Minimal recurrence relations for connection coefficients between classical orthogonal polynomials: Continuous case

E. Godoya,*, A. Ronveauxb, A. Zarzoc, I. Areaa

aDepartamento de Matemática Aplicada, Escuela Técnica Superior de Ingenieros Industriales y Minas, Universidad de Vigo, 36200-Vigo, Spain
bMathematical Physics, Facultés Universitaires Notre-Dame de la Paix, B-5000 Namur, Belgium
cDepartamento de Matemática Aplicada, Escuela Técnica Superior de Ingenieros Industriales, Universidad Politécnica de Madrid, c/ José Gutiérrez Abascal 2, 28006 Madrid, Spain

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Abstract

We present a simple approach in order to compute recursively the connection coefficients between two families of classical (continuous) orthogonal polynomials (Hermite, Laguerre, Jacobi and Bessel), i.e. the coefficients $C_m(n)$ in the expression $P_n(x) = \sum_{m=0}^{\infty} C_m(n)Q_m(x)$, where $\{P_n(x)\}$ and $\{Q_m(x)\}$ belong to the aforementioned class of polynomials. This is done by adapting a general and systematic algorithm, recently developed by the authors, to the classical situation. Moreover, extensions of this method to some related connection problems and to the generalized linearization problem are given.

Keywords: Orthogonal polynomials; Connection problem; Linearization problem

AMS classification: 33C25; 65Q05

1. Introduction

Given two families of polynomials $\{P_n(x)\}_{n \in \mathbb{N}}$ and $\{Q_m(x)\}_{m \in \mathbb{N}}$, the so-called connection problem between them, i.e. the computation of coefficients $C_m(n)$ in the expression

$$P_n(x) = \sum_{m=0}^{n} C_m(n)Q_m(x),$$

Corresponding author. E-mail: egodoy@dma.uvigo.es.

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plays an important role in many problems in Pure and Applied Mathematics or in Mathematical Physics (see, e.g., [6, 9, 12, 15, 16]). One of the first examples of connection coefficients are the well known Stirling numbers of first and second kind [24]. Since then, a number of particular cases have been studied in the literature (see [6, 8, 11, 23, 25] and references therein) by using several methods, being none of them designed to be applied in a general way, i.e. for a large enough class of polynomials in both sides of connection problem (1).

Recently the authors [19] have proposed a systematic algorithm which allows the computation of the connection coefficients in a recurrent way. The following general assumptions on the \{P_n(x)\} and \{Q_m(x)\} families in (1) are required for this algorithm to work:

(a) The polynomial \(P_n(x)\) satisfies a differential equation

\[ \mathcal{L}_r[P_n(x)] := \sum_{i=0}^{r} A_i(x; n)n^{(i)}(x) = 0 \quad (p_n^{(i)}(x) \equiv \frac{d^i P_n(x)}{dx^i}), \]

where the coefficients \(A_i(x; n)\) are polynomials in \(x\) of fixed degree (i.e., independent of \(n\)).

(b) The family \(\{Q_m(x)\}\) verifies a finite \((h + 2)\)-term recurrence relation

\[ xQ_m(x) = \sum_{k=m-h}^{m+1} M_{m,k} Q_k(x), \quad (2) \]

where \(M_{m,k}\) are \(x\)-independent coefficients. Moreover, this \(Q_m\)-family has to verify a finite structure relation,

\[ p(x)Q'_m(x) = \sum_{k=m-s-1}^{m+i-1} N_{m,k} Q_k(x) \quad (t \equiv \frac{d}{dx}), \quad (3) \]

where \(N_{m,k}\) are constants, \(s\) is a fixed integer and \(p(x)\) is a \(t\)-degree polynomial.

**Remark 1.** The above requirements for the \(\{P_n(x)\}\) and \(\{Q_m(x)\}\) families are expressed in terms of the derivative operator, giving rise to the so-called “continuous case”. However, the algorithm can be also applied if the derivative is replaced by a difference operator, either in an uniform lattice (“discrete case”, see [5, 21, 22]) or in a non-uniform one, as in [4] which deals with the exponential lattice. Moreover, it could be also used if more general linear operators (including Hahn operator) are considered.

