Weighted $\text{BMO}$ and Carleson measures on spaces of homogeneous type

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Received 4 May 2007
Available online 31 December 2007
Submitted by R.H. Torres

Abstract
In this paper we characterize the weighted $\text{BMO}(\omega)(X)$, with $X$ a space of homogeneous type, through an adequate weighted Carleson measure. As a byproduct we can define the weighted Triebel–Lizorkin space $\dot{F}^{0,2}_{\infty}(\omega)(X)$ and obtain the identification with the above space.

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Keywords: Carleson measures; BMO; Weights; Triebel–Lizorkin; Spaces of homogeneous type

1. Introduction

Given a weight $\omega$ on a space of homogeneous type $(X, d, \mu)$ (see the definitions below) let us consider the space $\text{BMO}(\omega) = \text{BMO}(\omega)(X)$ of functions whose oscillation, when averaged over balls, is controlled by $\omega$, measuring their degree of smoothness. More precisely, a locally integrable function $f$ belongs to $\text{BMO}(\omega)$ if there is a constant $C$ such that the inequality

$$\frac{1}{\omega(B)} \int_B |f(y) - m_B(f)| \, d\mu(y) \leq C \tag{1.1}$$

holds for every ball $B \subset X$, where $m_B(f)$ denotes the average of $f$ over $B$ with respect to the measure $\mu$, and $\omega(B) = \int_B \omega(x) \, d\mu(x)$.

If we set $\|f\|_{\text{BMO}(\omega)}$ as the infimum of the constants $C$ appearing in (1.1), $\text{BMO}(\omega)$ becomes a Banach space modulo constants.

* The authors were supported by Consejo Nacional de Investigaciones Científicas y Técnicas de la República Argentina and Universidad Nacional del Litoral, Santa Fe, Argentina.
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The first appearance of this kind of weighted spaces goes back to [7] and [14]. In particular, the last authors introduced $\text{BMO}(\omega)(\mathbb{R}^n)$ as the natural space where weighted $L^\infty$ functions are mapped by the Hilbert transform on the line, generalizing the well-known $\text{BMO}$ space of John and Nirenberg.

For the case $\omega = 1$ Fefferman and Stein showed in [6] the tight connection between $\text{BMO}(\mathbb{R}^n)$ and Carleson measures. Let us recall that a measure $v$ on $X^+ := X \times (0, \infty)$ is said to be a Carleson measure if there is a constant $C$ such that for any ball $B(x_0, r) \subseteq X$,

$$v(B(x_0, r) \times (0, r)) \leq C \mu(B(x_0, r)).$$

Let $[d\nu]_C$ denote the infimum of the constants $C$ appearing in the above inequality. A modern statement of the result of Fefferman and Stein is in the book by Stein (see [15, Theorem 3, p. 159]) as follows:

**Theorem 1.1.** Let $\phi \in S(\mathbb{R}^n)$ with $\int \phi = 0$.

(a) Suppose $f \in \text{BMO}(\mathbb{R}^n)$, then

$$d\nu_f = \left| f \ast \phi_t(x) \right|^2 \frac{dx \, dt}{t},$$

is a Carleson measure and $[d\nu_f]_C \leq C \| f \|_{\text{BMO}}^2$.

(b) Conversely, suppose $\phi$ is non-degenerate. If

$$\int \frac{|f(x)|}{1 + |x|^{n+1}} \, dx < \infty$$

and $d\nu_f$ defined in (1.2) is a Carleson measure, then $f \in \text{BMO}$ and $\| f \|_{\text{BMO}}^2 \leq C[d\nu_f]_C$.

Here $\phi$ non-degenerate means that $\hat{\phi}$ does not vanish identically on any ray emanating from the origin, that is, for every $\xi \neq 0$ there exists $t > 0$ with $\phi(t\xi) \neq 0$ and, as usual $\phi_t(x) = t^{-n} \phi(t^{-1}x)$.

It is a well-known result that the homogeneous Triebel–Lizorkin space $F^{0,2}_\infty(\mathbb{R}^n)$, defined as the family of distributions, $f \in S'/\mathcal{P}— S'$ the set of tempered distributions and $\mathcal{P}$ the set of polynomials—such that $d\nu_f$ defined as in (1.2) is a Carleson measure ($\phi$ as in Theorem 1.1), coincides with $\text{BMO}(\mathbb{R}^n)$ (see [4]).

As it was pointed out in [12], even though the statement in Theorem 1.1 is very close to the above identification, part (b) of the theorem should be proved under the more general setting of distributions in $S'/\mathcal{P}$ instead of the integrability condition (1.3).

Harbour, Salinas and Viviani extended the result in Theorem 1.1 to the more general weighted spaces $\text{BMO}_\varphi(\omega)(\mathbb{R}^n)$ [12, Theorem 2.5], that is, the space of functions $f$ such that

$$\frac{1}{\omega(B)} \int_B \left| f(y) - m_B(f) \right| d(y) \leq C \varphi(|B|)$$

holds for every ball $B \subset \mathbb{R}^n$ and some constant $C$, under appropriate assumptions of the function $\varphi$ and the weight $\omega$. Their main result was proved for distributions in $S'/\mathcal{P}$, removing in this way the integrability condition (1.3), and obtained as a corollary, when $\varphi = 1$, the identification between $\text{BMO}_\varphi(\omega)(\mathbb{R}^n)$ and a weighted version of $F^{0,2}_\infty(\mathbb{R}^n)$. To prove their main theorem the authors established a duality inequality between generalized Carleson measures and tent spaces by means of an adequate atomic decomposition of those spaces.

We remark that Bui and Taibleson defined in [1] weighted $F^{s}_{2q}$ spaces in the Euclidian setting. However, as it is shown in [12], for $s = 0$ and $q = 2$, their definition does not coincide with the usual weighted space $\text{BMO}(\omega)$ since, at least for weights in the Muckenhoupt class $A_1$, is the un-weighted $\text{BMO}$ space.

The aim of this work is to prove in the context of spaces of homogeneous type the characterization of the weighted space $\text{BMO}_\varphi(\omega)(X)$ through a Carleson measure related to the weight and, then, define a weighted version of the Triebel–Lizorkin space, say $F^{02}_\infty(\omega)(X)$ and obtain, as a consequence of the above result, the identification between both spaces. Our proof recovers the stopping-time technique in the book of Stein [15] which makes it different from the one given in [12] using atomic decomposition on tent spaces. Our main result not only generalizes Theorem 1.1 to the new environment of spaces of homogeneous type, but also is achieved under the general setting of distributions,
thus removing condition (1.3). In the process we prove in Section 4 norm inequalities for a square non-tangential function which are of independent relevance.

This paper is organized in the following way: in Section 2 are the previous definitions of the main theorem. Section 3 is devoted to the definition of an atomic space whose dual is imbedded in BMO function which are of independent relevance.

We mention that the constants C appearing along the proofs may change from step-to-step.

2. Previous definitions and main theorem

Given a set $X$ and a real valued function $d(x, y)$ defined on $X \times X$, we say that $d$ is a quasi-distance on $X$ if there exists a positive constant $A$ such that for all $x, y, z \in X$ it verifies

$$d(x, y) \geq 0 \quad \text{and} \quad d(x, y) = 0 \quad \text{if and only if} \quad x = y,$$

$$d(x, y) = d(y, x),$$

$$d(x, y) \leq A[d(x, z) + d(z, y)].$$

In a set $X$ endowed with a quasi-distance $d(x, y)$, the balls $B_d(x, r) = \{y: d(x, y) < r\}$ form a basis of neighborhoods of $x$ for the topology induced by the uniform structure on $X$.

Let $\mu$ be a positive measure on a $\sigma$-algebra of subsets of $X$ which contains the open set and the balls $B_d(x, r)$. The triple $(X, d, \mu)$ is a space of homogeneous type if there exists a finite constant $K > 0$ such that

$$\mu(B_d(x, 2Ar)) \leq K \mu(B_d(x, r)),$$

for all $x \in X$ and $r > 0$. Macías and Segovia in [13] showed that it is always possible to find a quasi-distance $\delta(x, y)$ equivalent to $d(x, y)$ and $0 < \theta \leq 1$, such that

$$|\delta(x, y) - \delta(x', y)| \leq Cr^{1-\theta}\delta(x, x')^\theta \tag{2.1}$$

holds whenever $\delta(x, y) < r$ and $\delta(x', y) < r$. If $d$ satisfies (2.1), then $X$ is said to be of order $\theta$. Furthermore, $X$ is a normal space if $A_1r \leq \mu(B_d(x, r)) \leq A_2r$ for every $x \in X$ and $r > 0$ and some positive constants $A_1$ and $A_2$.

In this work $X := (X, d, \mu)$ means a normal space of homogeneous type of order $\theta$. $A$ denotes the constant of the triangular inequality associated to $d$ and $K$ the duplication constant associated to the measure $\mu$.

The class of test functions on $X$, the notion of approximation to the identity and the derived operators that we consider in the sequel are defined as in [2,9,10].

