4) - \gamma(f_3)] + f'_{2,S_j}[P(f_2) - D(f_2)] \\ &\qquad \times \{b(f_2. - f_1) - h e^{\alpha(f_2)}[\gamma(f_4) - \gamma(f_2)]\} + \frac{s(1-\beta)}{\beta}\Bigg) \end{aligned} \tag{44}$$

Also, (40) $\iff l = l'_{Q_j}.f_4/f'_{4,Q_j}$ or

$$L = \frac{l}{f_4} = \frac{l'_{Q_j}}{f'_{4,Q_j}} \tag{45}$$

where $L$ is given by (37) and $l'_{Q_j}$ is given by (42). Recalling (32) and the definition of $P(t)$ as introduced in (iv) Section 1, we have

$$P(f_1) = 1/t_{1j} \tag{46}$$

$$P(f_2) = \frac{S_j + \beta\int_{f_1}^{f_2} D(t)\,dt}{f_2} = \frac{\beta\int_0^{f_2} D(t)\,dt}{f_2} \tag{47}$$

and

$$P(f_3) = \frac{S_j + \beta\int_{f_1}^{f_2} D(t)\,dt + Q_j}{f_3} = \frac{\beta\int_0^{f_2} D(t)\,dt + Q_j}{f_3} \tag{48}$$

The two eqs. (44) and (45) can, now, be used to determine the optimal values of $Q_j$ and $S_j$. Then the minimum total cost can be determined from (45). This will be illustrated in the following example.

## 5. Illustrative example for the (FTL) case, numerical verification and concluding remarks

The application of the above theoretical results are illustrated in the following example. Consider a production lot size inventory model with linear demand rate function given by

$$D(t) = 2at + d \quad t \geqslant 0$$

The parameter "$a$" represents the rate of change in the demand rate. The case $a > 0$ implies an increasing demand rate, the case $a < 0$ implies a decreasing demand rate and the case a $= 0$ allows the possibility for a constant demand rate where then $D(t) = d,\ \forall t \geqslant 0$. Note also that $d = D(0)$ represent the demand rate at time $t = 0$. And a deterioration rate given by

$$\theta(t) = \frac{a_1}{b_1 - b_2 t}, \quad t \geqslant 0,\ \ b_1 \geqslant a_1 \geqslant 0 \ \text{ and } \ b_1 > b_2 \geqslant 0$$

Here $b_1$ is to be taken sufficiently larger than $b_2$ in order to keep $\theta(t) \geqslant 0$ (viz $b_1/b_2 \geqslant T_{4j}$). Note that $\theta(t)$ is an increasing function of $t$. The parameters $a_1$, $b_1$ and $b_2$ are just function parameters so that $a_1/b_1$ represent the deterioration rate at time $t = 0$. If $b_2 = 0$ then $\theta(t) = a_1/b_1,\ \forall t \geqslant 0$ which means that we have a constant rate of deterioration. If $a_1 = 0$ then $\theta(t) = 0,\ \forall t \geqslant 0$ which corresponds to the without deterioration case.

Next we calculate the theoretical functions $\alpha(t)$, $\gamma(t)$ and $S_j$, $Q_j$, $\int P(t)\,\mathrm{d}t$, $f_i$, $f'_{i,S_i}$, $f'_{i,Q_i}$ ($i = 1$, 2, 3 and 4) and $l$ as they defined in the previous sections for the above demand and deterioration rates.

Now, from (9) we have

$$\alpha(t) = \int \frac{a_1}{b_1 - b_2 t}\,\mathrm{d}t = \ln(b_1 - b_2 t)^{-a_1/b_2}$$

(All constants of the indefinite integrals are ignored since we are going to use them in the definite case.)

$$\mathrm{e}^{\alpha(t)} = \mathrm{e}^{\ln(b_1 - b_2 t) - a_1/b_2} = (b_1 - b_2 t)^{-a_1/b_2}$$

$$\gamma(t) = \int \mathrm{e}^{-\alpha(t)}\,\mathrm{d}t = \int \mathrm{e}^{\ln(b_1 - b_2 t)^{a_1/b_2}}\,\mathrm{d}t = \int (b_1 - b_2 t)^{a_1/b_2}\,\mathrm{d}t = c_1 (b_1 - b_2 t)^{(b_2 + a_1)/b_2}$$

$$\text{where } c_1 = \frac{-1}{a_1 + b_2}, \quad \gamma(t)\mathrm{e}^{\alpha(t)} = c_1 (b_1 - b_2 t)$$

Now, from (32) we have $S_j = \beta \int_0^{f_1} D(t)\,\mathrm{d}t = \beta[a f_1^2 + d f_1]$ from which we have

$$f_1 = \frac{-\beta d \pm \sqrt{\Delta_1}}{2\beta a} \tag{49}$$

where $\Delta_1 = \beta^2 d^2 + 4\beta a S_j$.