At this point, let us briefly describe our method (see [19] for details). The action of the \(r\)th order differential operator \(\mathcal{L}_r\) on both sides of the connection problem (1), gives rise to the relation:

\[ \sum_{m=0}^{n} C_m(n) \mathcal{L}_r[Q_m(x)] = 0, \quad (4) \]

which contains terms of the form \(x^i Q_{m}^{(i)}(x)\), where \(i\) runs from 0 to \(r\) and \(j\) depends upon the degree of the polynomial coefficients characterizing the operator \(\mathcal{L}_r\). Then, the appropriate (and possibly repeated) use of properties (2), (3) allows us to express all terms appearing in the latter sum as a linear combination (with constant coefficients) of the polynomials \(Q_i(x)\) themselves. Thus, if the family \(\{Q_m(x)\}\) satisfies (2), (3), it is always possible to transform (4) in a relation of the
form: \( \sum_{m=0}^{K} C_m(n) R_m(x) = 0 \). Here, \( K \) is a positive integer whose specific value depends on the operator \( \mathcal{L}_x \) and also on the relations (2), (3) (see [19]), and \( R_m \) denotes a linear operator with constant (x-independent) coefficients acting on the index \( m \). Then, a simple shift of indices leads to the expression

\[
\sum_{m=0}^{K} M_m[C_m(n)] Q_m(x) = 0 \quad (m=0, \ldots, K).
\]

Here, as \( R_m \) above, \( M_m \) denotes a linear operator with constant coefficients acting on \( m \). So, in this way, a linear system of equations satisfied by the connection coefficients is obtained. Finally, due to its particular structure (see [19]), from this linear system a recurrence relation (in the index \( m \) only) for the \( C_m(n) \)-coefficients can be easily devised.

**Remark 2.** The algorithm we have just described can be considered as a proof of the following statement: “The aforementioned requirements (a) and (b) are sufficient conditions for the connection coefficients \( C_m(n) \) in (1) to verify a recurrence relation in the index \( m \) only”. For a detailed discussion on how the connection coefficients are affected by a hierarchy of nested assumptions on the families \( \{ P_m(x) \} \) and \( \{ Q_m(x) \} \), see [21, Section 2].

In the context of the connection problems we are dealing with, one of the most important cases appears when the family \( \{ P_m(x) \} \) in (1) happens to be orthogonal on an interval \( I \subseteq \mathbb{R} \) with respect to a positive measure \( d\mu(x) \). If this is so, the coefficient \( C_m(n) \) is the \( m \)th Fourier coefficient of \( P_m(x) \) with respect to the orthogonal polynomial basis \( \{ Q_m(x) \} \) and hence can be expressed as the following integral

\[
C_m(n) = \frac{1}{d^2_m} \int_I P_m(x) Q_m(x) \, d\mu(x) \quad \left( d^2_m = \int_I Q_m^2(x) \, d\mu(x) \right).
\]

This orthogonality property does not imply that the family \( \{ Q_m(x) \} \) satisfies relations (2), (3) and vice versa (i.e., in general, a polynomial family which verifies (2), (3) need not to be an orthogonal family). There exists, however, a wide class of polynomials for which both orthogonality and relations (2), (3) holds simultaneously. These are the so-called “semi-classical orthogonal polynomials” [10, 14]. In particular, a semi-classical orthogonal family always satisfies a three-term recurrence relation (i.e., (2) with \( h=1 \)) and a structure relation (3), where the integer \( s \) is now a number characterizing the class [14] of the family. Of course, if \( Q_m(x) \) in (1) is semi-classical, the recurrence relation for the \( C_m(n) \) coefficients given by our algorithm is, in fact, a recurrence for the integrals (6) which, as pointed out above, only involves the index \( m \).

On the other hand, the semi-classical class of orthogonal polynomials contains as (possibly) the most important particular case the well known classical orthogonal polynomials of Jacobi, Laguerre, Hermite and Bessel, for which one has \( s=0 \) in (3) and also \( t \leq 2 \). These four families are the main aim of this work.