Given $0 < \beta \leq \theta, \gamma > 0$, $x_0 \in X$ and $l > 0$, a function $f$ defined in $X$ is a smooth molecule of type $(x_0, l, \beta, \gamma)$, if there exists a constant $C > 0$ such that

$$|f(x)| \leq Cl^{\gamma\theta}/(l + d(x, x_0))^{1+\gamma\theta}, \tag{2.2}$$

$$|f(x) - f(x')| \leq C\left(d(x,x')/l + d(x, x_0)\right)^{\beta\gamma\theta}/(l + d(x, x_0))^{1+\gamma\theta}, \tag{2.3}$$

for $d(x, x') \leq \sum_{l=1}(l + d(x, x_0))$, and

$$\int f(x)d\mu(x) = 0 \tag{2.4}$$

hold for every $x \in X$. The set $\mathcal{M}(x_0, l, \beta, \gamma)$ of all smooth molecules of type $(x_0, l, \beta, \gamma)$ is a Banach space with the norm $\|f\|_{\mathcal{M}(x_0,l,\beta,\gamma)} = \inf[C: 2.2 \text{ and } 2.3 \text{ hold}].$ Fixing $x_0 \in X$ and $l = 1$, it is easy to see that the space $\mathcal{M}(\beta, \gamma) = \mathcal{M}(x_0, l, \beta, \gamma)$ coincides with $\mathcal{M}(x_1, r, \beta, \gamma)$ with equivalence of norms for all $x_1 \in X$ and $r > 0$. Furthermore, $\mathcal{M}(\beta, \gamma)$ is a Banach space.

It is known that the space $\mathcal{M}(\beta_1, \gamma)$ is not dense in $\mathcal{M}(\beta_2, \gamma)$ if $\beta_1 > \beta_2$. To overcome this problem in [11] the authors considered the space $\mathcal{M}(\beta, \gamma)$ which is the completion of $\mathcal{M}(\epsilon, \epsilon)$, for a fix $\epsilon \leq \theta$, in $\mathcal{M}(\beta, \gamma)$ when $0 < \beta, \gamma < \epsilon$. 

As usual, the dual space \((\mathcal{M}(\beta, \gamma))^\prime\) is the set of all continuous linear functionals \(\mathcal{L} : \mathcal{M}(\beta, \gamma) \to \mathbb{C}\). The pairing \(\langle \mathcal{L}, h \rangle\) will denote the natural application of \(\mathcal{L}\) to an element \(h \in \mathcal{M}(\beta, \gamma)\). By \(\|\mathcal{L}\|_{(\mathcal{M}(\beta, \gamma))^\prime}\) we mean the infimum of the constants \(C\) such that \(\|\langle \mathcal{L}, h \rangle\| \leq C\|h\|_{\mathcal{M}(\beta, \gamma)}\) for all \(h \in \mathcal{M}(\beta, \gamma)\).

A family \(\{S_t\}_{t>0}\) of linear integral operators is said to be an approximation to the identity of order \(\epsilon \in (0, \theta]\) if there exists \(C > 0\) such that for all \(t > 0\) and all \(x, x', y, y' \in X\), the kernel \(s(t, x, y)\) of \(S_t\) is a function from \(\mathbb{R}^+ \times X \times X\) into \(\mathbb{C}\) differentiable in the variable \(t\) and also satisfying

\[
|s(t, x, y)| \leq C \frac{t^\epsilon}{(t + d(x, y))^{1+\epsilon}}; \tag{2.5}
\]

\[
|s(t, x, y) - s(t, x', y)| \leq C \frac{d(x, x')^\epsilon}{(t + d(x, y))^{1+\epsilon}} \frac{t^\epsilon}{(t + d(x, y))^{1+\epsilon}}; \tag{2.6}
\]

for \(d(x, x') \leq \frac{1}{\epsilon^\delta} (t + d(x, y))\);

\[
|s(t, y, x) - s(t, y, y')| \leq C \frac{d(x, x')^\epsilon}{(t + d(x, y))^{1+\epsilon}} \frac{t^\epsilon}{(t + d(x, y))^{1+\epsilon}}; \tag{2.7}
\]

for \(d(x, x') \leq \frac{1}{\epsilon^\delta} (t + d(x, y))\);

\[
\left| \left[ s(t, x, y) - s(t, x', y) \right] - \left[ s(t, x, y') - s(t, x', y') \right] \right| \leq C \frac{d(x, x')^\epsilon}{(t + d(x, y))^{1+\epsilon}} \frac{d(y, y')^\epsilon}{(t + d(x, y))^{1+\epsilon}} \frac{t^\epsilon}{(t + d(x, y))^{1+\epsilon}}; \tag{2.8}
\]

for \(d(x, x') \leq \frac{1}{\epsilon^\delta} (t + d(x, y))\) and \(d(y, y') \leq \frac{1}{\epsilon^\delta} (t + d(x, y))\); and

\[
\int s(t, x, y) d\mu(y) = \int s(t, y, x) d\mu(y) = 1. \tag{2.9}
\]

That these kind of approximations to the identity actually exist it was shown in [3] in the discrete case and in [8] in the continuous one. Associated to the above approximation to the identity is the family

\[
\left\{ Q_t = -t \frac{d}{dt} S_t \right\}_{t>0} \tag{2.10}
\]

of integral operators satisfying \(\int_0^\infty Q_t d\frac{dt}{t} = I\) in \(L^2\), that is,

\[
\lim_{\delta \to 0} \left\| \frac{1}{\delta} \int_0^\delta Q_t f \frac{dt}{t} - f \right\|_2 = 0.
\]

Moreover, the kernel associated to \(Q_t, q_t(x, y) = -t \frac{d}{dt^3}(t, x, y)\) satisfies properties (2.5)–(2.8) and

\[
\int q_t(x, y) d\mu(y) = \int q_t(y, x) d\mu(y) = 0. \tag{2.11}
\]

Along this work the positive number \(\epsilon \leq \theta\) will denote the order of \(\{S_t\}_{t>0}\) and \(\{Q_t\}_{t>0}\).

Given \(f \in (\mathcal{M}(\beta, \gamma))^\prime\) the distribution \(Q_t f\) is well defined by \(\langle Q_t f, g \rangle = \langle f, Q_t^* g \rangle\) for all \(g \in \mathcal{M}(\beta', \gamma')\), \(0 < \beta', \gamma'\) and, moreover, \(Q_t f(y) = \{f, q_t(y, \cdot)\}\) is a smooth molecule of order \(\epsilon\).

A non-negative function \(\omega\) defined on a space of homogeneous type \(X\) is a weight in the Muckenhoupt class \(A_q\), \(q > 1\), if there exists a constant \(C\) such that for any ball \(B \subseteq X\),

\[
\left( \frac{1}{\mu(B)} \int_B \omega d\mu \right) \left( \frac{1}{\mu(B)} \int_B \omega^{-\frac{1}{q-1}} d\mu \right)^{q-1} \leq C,
\]

and \(\omega \in A_1\) if

\[M \omega(x) \leq C \omega(x),\]
a.e., where \( M \) is the Hardy–Littlewood maximal operator. A weight is in the class \( A_\infty \) if it belongs to some \( A_q, q \geq 1 \). It is easy to prove using Hölder’s inequality and the \( A_q \) condition that if \( \omega \) is a weight in \( A_q, 1 \leq q < \infty \), then there exists a constant \( C \) such that

\[
\left( \frac{\mu(E)}{\mu(B)} \right)^q \leq C \frac{\omega(E)}{\omega(B)} \tag{2.13}
\]

holds for every measurable set \( E \subset B \) and every ball \( B \) in \( X \).

Our definition of weighted Carleson measure on spaces of homogeneous type relies on the given in [12] in the Euclidian context and for \( \varphi = 1 \):

Given a weight \( \omega \) in \( A_\infty \), a measure \( dv \) on \( X^+ := X \times (0, \infty) \) is an \( \omega \)-Carleson measure, if there is a constant \( C \) such that

\[
\int_{T(B)} |dv| \leq C \omega(B), \tag{2.14}
\]

for any ball \( B \subset X \). Here \( T(B) \) means the tent over \( B = B(x_B, r) \), that is, \( T(B) = \{ (y, t) : d(y, x_B) + t < r \} \).

The infimum of the constants \( C \) appearing in (2.14) will be denoted by \([dv]_\omega\).

We now state the main theorem:

**Theorem 2.1.** Let \( X \) be a normal space of homogeneous type of order \( \theta \), \( \{Q_t \}_{t>0} \) be a family as in (2.10), \( \omega \) be a weight in \( A_q \) with \( q < 1 + \epsilon \). For any \( 0 < \beta, \gamma < \epsilon \) the following statements hold:

(a) If \( f \in (\mathcal{M}(\beta, \gamma))' \) is such that

\[
dv^Q := |Q_t f(y)|^2 \frac{\mu(B(y, t))}{\omega(B(y, t))} d\mu(y) \frac{dt}{t} \tag{2.15}
\]

is an \( \omega \)-Carleson measure, then \( f \in BMO(\omega) \) and \( \|f\|_{BMO(\omega)} \leq C [dv^Q]_{\omega}^{1/2} \).

(b) If \( f \in BMO(\omega) \), then \( f \in (\mathcal{M}(\beta, \gamma))' \) and \( dv^Q \) is an \( \omega \)-Carleson measure. Moreover \( [dv^Q]_{\omega}^{1/2} \leq C \|f\|_{BMO(\omega)} \).

In view of the previous theorem we can introduce a weighted version of the Triebel–Lizorkin space \( F^{0, 2}_\infty \) on \( X \) as the set of distributions \( f \in (\mathcal{M}(\beta, \gamma))' \), with \( 0 < \beta, \gamma < \epsilon \) such that \( dv^Q \) defined in (2.15) is an \( \omega \)-Carleson measure for some family \( \{Q_t \}_{t>0} \) as in (2.11), (5.12)–(5.14). In this space, say \( F^{0, 2}_\infty (\omega) \), we can define a norm as

\[
\|f\|_{F^{0, 2}_\infty (\omega)} = [dv^Q]_{\omega}^{1/2}. \tag{2.16}
\]

The good definition of the above norm (modulo constants) relies on its independence of the choice of the family \( \{Q_t \}_{t>0} \) which follows immediately from the identification between \( F^{0, 2}_\infty (\omega) \) and \( BMO(\omega) \), with equivalence of norms, obtained from definition (2.16) and our main Theorem 2.1.