(Negative values of (49) are to be rejected.)

From (29) and (30) we have

$\int P(t_j)\,\mathrm{d}t_j = \int (1 - r)\,\mathrm{d}i = (1 - r)i$, from which, (20) and (32), we have

$$S_j = \int_{f_1}^{f_2} P(u)\,\mathrm{d}u - \int_{f_1}^{f_2} D(u)\,\mathrm{d}u$$

$$S_j = (1-r)(f_2 - f_1) - af_2^2 - df_2 + af_1^2 + df_1$$

or (recall that $af_1^2 + df_1 = 1/\beta S_j$ from above equations)

$$af_2^2 - (d+1-r)f_2 + (1-r)f_1 + S_j\left(1 - \frac{1}{\beta}\right) = 0$$

from which we have

$$f_2 = \frac{(d+1-r) \pm \sqrt{\Delta_2}}{2a} \tag{50}$$

where $\Delta_2 = (d+1-r)^2 - 4a\{(1-r)f_1 + S_j(1-(1/\beta))\}$.

(Values of $f_2 < f_1$ or $f_2 < 0$ are to be rejected.)

Substituting in (31) we obtain

$$f_3 = f_2 + t_{1j}\frac{Q_j^{1-r}}{1-r} \tag{51}$$

Rewriting (21) as

$$\int_{f_2}^{f_3} P(u)e^{\alpha(u)}\,du = \int_{f_2}^{f_4} D(u)e^{\alpha(u)}$$

or

$$(1-r)\int_{f_2}^{f_3}(b_1 - b_2u)^{\frac{-a_1}{b_2}}\,du = \int_{f_2}^{f_4}(2au+d)(b_1-b_2u)^{-\frac{a_1}{b_2}}\,du$$

To facilitate calculations we can assume (without loss of generality) that $-a_1/b_2 = m$ where $m$ is an integer value. For instance, let us consider (for our Example) that $m = -1$, then the last relation leads to

$$(1-r)\ln\frac{f_3}{f_2} = 2a(f_4 - f_2) + c_2\ln\frac{f_4}{f_2} \tag{52}$$

where $c_2 = (db_2 + 2ab_1)/b_2$ from which, (50) and (51) we can find $f_4$.

Now, from the above relations we have

$$f'_{1,S_j} = \pm\frac{1}{\sqrt{\Delta_1}}, \quad f'_{3,S_j} = f'_{2,S_j} = \pm\left\{\left(1-\frac{1}{\beta}\right) + (1-r)f'_{1,S_j}\right\}\Big/\sqrt{\Delta_2}$$

(The choice of + or − sign, here, must coincide with our choices in (49) and (50).) From (52) we have

$$f'_{4,S_j} = f'_{2,S_j}\frac{(1-r)(f_2-f_3) + (2af_2+c_2)f_3}{f_3f_2(2af_4+c_2)}\cdot f_4$$

$f'_{3,Q_j} = f'_{2,Q_j} + t_{1j}Q_j^{-r} = t_{1j}Q_j^{-r}$ (Since $f'_{2,Q_j} = 0$ by (31)). Again (52) implies

$$f'_{4,Q_j} = f'_{3,Q_j}\frac{(1-r)f_4}{f_3(2af_4+c_2)}$$

From (47) and (48) we, respectively, have

$$P(f_2) = \beta(af_2 + d), \quad \text{and} \quad P(f_3) = \frac{\beta(af_2^2 + df_2) + Q_j}{f_3}$$

Finally, from (37) and with some algebra we find

$$\begin{aligned} l = {} & \frac{-2ab_2c_1h}{3}f_4^3 + \frac{c_1h(2ab_1 - db_2)}{2}f_4^2 + c_1b_1dhf_4 + \frac{c_1h(1-r)}{3}f_3^2 + c(1-r)f_3 \\ & + \left(\frac{2ab_2c_1h}{3} - \frac{2ab}{3}\right)f_2^3 + \left[-\frac{c_1h(2ab_1 - db_2)}{2} - \frac{c_1h(1-r)}{2b_2} + \frac{b(1-r)}{2} - \frac{db}{2}\right]f_2^2 \\ & - c_1b_1dhf_2 + \left[\frac{2ab}{3} - \frac{2\beta ab}{3}\right]f_1^3 + \left[\frac{bd}{2} - \frac{(1-r)b}{2} - \frac{\beta db}{2} + s(1-\beta)a\right]f_1^2 \\ & + [s(1-\beta)d - c_1(1-r)]f_1 + k \end{aligned} \tag{53}$$

Now, the above results are to be substituted in (44) and (45) in order to get the solution of the given example.