Notice that, when the algorithm above described is applied, one is obliged to use \( \{ Q_m(x) \} \) as expanding basis (see Eq. (5)) since only relations (2), (3) are available. However, the classical
polynomials satisfy a number of special properties which can be used to simplify computations. In particular, there exists a derivative representation, i.e. an expression of type

\[ Q_m(x) = E_m Q'_{m+1}(x) + F_m Q'_m(x) + G_m Q'_{m-1}(x), \]

which is a consequence of the orthogonality of the \( \{Q_m(x)\} \) family [13, 16].

After Section 2, where notations and some basic properties are introduced, in Section 3 the connection problem (1) is considered when both \( \{P_n(x)\} \) and \( \{Q_m(x)\} \) are classical families. Our algorithm is adapted to the classical case by taking into account the latter expression which allows us to consider as expanding basis, not only the \( \{Q_m(x)\} \) one, but also its first (\( \{Q'_m(x)\} \)) or second (\( \{Q''_m(x)\} \)) derivatives. These three possibilities give rise to three recurrence relations for the connection coefficients which are, in general, of different lengths. Then, the recurrence relation of minimal order can be chosen. From now on, the word “minimal” will be used in this sense.

Closely related to (1) there are other connection problems which appear very often in applications. In particular, two of them are of special interest, namely

\[ P_n(\phi(t)) = \sum_{m=0}^{n} C_m(n) Q_m(t), \]

where \( \phi(t) \) is a polynomial in \( t \) of degree \( \lambda \), and

\[ (ax + b)^k P_n(x) = a^k \sum_{m=0}^{n+k} C_m(n) Q_m(x). \]

In Section 4 these two problems are shown to be treatable with our algorithm when both \( \{P_n(x)\} \) and \( \{Q_m(x)\} \) are classical families.

The last section (Section 5) shows how our algorithm works also for solving recurrently the so-called generalized linearization problem, i.e., the computation of coefficients \( L_{ijk} \) in the expression

\[ P_i(x) Q_j(x) = \sum_{k=0}^{i+j} L_{ijk} R_k(x). \]

This is done also, when the polynomials \( P_i(x) \), \( Q_j(x) \) and \( R_k(x) \) are classical.

It should be finally mentioned that our goal here is not to exploit all possible situations covered by our algorithm within the classical class of orthogonal polynomials, but to emphasize its systematic character and its simplicity, which allow one to implement it in any computer algebra system (here the Mathematica [29] symbolic language has been used). Let us recall that problem (1) contains already \( 4 \times 3 \) situations with \( P_n(x) \) and \( Q_m(x) \) being not in the same family. Moreover, the families of Jacobi, Laguerre and Bessel depend on two, one and one parameters, respectively, giving rise to three new cases when both \( P_n(x) \) and \( Q_m(x) \) belong to these three families.

2. Notations and basic properties

Let \( \{P_n(x)\} \) be any family of classical polynomials orthogonal with respect to a weight \( \rho(x) \) in some interval \( I \subseteq \mathbb{R} \). Then, the function \( \rho(x) \) must be (see, e.g., [15]) a solution of the Pearson
differential equation

\[[\sigma(x)\rho(x)'] = \tau(x)\rho(x),\]  \hspace{1cm} (10)

being \(\sigma(x)\) and \(\tau(x)\) polynomials of degree at most two and one, respectively. In what follows monic polynomials will be considered.

Besides the second order differential equation

\[\mathcal{L}_2,\nu[P_n(x)] := \sigma(x)P''_n(x) + \tau(x)P'_n(x) + \lambda_n P_n(x) = 0,\]  \hspace{1cm} (11)

where \(\lambda_n = -n\tau' - (1/2)n(n - 1)\sigma'',\) these polynomials satisfy a number of structural properties which in turn provide characterizations of them (see, e.g., [3, 7, 13, 15]). We shall need here three of those properties.