3. The atomic space \( H^1_q (\omega) \) and \( BMO(\omega) \)

Given a weight \( \omega \in A_q, 1 < q < \infty \) we say that a function \( a \) is a \((q, \omega)\)-atom, if \( a \) is supported in a ball \( B \), has zero average and

\[
\omega(B)^{1/q} \|a\|_{L^q(\omega)} \leq 1. \tag{3.1}
\]

We thus define the atomic space \( H^1_q (\omega) \) as the set of distributions \( f \in (\mathcal{M}(\beta, \gamma))' \) which can be written—in the distribution sense—as \( f = \sum_{j \in J} b_j \) with \( \{b_j\}_{j \in J}, J \subset N \), a sequence of multiples of \((q, \omega)\)-atoms such that the quantity

\[
\Lambda(\{b_j\}) := \sum_{j \in J} \omega(B_j)^{1/q} \|b_j\|_{L^q(\omega)} < \infty,
\]
where $B_j$ is a ball containing the support of $b_j$. A quasi-norm in this space is defined by

$$[f]_{H^1_q(\omega)} = \inf \Lambda(\{b_j\})$$

where the infimum is taken over all the decompositions of $f$. Notice that a function $f \in L^q(\omega)$ supported in $B_j$ and with zero average belongs to $H^1_q(\omega)$ if and it satisfies (3.1), then $[f]_{H^1_q(\omega)} \leq 1$.

We will need the following alternative characterization on spaces of homogeneous type of the space $BMO(\omega)$ defined in (1.1). Its proof follows from [5, Theorem 2.3, p. 113] taking $a(B) = \omega(B)/\mu(B)$.

**Lemma 3.2.** Let $X$ be a space of homogeneous type, $1 \leq p < \infty$ and $\omega \in A_p$. Then $f \in BMO(\omega)$ if and only if there exists a constant $C$ such that

$$\frac{1}{\omega(B)} \int_B |f(y) - m_B(f)|^r \omega(y)^{1-r} \, d\mu(y) \leq C,$$

for all ball $B$ and each $1 \leq r \leq p'$, $r < \infty$. Moreover, $\|f\|_{BMO(\omega)}^r$ is equivalent to the infimum of the constants $C$ appearing in (3.3).

With the above lemma we can prove the following standard result.

**Lemma 3.4.** Let $L$ be a functional in the dual of $H^1_q(\omega)$, then there exists $h \in BMO(\omega)$ such that

$$L(f) = \int h(x) f(x) \, d\mu(x),$$

for any $f \in L^q(\omega)$ with compact support and zero average. Moreover

$$\|h\|_{BMO(\omega)} \leq [\mathcal{L}] := \inf \left\{ C : \sup_{[f]_{H^1_q(\omega)} \neq 0} \frac{|\mathcal{L}(f)|}{[f]_{H^1_q(\omega)}} \leq C \right\}. \tag{3.5}$$

**Proof.** For any ball $B$, $\mathcal{L}$ defines a bounded linear functional on $L^q_0(B, \omega)$, the subspace of functions in $L^q(\omega)$ with zero average and support in $B$. In fact, for such a function $f$ we have

$$|\mathcal{L}(f)| \leq C[f]_{H^1_q(\omega)} \leq C \omega(B)^{1/q'} \|f\|_{L^q(\omega)}.$$

By the M. Riesz representation theorem we know that there exists a function $h_B \in L^q(\omega)$ with support in $B$ such that

$$\mathcal{L}(f) = \int_B h_B f = \int_B \left( h_B - m_B(h_B) \right) f, \quad f \in L^q_0(B, \omega).$$

Moreover,

$$\left( \frac{1}{\omega(B)} \int_B |h_B - m_B(h_B)|^q \omega^{1-q} \right)^{1/q'} \leq C. \tag{3.6}$$

Considering now an increasing sequence of balls, a function $h$ may be defined modulo constants satisfying (3.6) for any ball. Since $\omega$ is in $A_q$, by Lemma 3.2, such inequality implies $h \in BMO(\omega)$ and gives an equivalent norm. In this way, (3.5) is obtained. □

The next estimates follow easily from (2.5) to (2.7) and will be useful later: if $\omega \in A_q$, $q > 1$, $f \in L^q(\omega)$ has null mean, $\text{supp } f \subset B = B(x_0, r)$ and $t + d(y, x_0) > 2Ar$, then

$$\max(|S_t f(y)|, |Q_t f(y)|) \leq C \frac{t^e}{(t + d(y, x_0))^{1+\epsilon} \|f\|_1} \leq C \frac{t^e}{(t + d(y, x_0))^{1+\epsilon} \left( \omega^{-q'/q}(B) \right)^{1/q'} \|f\|_{L^q(\omega)}} \leq C \frac{t^e}{(t + d(y, x_0))^{1+\epsilon} \mu(B) \omega^{-1/q}(B)^{1/q} \|f\|_{L^q(\omega)}}, \tag{3.7}$$
the last estimate follows from (2.12) and (2.6). Then
\[
\max\left(\|S_t f(y)\|, \|Q_t f(y)\|\right) \leq C \frac{r^\epsilon t^\epsilon}{(t + d(y, x_0))^{1+2\epsilon}} \mu(B)\left(\omega(B)\right)^{-1/q} \|f\|_{L^q(\omega)}.
\] (3.8)

**Lemma 3.9.** The molecular space \(\hat{\mathcal{M}}(\beta, \gamma)\) is a dense subspace of \(H^1_q(\omega)\) for \(0 < \beta, \gamma \leq \epsilon, \omega \in A_q\) and \(1 < q < 1 + \epsilon\).

**Proof.** The proof follows from the idea in [12] and [16]. Given a ball \(B_0 = B(x_0, r)\), a function \(g \in L^q(\omega)\) with zero integral can be split, pointwise and in the sense of \(\hat{\mathcal{M}}(\beta, \gamma)\) as
\[
g = \sum_{k \geq 0} (g - m_k) \chi_{E_k} + \sum_{k \geq 0} \beta_k R_k.
\] (3.10)

where \(E_0 = B_0, E_k = B(x_0, (2A)^k r) - B(x_0, (2A)^{k-1} r) = B_k - B_{k-1}, m_k = \mu(E_k)^{-1} \int_{E_k} g, \beta_k = \sum_{i \geq k+1} m_i \mu(E_i) = \int_{B_k} g\) and, finally, \(R_k = \mu(E_{k+1})^{-1} \chi_{E_{k+1}} - \mu(E_k)^{-1} \chi_{E_k}\).

Clearly, each term in (3.10) is a multiple of an atom. Moreover, if \(g \in \hat{\mathcal{M}}(\beta, \gamma)\), then it belongs to \(L^q(\omega)\) and has zero average and by the above decomposition \(g \in H^1_q(\omega)\). Therefore \(\hat{\mathcal{M}}(\beta, \gamma)\) is a subspace of \(H^1_q(\omega)\).

Also by (3.10), to show the density of the molecular space in the atomic space it is enough to approximate functions of \(L^q(\omega)\) with compact support and zero average by molecules in the quasi-norm of \(H^1_q(\omega)\).

Let then \(b\) be such a function and \(\{S_t\}_{t>0}\) be an approximation to the identity of order \(\epsilon\). It is easy to prove from properties (2.5) to (2.9) that \(S_t b - S_1 b\) belongs to \(\mathcal{M}(\epsilon, \epsilon)\) and then to \(\hat{\mathcal{M}}(\beta, \gamma)\) for \(0 < t < \infty\) and \(0 < \beta, \gamma \leq \epsilon\).

Moreover, we will show next that
\[
\lim_{t \to 0} \|S_t b - b\|_{H^1_q(\omega)} = 0 \quad \text{and} \quad \lim_{t \to 0} \|S_1 b\|_{H^1_q(\omega)} = 0.
\] (3.11)

It is worth mentioning that the above results mean that
\[
\lim_{t \to 0} \left\| \int \frac{Q_t b}{u} \right\|_{H^1_q(\omega)} = 0,
\] (3.12)

for \(b \in L^q(\omega)\) with null mean.

To prove (3.11) we first apply the decomposition (3.10) to \(g = S_t b - b\), with \(B_0 = (2A)B\) for \(B\) a ball of radius \(r\) containing the support of \(b\). We denote \(m_k^i\) and \(\beta_k^i\) the corresponding coefficients.