### 5.1. Numerical verification

The above illustrative example has been verified for a wide range of the model parameters from which we have chosen the following set of values.

| | |
|---|---|
| The slope of the learning curve | $r = 0.075$ |
| Time required to produce the first unit in the first cycle | $t_{11} = 0.0015$ year |
| Percentage of backordered items | $\beta = 0.85$ |
| Unit production cost | $c = \$50$ |
| Unit shortage cost per year | $s = b = \$0.5$ |
| Unit holding cost per year | $h = \$0.1$ |
| Set-up cost per set-up | $k = \$200$ |
| Parameters of demand rate | $a = 250$ units/year and $d = 125$ units/year |
| Parameters of deterioration rate | $a_1 = b_2 = 10$ units/year and $b_1 = 1000$ units/year |

Note that both demand and deterioration rates are increasing functions of time. A Nonlinear Programming Package has been used to determine the optimal values of $S_j$, $Q_j$, $T_{1j}$, $T_{2j}$, $T_{3j}$, $T_{4j}$, and the corresponding total minimum cost for five successive cycles. Further, the total number of units (say $Q$) produced in each cycle, the time required for their production and the production rate are also calculated. The results are shown in Table 1. In the first cycle we have taken $t_{11} = 0.0015$ year which results in a total number of $Q = 134.35$. Then from (28) we found that the time required to produce the unit 135.35 (which is the first unit to be produced in the second cycle) is equal to $t_{135.35} = 0.0015(135.35)^{-0.075} = 0.001038$. The same procedure is repeated for the other cycles. The tabulated results indicate that the time required to produce the first unit in a cycle ($t_{1j}$ in column 2) decreases with the number of the cycle which is also reflected by an increasing production rate as shown in column 7 in Table 1. Such increase is, in fact, consistent with the large

Table 1
Optimal results under FTL for the illustrative example with: $r = 0.075$, $t_{11} = 0.0015$, $\beta = 0.85$, $a = 250$, $d = 125$, $a_1 = b_2 = 10$, $b_1 = 1000$, $c = 50$, $s = b = 0.5$, $h = 0.1$ and $k = 200$

| Cycle no. $j$ | Time required to produce the first unit $t_{1j}$ | No. of shortage units $S_j$ | No. of units produced in $[T_1, T_3]$ $Q_j$ | Total number of units produced $Q$ | Time required to produce $Q$ units | Production rate | Production start time $T_{1j}$ | End of shortage period $T_{2j}$ | End of production period $T_{3j}$ | Cycle end $T_{4j}$ | Minimum total cost |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.0015 | 81.37 | 29.55 | 134.35 | 0.150849 | 890.63 | 0.417398 | 0.537310 | 0.568247 | 0.892341 | 6784.41 |
| 2 | 0.001038 | 83.41 | 21.42 | 125.38 | 0.097843 | 1281.44 | 0.424561 | 0.506839 | 0.522494 | 0.872341 | 6427.53 |
| 3 | 0.000722 | 84.94 | 20.82 | 122.91 | 0.066876 | 1837.88 | 0.429875 | 0.486355 | 0.496751 | 0.859469 | 6379.62 |
| 4 | 0.000503 | 84.98 | 20.53 | 120.98 | 0.045914 | 2634.93 | 0.430012 | 0.468673 | 0.475926 | 0.849959 | 6332.22 |
| 5 | 0.000351 | 87.58 | 19.52 | 120.18 | 0.031816 | 3774.20 | 0.438946 | 0.465949 | 0.470765 | 0.849063 | 6283.63 |

increase of the demand rate ($a = 250$ and $d = 125$) and the large decrease in the time $(t_{1j})$ required to produce the first unit for successive cycles. Note that the results show some kind of settlement in the number of units produced ($Q$ in column 5) and a decrease in the minimum of the total relevant cost (last column) which coincides with the main objective.

To sum up, in this paper, we have formulated and solved a (GPLS) model for two cases. The first (in general) the effects of learning are not considered. In the second (FTL) is incorporated. In both cases, the production, demand and deterioration rates are functions of time. Shortages are allowed but are partially backlogged. Though the problem in the (FTL) case is a constrained problem as in the general case, but it has been reduced to unconstrained one of two decision variables. Namely $S_j$ and $Q_j$ as they defined above. An illustrative example for the FTL case which explains the application of the theoretical results and a numerical verification of this illustrative example are also given. The numerical results clearly reflected the incorporated learning effects in the proposed model. This seems to be the first time where such models are introduced and verified.

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