First, the monic sequence \(\{P_n(x)\}\) verifies a three-term recurrence relation [7]

\[xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \quad (n=0,1,\ldots; \quad C_n \neq 0; \quad P_{-1}(x) \equiv 0, \quad P_0(x) \equiv 1).\]  \hspace{1cm} (12)

Second, one has the so-called structure relation (see, e.g., [3])

\[\sigma(x)P'_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \quad (n=0,1,\ldots; \quad P_{-1}(x) \equiv 0, \quad P_0(x) \equiv 1).\]  \hspace{1cm} (13)

And third, there exists the not so well known derivative representation [13, 15]

\[P_n(x) = \frac{1}{n+1} P'_{n+1}(x) + F_n P'_n(x) + G_n P'_{n-1}(x), \quad (n=0,1,\ldots; \quad P_{-1}(x) \equiv 0, \quad P_0(x) \equiv 1).\]  \hspace{1cm} (14)

Remark 3. All of the \(x\)-independent coefficients appearing in the above three structural properties can be expressed [13, 15, 26, 27, 30] in terms of the polynomials \(\sigma\) and \(\tau\) which characterize the Pearson weight equation (10). In particular, these expressions for the coefficients \(B_n\) and \(C_n\) in (12) have been given in [26, 30]. The same result but for the coefficients \(\alpha_n, \beta_n\) and \(\gamma_n\) in (13) can be found in [27, 30]. Finally, coefficients \(F_n\) and \(G_n\) in (14) can be computed in terms of \(\sigma\) and \(\tau\) by mimicking the technique developed in [27, 30] (see also [13] where a completely different method is used).

For the sake of completeness, an Appendix has been included at the end of the manuscript giving these expressions. Moreover, the Appendix also includes two tables (see Tables 3 and 4) where the specific values of these coefficients are collected for each monic classical orthogonal family.

Concerning notations for the connection problem (1), the family \(\{P_n(x)\}\) will satisfy Eqs. (11)-(14), while for the family \(\{Q_m(x)\}\) the upper bar notation will be used; i.e., this family will satisfy Eqs. (11)-(14), with overlined coefficients \((\bar{\sigma}, \bar{\tau}, \bar{\lambda}_m), (\bar{B}_m, \bar{C}_m), (\bar{\alpha}_m, \bar{\beta}_m, \bar{\gamma}_m)\) and \((\bar{F}_m, \bar{G}_m)\), respectively.

Remark 4. As we have just pointed out, throughout the paper the normalization for both \(\{P_n(x)\}\) and \(\{Q_m(x)\}\) families in (1) will be to consider monic polynomials. Notice that this can be done without loss of generality in what concern the connection problems. This is so because if other normalizations are considered, say \(\tilde{P}_n(x) = N_n P_n(x)\) and \(\tilde{Q}_m(x) = M_m Q_m(x)\), it is easy to check that the new connection coefficients \(\tilde{C}_m(n)\) between the families \(\{\tilde{P}_n(x)\}\) and \(\{\tilde{Q}_m(x)\}\) are given by \(\tilde{C}_m(n) = N_n^{-1} M_m C_m(n)\).
3. Minimal recurrence relations for connection coefficients between classical orthogonal polynomials

As in [19], the first step to obtain a recurrence relation for the connection coefficients consists in applying the differential operator \( L_{2,n} \), defined in (11), to both sides of the connection problem (1). This gives

\[
L_{2,n} [P_n] = \sum_{m=0}^{n} C_m(n) \{ \sigma(x) Q'_m(x) + \tau(x) Q'_m(x) + \lambda_n Q_m(x) \} = 0. \tag{15}
\]

Then, the required recurrence relation comes out (with a shift of indices) after expanding, in a linear constant coefficients combination of linearly independent polynomials, the expression:

\[
S_{m,n}(x) := \sigma(x) Q'_m(x) + \tau(x) Q'_m(x) + \lambda_n Q_m(x). \tag{16}
\]