Given \(x \in E_k, k \geq 1\), from (3.7) and \(\omega \in A_q\) it follows that
\[
\|S_t b(x)\| \leq C \|b\|_{L^q(\omega)} \frac{t^\epsilon}{(t + (2A)^k r)^{1+\epsilon}} \mu(B_0)^{-1/q} \omega(B_0).
\]

In this way,
\[
\|S_t b - b - m_k^i \chi_{E_k}\|_{L^q(\omega)} = \|S_t b \chi_{E_k}\|_{L^q(\omega)} \leq C(b, \omega, B_0) \frac{t^\epsilon}{(t + (2A)^k r)^{1+\epsilon}} \omega(B_k)^{1/q}, \quad k \geq 1.
\] (3.13)

Furthermore, for \(k = 0\) since \(\{S_t\}_{t>0}\) is an approximation to the identity and \(\omega \in A_q\), then
\[
\|S_t b - b\chi_{E_0}\|_{L^q(\omega)} \to 0 \quad \text{when} \; t \to 0.
\] (3.14)

Since \(\omega \in A_q\), then
\[
\omega(B_k) \leq C \left(\frac{\mu(B_k)}{\mu(B_0)}\right)^q \omega(B_0);
\] (3.15)

thus setting \(h_k^i = (S_t b - b - m_k^i \chi_{E_k}\), it follows that
\[ \sum_{k=0}^{\infty} \omega(B_k)^{1/q} \left\| h_k^t \right\|_{L^q(\omega)} \leq 2\omega(B_0)^{1/q} \left\| (S_1 b - b) \chi_{E_0} \right\|_{L^q(\omega)} + C(b, \omega, B_0)t^\epsilon \sum_{k=1}^{\infty} \frac{\omega(B_k)}{((2A)^k r)^{1+\epsilon}} \]

\[ \leq C(b, \omega, B_0) \left( \left\| (S_1 b - b) \chi_{E_0} \right\|_{L^q(\omega)} + t^\epsilon \sum_{k=1}^{\infty} \frac{1}{((2A)^k r)^{1+\epsilon-q}} \right) ; \]

thus

\[ \Lambda(\{h_k^t\}) \to 0 \text{ for } t \to 0 \text{ and } 1 < q < 1 + \epsilon. \quad (3.16) \]

Also, by (3.7)

\[ |\beta_k^t| = \left| \int_{B_k^t} S_1 b(y) d\mu(y) \right| \leq C(b, \omega, B) \frac{t^\epsilon}{((2A)^k r)^\epsilon}. \]

From the above inequality, the definition of \( R_k \) and (3.15) it follows that

\[ \sum_{k \geq 0} \omega(B_k)^{1/q} \left\| \beta_k^t R_k \right\|_{L^q(\omega)} \leq C t^\epsilon \sum_{k \geq 0} \frac{1}{((2A)^k r)^\epsilon} \mu(B_k) \leq C t^\epsilon \sum_{k=1}^{\infty} \frac{1}{((2A)^k r)^{1+\epsilon-q}}. \]

In this way,

\[ \Lambda(\{\beta_k^t R_k\}) \to 0 \text{ for } t \to 0 \text{ and } q < 1 + \epsilon, \quad (3.17) \]

and the left-hand side of (3.11) is proved. To prove the right-hand side we apply the decomposition (3.10) to \( g = S_{1/t} b \), \( \tilde{m}_k = \mu(E_k) r^{-1} \int_{E_k} g \) and \( \tilde{\beta}_k = \int_{B_k^t} g \). Applying (3.8) to \( S_{1/t} b \) for \( k \geq 1 \) we get

\[ \| S_{1/t} b \chi_{E_k} \|_{L^q(\omega)} \leq C(b, \omega)^r \ominus \min \left( \frac{1}{t^\epsilon((2A)^k r)^{(1+2\epsilon)}}, t^{1+\epsilon} \right) \omega(B_k)^{1/q}. \]

Thus,

\[ \| S_{1/t} b \chi_{E_k} \|_{L^q(\omega)} \leq C(b, \omega)^r \omega(B_k)^{1/q} r^\epsilon \frac{t^\delta}{((2A)^k r)^{1+\epsilon-\delta}} \text{ for } k \geq 1 \text{ and any } 0 < \delta < \epsilon. \quad (3.18) \]

For \( k = 0 \), applying (2.5) to \( S_{1/t} \) we get that

\[ \| S_{1/t} b \chi_{E_0} \|_{L^q(\omega)} \leq C(b, \omega, r)t \omega(E_0)^{1/q}. \quad (3.19) \]

Denoting \( \tilde{h}_k^t = (S_{1/t} b - \tilde{m}_k) \chi_{E_k} \) we get from (3.15) that

\[ \sum_{k \geq 0} \omega(B_k)^{1/q} \left\| \tilde{h}_k^t \right\|_{L^q(\omega)} \leq C \omega(E_0) \left( t + r^{\epsilon} t^\delta \sum_{k \geq 1} \frac{1}{((2A)^k r)^{1+\epsilon-\delta-q}} \right) \]

\[ \leq C \omega(E_0)(t + r^{\epsilon} t^\delta) \to 0 \text{ when } t \to 0, \]

if \( \delta \) is chosen small enough such that \( q < 1 + \epsilon - \delta \). Thus

\[ \Lambda(\{\tilde{h}_k^t\}) \to 0 \text{ when } t \to 0. \quad (3.20) \]

Also, from (3.7) applied to \( S_{1/t} \) it is deduced that

\[ |\tilde{\beta}_k^t| \leq C \max \left( \frac{1}{((2A)^k r)^\epsilon}, t^\epsilon \right). \]

Thus, applying (3.15) we have
\[
\sum_{k \geq 0} \omega(B_k)^{1/q} \| \tilde{\beta}_k R_k \|_{L^q(\omega)} \leq C t^\epsilon \sum_{(2A)^k r \leq 1/t} \frac{\omega(B_k)}{\mu(B_k)} + C \sum_{(2A)^k r > 1/t} \frac{1}{((2A)^k r)^\epsilon} \frac{\omega(B_k)}{\mu(B_k)} \\
\leq C t^\epsilon \sum_{(2A)^k r \leq 1/t} ((2A)^k r)^{q-1} + C \sum_{(2A)^k r > 1/t} \frac{1}{((2A)^k r)^{1+\epsilon-q}} \\
\leq C t^{1+\epsilon-q},
\]  

(3.21)

if \(1 < q < 1 + \epsilon\). Thus

\[\Lambda(\{\tilde{\beta}_k R_k\}) \to 0\quad \text{for } t \to 0,\]

and the right-hand side of (3.10) is proved. \(\square\)

4. A non-tangential square function \(g_Q\)

Let us define the following non-tangential square function \(g_Q(f)\) by

\[
g_Q(f)(x) = \left( \int \int_{\Gamma(x)} \frac{|Q_t f(\tilde{y})|^2}{\mu(B(y,t))} \frac{dt}{t} \right)^{1/2},
\]  

(4.1)

where \(\Gamma(x)\) denotes the cone \(\{(y,t) \in X^+: d(y,x) < t\}\).

Since \(X\) is a normal space,

\[
g_Q(f)(x) \simeq \left( \int \int_{\Gamma(x)} \frac{|Q_t f(\tilde{y})|^2}{t^2} d\mu(y) dt \right)^{1/2}.
\]

Let us denote \(B = L^2(X^+, d\mu(y) dt/t)\), of measurable functions \(a : X^+ \to \mathbb{C}\) with norm \(|a|_B = (\int_{X^+} |a(y,t)|^2 d\mu(y) dt/t)^{1/2} < \infty\), \(\mathcal{M}(X)\) the set of measurable functions defined on \(X\) valued in \(\mathbb{C}\) and \(\mathcal{M}(X, B)\) the set of Bochner-measurable functions \(h : X \to B\). The space \(L^p(X, \mathcal{B}(\omega))\) is the set of \(h \in \mathcal{M}(X, B)\) with finite norm

\[
\|h\|_{L^p(X, \mathcal{B}(\omega))} = \left( \int_X |h(x)|^p_B \omega(x) d\mu(x) \right)^{1/p}.
\]

When \(\omega = 1\) we will simply name the space as \(L^p(X, B)\).

**Theorem 4.1.** If \(1 < p < \infty\) and \(\omega \in A_p\), then \(g_Q\) is bounded in \(L^p(\omega)\). More precisely, there is a constant \(C\) such that

\[
\|g_Q(f)\|_{L^p(\omega)} \leq C \|f\|_{L^p(\omega)},
\]  

(4.2)

for all \(f \in L^p(\omega)\).

**Proof.** Let consider an operator \(\tilde{S} : \mathcal{M}(X) \to \mathcal{M}(X, B)\) in the following way: Let \(\phi\) be a non-negative infinitely differentiable function on \(\mathbb{R}_+\) such that \(\phi(s) = 1\) for \(0 < s < 1\) and \(\phi(s) = 0\) for \(s \geq 2\). The function \(\phi_t(x, y) = \frac{1}{t} \phi\left(\frac{d(x, y)}{t}\right)\) satisfies

\[
0 \leq \phi_t(x, y) \leq 1/t,
\]

\[
\phi_t(x, y) = 1/t \quad \text{for } d(x, y) \leq t \quad \text{and} \quad \phi_t(x, y) = 0 \quad \text{for } d(x, y) > 2t,
\]

\[
|\phi_t(x, y) - \phi_t(x', y)| \leq C \frac{d(x, x')^\epsilon}{t^{1+\epsilon}} X_{[0,4A]} \left( \max(d(x, y), d(x', y)) \right).
\]  

(4.3)

for \(d(x, y) > 2Ad(x', x')\), and all \(\epsilon\) such that \(0 < \epsilon \leq \theta\).
We now define
\[
\tilde{S}f(x) = \left\{ \tilde{S}(y,t) f(x) \right\}_{(y,t) \in X^+},
\]
which has associated kernel
\[
\tilde{K}(x, z) = \left\{ t^{1/2} \phi_t(x, y) q_t(y, z) \right\}_{(y,t) \in X^+}.
\]
By Fubini's Theorem and the Littlewood–Paley characterization of $L^p$ obtained in [3] in the setting of spaces of homogeneous type, also holding—by Theorem 5.1—for the family $\{Q_t\}_{t > 0}$, $\tilde{S}$ is bounded from $L^2(X)$ into $L^2(X, B)$. More precisely
\[
\|\tilde{S}f\|_{L^2(X, B)}^2 = \int_X \left( \int_0^\infty \left( \int_X |\phi_t(x, y)|^2 |Q_t f(y)|^2 dt d\mu(y) \right) d\mu(x) \right)^{1/2} \leq C \|f\|_{L^2}^2.
\]
(4.4)

We claim that $g_Q(f)(x) \leq |\tilde{S}f(x)|_B$ for every $x \in X$. In fact
\[
|\tilde{S}f(x)|_B = \left( \int_0^\infty \left( \int_X |\phi_t(x, y)|^2 |Q_t f(y)|^2 dt d\mu(y) \right)^{1/2} \right)^2 \leq C \|f\|_{L^2}^2.
\]
Therefore to prove that $g_Q$ is bounded on $L^p(\omega)$ it is enough to show that $\tilde{S}$ is bounded from $L^p(\omega)$ to $L^p(X, B)(\omega)$. But, in view of (4.4) and the theory of vector valued singular integrals the goal will be achieved by proving that the kernel $\tilde{K}$ of $\tilde{S}$ is a standard vector valued Calderón–Zygmund kernel. More precisely, there exists a constant $C$ such that
\[
|\tilde{K}(x, z)|_B \leq C \frac{1}{d(x, z)};
\]
(4.5)
\[
|\tilde{K}(x, z) - \tilde{K}(x', z)|_B \leq C \frac{d(x, x')^{\epsilon/2}}{d(x, z)^{1+\epsilon/2}},
\]
(4.6)
if $d(x, z) > 2Ad(x, x')$;
\[
|\tilde{K}(z, x) - \tilde{K}(z, x')|_B \leq C \frac{d(x, x')^{\epsilon/2}}{d(x, z)^{1+\epsilon/2}}
\]
(4.7)
if $d(x, z) > 4A^2d(x, x')$, and $\epsilon$ is the order of the approximation to the identity.

Let us first check (4.5)
\[
|\tilde{K}(x, z)|_B^2 \leq \int_X \left( \int_0^\infty |\phi_t(x, y)|^2 |q_t(y, z)|^2 dt d\mu(y) \right) \int_0^\infty \left( \int_0^\infty \frac{t^{2\epsilon}}{(t+ d(y, z))^{2+2\epsilon}} dt \right) d\mu(y)
\]
\[
= J_1 + J_2 + J_3,
\]
(4.8)

where the partition of $X^+$ considered is
\[
A_1 = \{ y : d(y, z) > 2Ad(x, z) \},
\]
\[
A_2 = \{ y : d(y, z) < d(y, z) \leq 2Ad(x, z) \},
\]
\[
A_3 = \{ y : d(y, z) \leq 1/(2A)d(x, z) \}.
\]
Since for $y \in A_1$ is $d(x, z) \leq \frac{1}{2A} d(y, z) \leq d(x, y) \leq 2Ad(y, z)$, then

$$J_1 \leq C \int_{A_1 \cap \{d(x,y)/2\}} \int_{d(x,y)>2Ad(x,z)} \frac{1}{t^3} \frac{d\mu(y)}{t} \leq C \int_{d(x,y)>2Ad(x,z)} \frac{1}{d(x,y)^3} d\mu(y) \leq C \frac{1}{d(x,z)^2}. \quad (4.9)$$

For $y \in A_2$ is $d(y, x) \leq 3A^2d(x, z)$ and $d(y, z) \sim d(x, z)$, then

$$J_2 \leq C \int_{A_2} \left( \int_{d(x,y)/2} \int_{d(y,z)^2+\epsilon} \frac{1}{t^3} \frac{d\mu(y)}{t} + \int_{3A^2d(x,z)} \frac{1}{d(x,y)^3} d\mu(y) \right) dt d\mu(y) = J_{2a} + J_{2b}.$$

But,

$$J_{2a} \leq C \int_{A_2} \int_{d(x,y)/2} \frac{1}{t^{1-\epsilon}} \frac{1}{d(y,z)^{2+\epsilon}} \frac{d\mu(y)}{t} \leq C \frac{1}{d(x,z)^{2+\epsilon}} \int_{d(x,y)<3A^2d(x,z)} \frac{1}{d(y,z)^{1-\epsilon}} d\mu(y) \leq C \frac{1}{d(x,z)^2},$$

and

$$J_{2b} \leq C \frac{1}{d(x,z)^3} \int_{d(x,y)\leq 3A^2d(x,z)} d\mu(y) \leq C \frac{1}{d(x,z)^2}. \quad (4.10)$$

Thus, from the above inequalities it follows that

$$J_2 \leq C \frac{1}{d(x,z)^2}. \quad (4.10)$$

Finally for $y \in A_3$ is $d(y, z) \leq \frac{1}{2A} d(x, z)$ and $d(x, z) \sim d(x, y)$. Thus

$$J_3 \leq \int_{A_3 \cap \{d(x,y)/2\}} \int_{d(x,y)/2} \frac{1}{t^3} \frac{d\mu(y)}{t} \leq C \frac{1}{d(x,z)^2}, \quad (4.11)$$

and the proof of (4.5) is finished.

To prove (4.6) let consider $d(x, z) > 2Ad(x, x')$, denote $a = \min(d(x, y), d(x' y))$ and define $B = \{d(x, y) > 2Ad(x, x')\}$. Then

$$\left| \tilde{K}(x, z) - \tilde{K}(x', z) \right|_B^2 = \int_X \int_0^\infty \left| \phi_t(x, y) - \phi_t(x', y) \right|^2 \left| q_t(y, z) \right|^2 dt \, d\mu(y) \leq C \int_B \int_a/2^2 \frac{d(x,x')^{2\epsilon}}{t^2} \frac{1}{(t + d(y,z))^{2+2\epsilon}} dt \, d\mu(y) + 2C \int_B \int_a/2^2 \frac{t^{2\epsilon}}{(t + d(y,z))^{2+2\epsilon}} dt \, d\mu(y) = I + II. \quad (4.12)$$

For $y \in B$ is $d(x', y) > d(x, y)/2A$ and then $a > d(x, y)/2A$. Denoting

$$B_1 = B \cap \left\{ y: d(x, y) \geq d(x, z)/2A \right\},$$

we have
\[ I = C d(x, x')^{2e} \int_{B_1} \int_t^{\infty} \frac{1}{(t + d(y, z))^{2+2e}} \frac{dt}{t^2} d\mu(y) = I_1 + I_2. \] (4.13)

For \( y \in B_1 \) is \( d(y, z) \leq 3A^2d(y, x) \) and then

\[ I_1 \leq C d(x, x')^{2e} \int_{B_1} \int_t^{\infty} \frac{1}{t^{3+2e}} \frac{dt}{t} d\mu(y) \leq C \frac{d(x, x')^{2e}}{d(x, z)^{2+2e}}. \] (4.14)

For \( y \in B \setminus B_1 \) is \( d(y, z) \sim d(x, z) \) and

\[ I_2 \leq C d(x, x')^{2e} \int_{B \setminus B_1} \int_t^{\infty} \frac{1}{t^{1+e}} \frac{dt}{t} d\mu(y) \leq C \frac{d(x, x')^{2e}}{d(x, z)^{2+2e}}. \] (4.15)

Then

\[ I_{2a} \leq C \frac{d(x, x')^{2e}}{d(x, z)^{2+2e}} \int_{B \setminus B_1} \int_t^{\infty} \frac{1}{t^{1+e}} \frac{dt}{t} d\mu(y) \leq C \frac{d(x, x')^{2e}}{d(x, z)^{2+2e}}. \] (4.16)

and

\[ I_{2b} \leq C \frac{d(x, x')^{2e}}{d(x, z)^{3+2e}} \int_{\{d(y, z) \leq d(x, z)\}} d\mu(y) \leq C \frac{d(x, x')^{2e}}{d(x, z)^{2+2e}}. \] (4.17)

In this way, from (4.13)–(4.17) it follows that

\[ I \leq C \frac{d(x, x')^{2e}}{d(x, z)^{2+2e}}. \] (4.18)

To estimate II, we notice that if \( y \in B^c \), then \( d(x', y) \leq 3A^2d(x, x') \) so that

\[ II \leq \left\{ \int \int_t^{\infty} + \int \int_t^{\infty} \right\} \frac{1}{t^{1-2e}} \frac{1}{(t + d(y, z))^{2+2e}} \frac{dt}{t} d\mu(y) = II_1 + II_2. \] (4.19)

Since the above two integrals are similar the estimate for II_1 will also hold for II_2.

If \( y \in B^c \), then \( d(x, y) < d(x, z) \). We consider the set

\[ (B^c)_1 = B^c \cap \{ y : d(x, z) \leq 2Ad(x, y) \} \]

and notice that for \( y \in (B^c)_1 \) is \( d(x, z) \sim d(x, y) \) and \( d(y, z) \leq 4A^2d(x, y) \) and for \( y \in B^c \setminus (B^c)_1 \) is \( d(x, z) \sim d(y, z) \) and \( d(x, y) \leq d(x, z)/(2A) \). Thus,

\[ II_1 \leq C \left\{ \int \int_t^{\infty} \frac{1}{t^{1-2e}} d\mu(y) + \int \int_t^{\infty} \frac{1}{t^{1-2e}} d\mu(y) \right\} \]

\[ = II_{1a} + II_{1b} + II_{1c}. \] (4.20)

Then

\[ II_{1a} \leq C \int \frac{1}{d(x, y)^{2+2e}} d\mu(y) \leq C \frac{d(x, x')^{2e}}{d(x, z)^{2+2e}}. \] (4.21)

On the other hand, for any \( 0 < \delta < \epsilon \) it is \( \epsilon < 1 + \delta \), thus we have
II_{1b} \leq C \int_{B^c \setminus (B^c)_{1}} \frac{1}{d(x,y)^{1+\delta - \epsilon}} \frac{dt}{d(y,z)^{2+\epsilon - \delta}} d\mu(y) \leq C \int_{B^c} \frac{1}{d(x,y)^{1+\delta - \epsilon}} d\mu(y)

\leq C \frac{d(x,x')^{\epsilon - \delta}}{d(x,z)^{2+\epsilon - \delta}}.