As pointed out in the introduction, the algorithm developed in [19] chooses the \( \{ Q_m(x) \} \) family as expanding basis, giving a recurrence of maximum order eight (cf. [19, Eq. (9)]) On expanding (16) in this basis, one should multiply this expression by \( \bar{\sigma}(x) \) twice in order to use (11) to eliminate \( \sigma(x) Q'_m(x) \) and after that (13) and (12). However, the existence of the derivative representation (14) allows us to consider as expanding basis, not only the \( \{ Q_m(x) \} \) one, but also its first \( \{ Q'_m(x) \} \) or second \( \{ Q''_m(x) \} \) derivatives. Choosing the latter two bases the aforementioned multiplication by \( \bar{\sigma}(x) \) twice is not needed and then the order of the recurrence for the \( C_m(n) \)-coefficients is not increased artificially.

As shown below (see Section 3.1), the minimal recurrence relation (i.e. the shortest one in order) for the connection coefficients in (1) is attained by using the \( \{ Q'_m(x) \} \) basis and it is of maximum order four, with two exceptions. First in the Laguerre–Hermite and Hermite–Laguerre problems the order is reduced by one due to the fact that the degree of polynomials \( \sigma(x) \) and \( \bar{\sigma}(x) \) for these two families is less than or equal to one. Second, when the families \( \{ P_n(x) \} \) and \( \{ Q_m(x) \} \) in the connection problem (1) satisfy two differential equations, (11) and (11), respectively, with the same \( \sigma(x) \) (i.e. \( \sigma(x) = \bar{\sigma}(x) \)), then the order of the recurrence (see Section 3.2 below) is strongly reduced to two considering \( \{ Q'_m(x) \} \) instead of the aforementioned \( \{ Q''_m(x) \} \). All of these considerations are summarized in Table 1. Of course, further reductions in the order could appear if some specific values for the parameters are considered.

Let us now describe the algorithms when \( \{ Q'_m(x) \} \) or \( \{ Q''_m(x) \} \) is chosen as expanding basis for expression (16).

3.1. Using the \( \{ Q'_m(x) \} \) basis

To consider \( \{ Q'_m(x) \} \) as expanding basis one can proceed as follows. First, Eq. (14) and its derivative allow to write

\[
Q_m(x) = \sum_{j=m-2}^{m+2} a_{m,j} Q'_j(x). \tag{17}
\]
Table 1
Order of the ‘minimal’ recurrence relation satisfied by the connection coefficients between two families of classical orthogonal polynomials and the corresponding expanding basis. The column represents the \( \{P_n(x)\} \) monic family in Eq. (1) and the row the \( \{Q_n(x)\} \) monic family also with the notation of Eq. (1).

<table>
<thead>
<tr>
<th>{Q_n} family</th>
<th>{P_n} family ↓</th>
<th>Hermite</th>
<th>Laguerre</th>
<th>Jacobi</th>
<th>Bessel</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H_n(x) )</td>
<td>( P_n^{(a)}(x) )</td>
<td>order 3</td>
<td>order 4</td>
<td>order 4</td>
<td>order 4</td>
</tr>
<tr>
<td>( L_n(x) )</td>
<td>( Q_n^{(a)}(x) )</td>
<td>order 2</td>
<td>order 4</td>
<td>order 4</td>
<td>order 4</td>
</tr>
<tr>
<td>( P_n^{(a,b)}(x) )</td>
<td>( Q_n^{(a)}(x) )</td>
<td>order 4</td>
<td>order 4</td>
<td>order 2</td>
<td>order 4</td>
</tr>
<tr>
<td>( \Lambda_n^{(a)}(x) )</td>
<td>( Q_n^{(a,b)}(x) )</td>
<td>order 4</td>
<td>order 4</td>
<td>order 4</td>
<td>order 2</td>
</tr>
</tbody>
</table>

Second, from the derivative of (12) and (14) one has
\[
\tau(x)Q_m''(x) = \sum_{j=m-2}^{m+2} a_{m,j}^{(1)} Q_j''(x). \tag{18}
\]

And third, the second derivative of (12) and (14) gives
\[
\sigma(x)Q_m''(x) = \sum_{j=m-2}^{m+2} a_{m,j}^{(2)} Q_j''(x), \tag{19}
\]

where all the \( a_{m,j}^{(k)} \)-coefficients are constants easily computed.