Now taking \( \delta \to 0 \) we have

II_{1b} \leq C \frac{d(x,x')^{\epsilon}}{d(x,z)^{2+\epsilon}}. \quad (4.22)

Finally,

II_{1c} \leq C \int_{B^c \setminus (B^c)_{1}} \frac{1}{d(x,y)^{3}} d\mu(y) \leq C \int_{B^c} \frac{1}{d(x,y)^{1-2\epsilon}} d\mu(y) \leq C \frac{d(x,x')^{2\epsilon}}{d(x,z)^{2+2\epsilon}}. \quad (4.23)

From (4.19)–(4.23) and the observation made after (4.19) we have

II \leq C \frac{d(x,x')^{\epsilon}}{d(x,z)^{2+\epsilon}}. \quad (4.24)

Now, from (4.12), (4.18) and (4.24) we finally have (4.6).

To prove (4.7) we consider

\[ d(x,y) > 4A^2d(x,x'). \]

Taking into account that if \( t \geq \frac{d(y,z)}{2} \), then, by the triangular inequality \( d(x,y) + t > \frac{d(x,y) + d(y,z)}{2} > 2Ad(x,x') \), defining

\[ E = \{ y : d(y,z) \geq d(x,z)/2 \}, \]

and using the regularity condition (2.7) for \( q_t \), we have that

\[ \left| \tilde{K}(z,x) - \tilde{K}(z,x') \right|_B \leq \int \int_X \phi_t(z,y)^2 |q_t(y,x) - q_t(y,x')|^2 dt d\mu(y) \leq \frac{1}{t^{1+\delta - \epsilon}} \int \int_{E \cup E^c} \frac{t^{2\epsilon}}{(t + d(y,x))^{2+4\epsilon}} dt d\mu(y), \]

\[ = I_1 + I_2. \quad (4.25) \]

Notice that if \( y \in E \), then \( d(x,y) \leq 3A^2d(y,z) \); and thus

\[ I_1 \leq C \int_{E} \frac{1}{d(y,z)^{3+2\epsilon}} dt d\mu(y) \leq C \int_{E} \frac{1}{d(x,y)^{3+2\epsilon}} d\mu(y) \leq C \frac{d(x,x')^{2\epsilon}}{d(x,z)^{2+2\epsilon}}. \quad (4.26) \]

If \( y \in E^c \), then \( d(y,z) < \frac{d(x,y)}{2A} < d(x,y) \). Thus, for any \( 0 < \delta < \epsilon \) we have

\[ I_2 \leq C \int_{E^c} \left( \frac{1}{d(x,y)^{2+3\epsilon - \delta}} \int d(y,z) d\mu(y) + \int_{E^c} \frac{1}{d(y,z)^{1+\delta - \epsilon}} d\mu(y) \right) \]

\[ \leq C \int_{E^c} \left( \frac{1}{d(x,y)^{2+3\epsilon - \delta}} \int d(y,z) d\mu(y) + \frac{1}{d(x,y)^{1+\delta - \epsilon}} d\mu(y) \right) \]

\[ \leq C \int_{E^c} \left( \frac{1}{d(x,y)^{2+3\epsilon - \delta}} \frac{d(x,y)}{d(y,z)^{1+\delta - \epsilon}} + \frac{1}{d(x,y)^{3+2\epsilon}} d\mu(y) \right) \]

\[ \leq C \int_{E^c} \left( \frac{1}{d(x,y)^{2+3\epsilon - \delta}} \frac{d(x,y)}{d(y,z)^{1+\delta - \epsilon}} + \frac{1}{d(x,y)^{3+2\epsilon}} d\mu(y) \right) \]

\[ \leq C \frac{d(x,x')^{2\epsilon}}{d(x,z)^{2+2\epsilon}}. \]
\[ C \frac{d(x, x')^{2\epsilon}}{d(x, z)^{2+2\epsilon}}. \]  

From (4.25)–(4.27) we have thus proved (4.7) and finished the proof of the theorem. \( \square \)

The above lemma allows us to obtain the following one:

**Theorem 4.2.** If \( 1 < q < 1 + \epsilon \) and \( \omega \in A_q \), then there is a constant \( C > 0 \) such that

\[ \|g_Q(f)\|_{L^1(\omega)} \leq C \|f\|_{H^1_q(\omega)}, \]  

for all \( f \in H^1_q(\omega) \).

**Proof.** We first prove (4.28) for atoms \( a \) supported in a ball \( B(x_0, r) \).

Let first consider the case \( d(x, x_0) > 4A^2r \) and any \( y \) such that \( d(y, x) < t \). Under these conditions \( t + d(y, x_0) \geq d(x, x_0)/2A \geq 2Ar \). Moreover, since \( a \) has null mean, \( \omega \in A_q \) and \( X \) is normal, then by (3.8), it follows that

\[ \left| Q_i a(y) \right| \leq C \frac{(t + d(y, x_0))^{1+\epsilon}}{t^{1+\epsilon}} \omega(B_0)^{-1/q} \|a\|_{L^q(\omega)^2}. \]  

In this way,

\[ \int_{d(x, x_0)/2A}^{d(x, x_0)} \int_0^{d(y, x)/2A} \frac{d\mu(y)}{\mu(B(y, t))} dt \leq C \frac{r^{2+2\epsilon}}{d(x, x_0)^{2+2\epsilon}} \omega(B_0)^{-1/q} \|a\|_{L^q(\omega)^2}. \]  

On the other hand,

\[ \int_{d(x, x_0)/(2A)}^{d(x, x_0)} \int_{d(y, x)/2A}^{d(y, x)} \frac{d\mu(y)}{\mu(B(y, t))} dt \leq C \frac{r^{2+2\epsilon}}{d(x, x_0)^{2+2\epsilon}} \omega(B_0)^{-1/q} \|a\|_{L^q(\omega)^2}. \]  

From (4.30) and (4.31) we obtain for \( d(x, x_0) > 4A^2r \),

\[ \left( g_Q(a)(x) \right)^2 \leq C \frac{r^{2+2\epsilon}}{d(x, x_0)^{2+2\epsilon}} \omega(B_0)^{-1/q} \|a\|_{L^q(\omega)^2}. \]  

In this way,

\[ \int_{d(x, x_0) > 4A^2r} \left| g_Q(a)(x) \right| \omega(x) d\mu(x) \leq C \sum_{k=1}^{\infty} \int_{d(x, x_0)/(2A)^r} \frac{r^{1+\epsilon}}{d(x, x_0)^{1+\epsilon}} \omega(x) d\mu(x) \omega(B_0)^{-1/q} \|a\|_{L^q(\omega)^2} \]  

\[ \leq C \sum_{k=1}^{\infty} (2A)^{k(1+\epsilon)} \omega((2A)^k B_0) \omega(B_0)^{-1/q} \|a\|_{L^q(\omega)^2}. \]
\[
\sum_{k=1}^{\infty} \frac{1}{(2A)^{k(1+\epsilon)}} \left( \frac{\mu((2A)^kB_0)}{\mu(B_0)} \right)^q \omega(B_0)^{1-1/q} \|a\|_{L^q(\omega)}
\]

\[
\sum_{k=1}^{\infty} \frac{1}{(2A)^{k(1+\epsilon-q)}} \omega(B_0)^{1/q'} \|a\|_{L^q(\omega)}
\]

\[
C \omega(B_0)^{1/q'} \|a\|_{L^q(\omega)},
\]

if \( q < 1 + \varepsilon \).

On the other hand, by Theorem 4.1, if \( q > 1 \), then

\[
\int_{d(x,x_0) \leq 4A^2r} g_Q(a)(x) \omega(x) d\mu(x) \leq C \|g_Q(a)\|_{L^q(\omega)} \omega(B_0)^{1/q'} \|a\|_{L^q(\omega)} \omega(B_0)^{1/q'}.
\]

From (4.33) and (4.34) it then follows that

\[
\int_X g_Q(f)(x) \omega(x) d\mu(x) \leq C \|a\|_{L^q(\omega)} \omega(B_0)^{1/q'}.
\]

Let now consider \( f = \sum_{j \in J} a_j \) and use Minkowski’s inequality and (4.35) to show that

\[
\int_X |g_Q(f)(x)| \omega(x) d\mu(x) \leq \sum_j \int_X |g_Q(a_j)(x)| \omega(x) d\mu(x) \leq C \sum_j \|a_j\|_{L^q(\omega)} \omega(B_j)^{1/q'} \leq C \Lambda([a_j]);
\]

and taking the \( \inf \Lambda([a_j]) \) over all the decompositions of \( f \) we have finally proved (4.28). \( \square \)

**Remark 4.36.** It is worth observing that the estimates in Lemmas 4.1 and 4.2 rely on inequalities (2.5), (2.7), (3.7) and (3.8), and these last two, in turn, on the first ones. Thus both lemmas also hold for any family of operators \( \{\tilde{Q}_t\}_{t>0} \) satisfying (2.5) and (2.7).

### 5. Main lemmas

Given \( x_B \in X \) and \( r > 0 \) we denote \( B = B(x_B,r) \) the ball with center \( x_B \) and radius \( r \), \( T(B) = \{(y,t) \in X^+ : d(y,x_B) + t < r\} \) the tent over \( B \).

Given a measurable function \( F = F(y,t) \) on \( X^+ \) the \( \omega \)-Carleson function of \( F \) is defined by

\[
C^*(F)(x) = \sup_{B \ni x} \left( \frac{1}{\omega(B)} \int_{T(B)} |F(y,t)|^2 \frac{\mu(B(y,t))}{\omega(B(y,t))} d\mu(y) \frac{dt}{t} \right)^{1/2}.
\]

If \( F(y,t) = Q_t f(y) \), then, clearly, \( C^*(F) \in L^\infty \) if and only if \( dv_f \) is an \( \omega \)-Carleson measure and

\[
\|C^*(Q_{\cdot f}(\cdot))\|_{L^\infty} = \|dv_f\|_{\omega}.
\]

We also define the \( \omega \)-square function of \( F \) restricted to time \( \tau \) as

\[
G(F/\tau)(x) = \left( \int_{\Gamma^\tau(x)} |F(y,t)|^2 \frac{\mu(B(y,t))}{\omega(B(y,t))} d\mu(y) \frac{dt}{t} \right)^{1/2},
\]

with \( \Gamma^\tau(x) = \{(y,t) : d(y,x) < t < \tau\} \).

The stopping time \( \tau(x) \) of \( x \) is defined by

\[
\tau(x) = \sup \{ \tau > 0 : G(F/\tau)(x) \leq AC(F)(x) \},
\]

where \( A \) is a great enough constant independent of \( F \) and \( x \) to be chosen later.
Lemma 5.4. Given a weight \( \omega \) in \( A^\infty \) there exists \( C > 0 \) such that for any ball \( B \) of radius \( r \),
\[
\omega(\{x \in B: \tau(x) \geq r\}) > C \omega(B). 
\]

Proof. If \( \widetilde{B} = B(x_B, 3A^2 r) \), then \( \bigcup_{x \in B} \Gamma'(x) \subseteq T(\widetilde{B}) \). Thus by applying Tonelli’s Theorem and since \( \omega \) is doubling we get
\[
\frac{1}{\omega(B)} \int_B \mathcal{G}(F/r)(x)^2 \omega(x) \, d\mu(x) = \frac{1}{\omega(B)} \int \int \int_{\Gamma'(x)} |F(y,t)|^2 \mu(B(y,t)) \frac{\mu(B(y,t))}{\omega(B(y,t))^2} \, d\mu(y) \, dt \frac{\omega(x)}{t} \omega(x) \, d\mu(x) 
\]
\[
\leq \frac{1}{\omega(B)} \int \int_{T(\widetilde{B})} |F(y,t)|^2 \mu(B(y,t)) \frac{\mu(B(y,t))}{\omega(B(y,t))} \, d\mu(y) \, dt \frac{1}{t} 
\]
\[
\leq a \frac{1}{\omega(B)} \int \int_{T(\widetilde{B})} |F(y,t)|^2 \mu(B(y,t)) \frac{\mu(B(y,t))}{\omega(B(y,t))} \, d\mu(y) \, dt \frac{1}{t} 
\]
\[
\leq a \inf_{x \in B} C(F)(x)^2. \tag{5.5}
\]

On the other hand,
\[
\frac{1}{\omega(B)} \int_B \mathcal{G}(F/r)(x)^2 \omega(x) \, d\mu(x) \geq \frac{1}{\omega(B)} \int_{\{x \in B: \tau(x) < r\}} \mathcal{G}(F/r)(x)^2 \omega(x) \, d\mu(x) 
\]
\[
> \frac{A^2}{\omega(B)} \inf_{x \in B} C(F)(x)^2 \omega(\{x \in B: \tau(x) < r\}). \tag{5.6}
\]

From (5.5) and (5.6) it follows that
\[
\omega(\{x \in B: \tau(x) < r\}) < \frac{a}{A^2} \omega(B),
\]
and choosing \( A^2 > a \) we get
\[
\omega(\{x \in B: \tau(x) \geq r\}) > C \omega(B),
\]
with \( C = 1 - \frac{a}{A^2} \). \( \square \)

Given a function \( G(y,t) \) let us denote
\[
g(G)(x) = \left( \int \int \frac{|G(y,t)|^2}{\mu(B(y,t))} \, d\mu(y) \, dt \frac{1}{t} \right)^{1/2}. 
\]

Notice that if \( G(y,t) = Q_t f(y) \), then \( g(G)(x) = g_Q(f)(x) \) is the square function defined in (4.1).

Lemma 5.7. There is a constant \( C > 0 \) such that for any pair of measurable functions \( F(y,t) \) and \( G(y,t) \) on \( X^+ \),
\[
\int \int_{X^+} |F(y,t)||G(y,t)| \, d\mu(y) \, dt \frac{1}{t} \leq C \|C(F)\|_{L^\infty} \|g(G)\|_{L^1(\omega)}. \tag{5.8}
\]

Proof. Let \( H(y,t) \) be a non-negative and measurable function defined on \( X^+ \). By Lemma 5.4 and Tonelli’s Theorem we have that
\[
\int_0^\infty \int_X H(y,t) \omega(B(y,t)) \, d\mu(y) \, dt \frac{1}{t} \leq C^{-1} \int_0^\infty \int_X H(y,t) \omega(\{x \in B(y,t): \tau(x) \geq t\}) \, d\mu(y) \, dt \frac{1}{t}
\]
\[ \leq C^{-1} \int_X \int_0^\tau(x) \int \{ y \in X : d(y,x) < t \} H(y,t) d\mu(y) dt \omega(x) d\mu(x) \]

\[ = C^{-1} \int_X \left( \int_{\Gamma^{(1)}(x)} H(y,t) d\mu(y) \frac{dt}{t} \right) \omega(x) d\mu(x). \]  

(5.9)

Let now set \( H(y,t) = \frac{|F(y,t)G(y,t)|}{\omega(B(y,t))} \). From (5.9), Schwartz inequality and definition (5.3) it follows that

\[ \int_X \int \left| F(y,t)G(y,t) \right| d\mu(y) dt \leq C^{-1} \int_X \left( \int_{\Gamma^{(1)}(x)} |F(y,t)G(y,t)|^2 \frac{\mu(B(y,t))}{\omega(B(y,t))^2} d\mu(y) \frac{dt}{t} \right)^{1/2} \times \left( \int_{\Gamma^{(1)}(x)} \frac{|G(y,t)|^2}{\mu(B(y,t))} d\mu(y) \frac{dt}{t} \right)^{1/2} \omega(x) d\mu(x) \]

\[ \leq C^{-1} \int_X \|C(F)(x)\|_{L^\infty} \|g(G)(x)\|_{L^1(\omega)} \omega(x) d\mu(x) \]

\[ \leq AC^{-1} \int_X \|C(F)\|_{L^\infty} \|g(G)\|_{L^1(\omega)} \omega(x) d\mu(x) \]

and, thus, the claim of the lemma is proved. \( \square \)

Next we state a continuous version of a Calderón-type reproduction formula whose proof is in [2].

**Theorem 5.1.** Let \( \{S_t\}_{t > 0} \) be an approximation to the identity of order \( \epsilon \leq \theta \) and \( \{Q_t\}_{t > 0} \) be a family of operators as in (2.10). Then there exist families of operators \( \{\tilde{Q}_t\}_{t > 0} \) and \( \{\tilde{\tilde{Q}_t}\}_{t > 0} \) such that for all \( f \in \check{M}(\beta, \gamma) \) (\( f \in \check{M}(\beta, \gamma') \)), \( 0 < \beta, \gamma < \epsilon \),

\[ f = \int_0^\infty \tilde{Q}_t f \frac{dt}{t} \quad \text{and} \quad f = \int_0^\infty Q_t \tilde{\tilde{Q}_t} f \frac{dt}{t}. \]  

(5.11)

where the integral converges in \( \check{M}(\beta', \gamma') \) (\( \check{M}(\beta', \gamma') \)) for \( \beta' < \beta \) and \( \gamma' < \gamma \) (\( \beta < \beta' \) and \( \gamma < \gamma' \)). The integral also converges in \( L^p \) (\( 1 < p < \infty \)).

Moreover, \( \tilde{q}_t(x,y) \), the kernel of \( \tilde{Q}_t \) satisfies the following estimates: for each \( \epsilon' \), \( 0 < \epsilon' < \epsilon \), there exists a constant \( C \) such that

\[ |\tilde{q}_t(x,y)| \leq C \frac{t^{\epsilon'}}{(t + d(x,y))^{1+\epsilon'}}. \]  

(5.12)

\[ |\tilde{q}_t(x,y) - \tilde{q}_t(x',y)| \leq C \frac{d(x,x')^{\epsilon'}}{(t + d(x,y))^{1+\epsilon'}} \wedge \frac{t^{\epsilon'}}{2A(t + d(x,y))}, \quad \text{for} \quad d(x,x') \leq \frac{1}{2A} (t + d(x,y)), \]  

(5.13)

\[ \int \tilde{q}_t(x,y) d\mu(y) = \int \tilde{q}_t(y,x) d\mu(y) = 0, \quad \text{for all} \quad t > 0. \]  

(5.14)

The kernel \( \tilde{\tilde{q}}_t(x,y) \) of \( \tilde{\tilde{Q}_t} \) satisfies the above conditions except for interchanging \( x \) and \( y \) in (5.13).