Now, inserting (17)–(19) in Eq. (15) one obtains
\[
\sum_{m=0}^{n} C_m(n) \left\{ \sum_{j=m}^{m+2} \Omega_{m,j}(n) Q_j''(x) \right\} = 0,
\]

\[
\Omega_{m,j}(n) := a_{m,j}^{(2)} + a_{m,j}^{(1)} + \lambda_m a_{m,j},
\]

Finally, after an appropriate shift of indices, this latter expression provides the backward “minimal” recurrence relation of order four which can be written as
\[
\sum_{s=0}^{4} \Omega_{m+s,m+2}(n) C_{m+s}(n) = 0, \quad 0 \leq m \leq n - 1,
\]

being the initial conditions given by \( C_{n+s}(n) = 0 \) \((s = 1, 2, 3)\) and \( C_n(n) = 1\), since monic polynomials have been considered.
Remark 5. When considering the Laguerre–Hermite and Hermite–Laguerre connection problems it can be easily checked that the coefficient $\Omega_{m+4,m+2}(n)$ in (20) vanish and then the order of the minimal recurrence relation is reduced to three (see Table 1).

3.2. Using the \{Q'_m(x)\} basis

When the differential equations (11) and (11) satisfied by the polynomials $P_n(x)$ and $Q_m(x)$ in (1), respectively, are such that $\sigma(x) = \bar{\sigma}(x)$, then the minimal recurrence relation for the connection coefficients appears when $\{Q'_m(x)\}$ is the expanding basis for expression (16). For this reason, this basis should be used when dealing with the Laguerre–Laguerre, Jacobi–Jacobi and Bessel–Bessel connection problems (see Table 1).

The algorithm in this case is as follows. First, the derivative representation (14) gives

$$Q'_m(x) = \sum_{j=m-1}^{m+1} b_{m,j} Q'_j(x).$$

Second, the derivative of (12) and (14) allow to write

$$\tau(x) Q''_m(x) = \sum_{j=m-1}^{m+1} b_{m,j}^{(2)} Q'_j(x).$$

And third, from (11) one has $\sigma(x) Q'''_m(x) = -\bar{\tau}(x) Q'_m(x) = -\bar{\tau}(x) Q'_m(x) - \lambda_m Q'_m(x)$ and then, applying again the derivative of (12) and (14) one obtains

$$\sigma(x) Q''''_m(x) = \sum_{j=m-1}^{m+1} b_{m,j}^{(3)} Q'_j(x),$$

where $b_{m,j}^{(k)}$ are constants easily computed.

Insertion of (21)–(23) into (15) gives

$$\sum_{m=0}^{n} C_m(n) \left\{ \sum_{j=m-1}^{m+1} A_{m,j}(n) Q'_j(x) \right\} = 0,$$

$$A_{m,j}(n) = b_{m,j}^{(2)} + b_{m,j}^{(1)} + \lambda_m b_{m,j}.$$

Finally, after an appropriate shift of indices, this latter expression provides the backward "minimal" recurrence relation of order two which can be written as

$$\sum_{z=-1}^{1} A_{m+s,n}(n) C_{m+s}(n) = 0, \quad 1 \leq m \leq n,$$

being the initial conditions given by $C_{n+1}(n) = 0$ and $C_n(n) = 1$, since monic polynomials have been considered.
3.3. Examples

As illustration of the recurrences which these algorithms provide, we consider here two examples: the Bessel–Bessel and Jacobi–Jacobi connection problems.