We are now in position to prove our main theorem.
6. Proof of Theorem 2.1

To prove item (a) let us consider a distribution \( f \in (\mathcal{M}(\beta, \gamma))' \) such that \( dv_f = |Q_t f(y)|^2 \frac{\mu(B(y,t))}{\omega(B(y,t))} \frac{d\mu(y)}{t} \) is an \( \omega \)-Carleson measure, let consider a molecule \( h \in \mathcal{M}(\epsilon, \epsilon) \).

By Theorem 5.1

\[
\langle f, h \rangle = \int_0^\infty \langle \tilde{Q}_t Q_t f, h \rangle \frac{dt}{t} = \int_0^\infty \langle Q_t f, \tilde{Q}_t^* h \rangle \frac{dt}{t} = \int_0^\infty \int_X Q_t f(y) \tilde{Q}_t^* h(y) \frac{d\mu(y)}{t} \frac{dt}{t}.
\]

(6.1)

Notice that since the family \( \tilde{Q}_t^* \) built in Theorem 5.1 satisfies conditions (2.5) and (2.7) with control \( \epsilon' < \epsilon \), then Lemma 4.2 also applies to \( \tilde{g}(\omega) \), by Remark 4.36. From this observation and Lemma 5.7 applied to \( F(y, t) = Q_t f(y) \) and \( G(y, t) = \tilde{Q}_t^* h(y) \) it follows that

\[
|\langle f, h \rangle| \leq C \|C(Q_t f)\|_{L^\infty(\omega)} \|\tilde{g}(\omega)\|_{L^1(\omega)} \leq C [dv_f]_\omega [h]_{H_\omega^2(\omega)},
\]

(6.2)

for \( q < 1 + \epsilon' \), any arbitrary \( \epsilon' < \epsilon \), and \( \omega \) a weight in \( A_q \). Lemma 3.9 now shows that \( f \) defines a continuous linear functional on \( H_\omega^1(\omega) \), \( q < 1 + \epsilon \), and by Lemma 3.4 \( f \in BMO(\omega) \) and \( \|f\|_{BMO(\omega)} \leq C [dv_f]_\omega \). This finishes the proof of item (a).

We go now to the proof of item (b) of the theorem and consider \( f \in BMO(\omega) \). Let us accept for the moment that \( f \in (\mathcal{M}(\beta, \gamma))', \) for \( 0 < \beta, \gamma < \epsilon \) and \( q < 1 + \gamma \), and first show that \( [dv_f]_\omega \) is an \( \omega \)-Carleson measure. Let \( B = B(x_0, r) \) be a ball in \( X \). If \( f \) is split as

\[
f = (f - m_B f) \chi_B + (f - m_B f) \chi_{B^c} + m_B f = f_1 + f_2 + f_3,
\]

(6.3)

where \( \tilde{B} = B(x_0, 2A R) \), then it follows from (2.11) that \( Q_t f_3 = 0 \). On the other hand, Tonelli’s Theorem, the fact that \( \tilde{g}(\omega) \) is bounded on \( L^2(\omega^{-1}) \) because \( \omega^{-1} \in A_2 \), and (3.3) lead to the estimate

\[
\int \int_{T(B)} \left| Q_t f_1(y) \right|^2 \frac{\mu(B(y,t))}{\omega(B(y,t))} \frac{d\mu(y)}{t} \frac{dt}{t} \leq C \int \int_{T(B)} \left| Q_t f_1(y) \right|^2 \frac{\omega^{-1}(B(y,t))}{\mu(B(y,t))} \frac{d\mu(y)}{t} \frac{dt}{t} = C \int \int_{T(B)} \left| Q_t f_1(y) \right|^2 \frac{\omega^{-1}(B(y,t))}{\mu(B(y,t))} \frac{d\mu(y)}{t} \frac{dt}{t} \leq C \int \int_{B(x_0, A r)} \omega^{-1}(z) \int_0^\infty \left| Q_t f_1(y) \right|^2 \frac{d\mu(y)}{t} \frac{dt}{t} \frac{d\mu(z)}{t} = C \int \omega(B) \|f\|_{BMO(\omega)}^2.
\]

(6.4)

Denoting \( B_k = B(x_0, (2A)^k r) \) we also have

\[
\left| Q_t f_2(x) \right| \leq \sum_{k=2B_k - B_{k-1}}^\infty \int q_t(y, x) \left| f(x) - m_B f \right| d\mu(x)
\]

\[
\leq \sum_{k=2B_k - B_{k-1}}^\infty \int q_t(y, x) \left| f(x) - m_B f \right| d\mu(x)
\]
\[
\begin{align*}
+ \sum_{k=2}^{\infty} \left( \sum_{j=1}^{k} \frac{1}{\mu(B_j)} \right) \int_{B_j} \left| f(z) - m_{B_j} f \right| d\mu(z) 
\int_{B_k-B_{k-1}} \left| q_t(y,x) \right| d\mu(x) 
&= L_1 + L_2.
\end{align*}
\]

For \((y,t) \in T(B), q < (1+\epsilon)\) and using (3.15) we have
\[
L_1 \leq C \epsilon^2 \sum_{k=2}^{\infty} \frac{1}{(2A)^k r^{1+\epsilon}} \int_{B_k} \left| f(x) - m_{B_k} f \right| d\mu(x) \leq C \epsilon^2 \sum_{k=2}^{\infty} \frac{1}{(2A)^k r^{1+\epsilon}} \omega(B_k) \| f \|_{BMO(\omega)}
\]
\[
\leq C \frac{\omega(B)}{\mu(B)^q} \epsilon^2 \sum_{k=2}^{\infty} \frac{1}{(2A)^k r^{1+\epsilon-q}} \| f \|_{BMO(\omega)} \leq C (\omega(B))^q r^{1+\epsilon} \| f \|_{BMO(\omega)}.
\]

On the other hand, since \(q > 1\), then
\[
L_2 \leq C \epsilon^2 \sum_{k=2}^{\infty} \frac{\omega(B_j)}{\mu(B_j) (2A)^k r^{1+\epsilon}} \| f \|_{BMO(\omega)} \leq C \epsilon^2 \frac{\omega(B)}{\mu(B)^q} \sum_{k=2}^{\infty} \frac{1}{(2A)^k r^{1+\epsilon-q}} \sum_{j=1}^{k} \mu(B_j)^{q-1} \| f \|_{BMO(\omega)}
\]
\[
\leq C \epsilon^2 \frac{\omega(B)}{\mu(B)^q} \sum_{k=2}^{\infty} \frac{1}{(2A)^k r^{1+\epsilon-q}} \| f \|_{BMO(\omega)} \leq C \epsilon^2 \frac{\omega(B)}{r^{1+\epsilon}} \| f \|_{BMO(\omega)}.
\]

Thus, using Tonelli’s Theorem and the fact that \(\omega \in A_2\) we have
\[
\left\langle Q_t f_1 f_2(y) \right\rangle \leq \int_{T(B)} \int_{t} \left| Q_t f_1 f_2(y) \right| \frac{\mu(B(y,t))}{\omega(B(y,t))} d\mu(y) \frac{dt}{t}
\]
\[
\leq C \omega(B)^2 \int_{T(B)} \int_{t} \frac{t^{2\epsilon}}{r^{2+2\epsilon}} \frac{\omega^{-1}(B(y,t))}{\mu(B(y,t))} d\mu(y) \frac{dt}{t} \| f \|_{BMO(\omega)}^2
\]
\[
\leq C \omega(B)^2 \int_{T(B)} \int_{t} \frac{t^{2\epsilon}}{r^{2+2\epsilon}} \left( \int_{d(z,y)<t} \omega^{-1}(z) d\mu(z) \right) \frac{d\mu(y)}{\mu(B(y,t))} \frac{dt}{t} \| f \|_{BMO(\omega)}^2
\]
\[
\leq C \omega(B)^2 \int_{d(z,x_0)<2Ar} \omega^{-1}(z) \int_{0}^{r} \frac{t^{2\epsilon}}{r^{2+2\epsilon}} \frac{dt}{t} d\mu(z) \| f \|_{BMO(\omega)}^2
\]
\[
\leq C \omega(B)^2 \frac{\omega^{-1}(2A) B \| f \|_{BMO(\omega)}^2}{r^{2}}
\]
\[
\leq C \omega(B) \| f \|_{BMO(\omega)}^2.
\]

From (6.4) and (6.8) we get that \(d\nu\) is an \(\omega\)-Carleson measure and
\[
\| d\nu \| \omega \leq C \| f \|_{BMO(\omega)}^2.
\]

It remains to prove that \(f \in \mathcal{M}(\beta, \gamma)'\). To this end consider \(B_0 = B(x_0, 1)\) where \(x_0\) fix is as in (2.2) and (2.3), part \(f\) as in (6.3), and also consider \(\varphi \in \mathcal{M}(\beta, \gamma)\). Notice first that since \(\varphi\) has null mean then \(\langle f_3, \varphi \rangle = 0\). Also, by (2.2),
\[
\| f_1 \| \varphi \| f \|_{BMO(\omega)} \omega(B_0) = C \| \varphi \|_{\mathcal{M}(\beta, \gamma)} \| f \|_{BMO(\omega)}.
\]

On the other hand, in the same fashion as in (6.5)–(6.7) but replacing \(q_t(y,x)\) by \(\varphi(x)\), \(\epsilon\) by \(\gamma\), and \(t\) and \(r\) by \(1\) we have, as in (6.9),
\[
\| f_2 \| \varphi \| f \|_{BMO(\omega)}.\]

The proof of the theorem is therefore finished.
References