3.3.1. Bessel polynomials

In the connection problem \( Y_n^{(a)}(x) = \sum_{m=0}^{n} C_m(n) Y_m^{(b)}(x) \), the \( C_m(n) \)-coefficients satisfy the second order recurrence relation

\[
(b + 2m)(1 + b + 2m)(2 + b + 2m)^2(3 + b + 2m)(-1 + m - n) \\
\times (a + m + n)C_{m-1}(n) + 2m(1 + b + 2m)(2 + b + 2m)(3 + b + 2m) \\
\times [(2 + b)(b - a) + 2m(m + b + 1) - 2n(n + a + 1)]C_m(n) \\
+ 4m(1 + m)(b + 2m)(1 - a + b + m - n)(2 + b + m + n)C_{m+1}(n) = 0,
\]
valid for \( 1 \leq m \leq n \), with the initial conditions \( C_{n+1}(n) = 0 \) and \( C_n(n) = 1 \). The solution is

\[
C_m(n) = 2^{m-n} \binom{n}{m} \frac{(b - a - n + m + 1)_{n-m}}{(a + n + m + 1)_{n-m}(b + 2m + 2)_{n-m}}, \quad 0 \leq m \leq n,
\]
where \( (A)_n = A(A + 1)(A + 2)\ldots(A + n - 1) \) denotes the Pochhammer symbol. From a completely different approach, this expression has been already obtained in [2, Eq. (8.2)] (where there is an incorrect sign).

3.3.2. Jacobi polynomials

In the connection problem \( P_n^{(a,b)}(x) = \sum_{m=0}^{n} C_m(n) P_m^{(c,d)}(x) \), the \( C_m(n) \)-coefficients satisfy the second order recurrence relation

\[
(c + d + 2m)(1 + c + d + 2m)(2 + c + d + 2m)^2(3 + c + d + 2m) \\
\times (1 - m + n)(a + b + m + n)C_{m-1}(n) \\
\times 2m(1 + c + d + 2m)(2 + c + d + 2m)(3 + c + d + 2m) \\
\times [(2 + c + d)(bc - ad) + m(m + c + d + 1)(2b + c - 2a - d) \\
+ n(n + a + b + 1)(c - d)]C_m(n) \\
+ 4m(1 + m)(1 + c + m)(1 + d + m)(c + d + 2m) \\
\times (1 - a - b + c + d + m - n)(2 + c + d + m + n)C_{m+1}(n) = 0.
\]
valid for \( 1 \leq m \leq n \) with the initial conditions \( C_{n+1}(n) = 0 \) and \( C_n(n) = 1 \). Following a different approach, the solution of this recurrence has been expressed in [6, 11] in terms of heavy hypergeometric series up to a \(_3F_2\).
4. Two related connection problems

As pointed out in the introduction, in this section we consider the two connection problems (7) and (8) when both families \{P_n(x)\} and \{Q_m(x)\} belong to the classical class of orthogonal polynomials.

4.1. The connection problem: \( P_n(\phi(t)) = \sum_{m=0}^{n_i} C_m(n)Q_m(t) \)

If the family \{P_n(x)\} is classical, then the polynomial \( P_n(x) \) in (7) is a solution of the second order differential equation (11) and so, if the change of variable \( x = \phi(t) \) (being \( \phi(t) \) a polynomial in \( t \) of degree \( \lambda \)) is considered, the new polynomial \( P_n(\phi(t)) \) becomes a solution of the second order equation \( \mathcal{M}_{2,n}[P_n(\phi(t))] = 0 \), where (with the notation \( D_t \equiv d/dt \)),

\[
\mathcal{M}_{2,n}[P_n(\phi(t))] = \left( (\phi'(t))^2 \mathcal{Z}(t) + (\phi''(t))(t)\mathcal{S}(t) - \phi'(t)\mathcal{D}(t) + \lambda(\phi'(t))^3I \right),
\]

being \( \mathcal{S}(t) = \sigma(\phi(t)), \mathcal{T}(t) = \tau(\phi(t)) \) and \( \sigma \) and \( \tau \) the polynomials appearing in (10), (11).

In these conditions, the first step of the above described algorithms can be taken by applying this \( \mathcal{M}_{2,n} \) operator to both sides of (7). This gives:

\[
\mathcal{M}_{2,n}[P_n(\phi(t))] = \sum_{m=0}^{n_i} C_m(n)\mathcal{M}_{2,n}[Q_m(t)] = 0,
\]

which is an expression similar to (15). Then, as Eq. (16) before, the quantity

\[
\mathcal{M}_{2,n}[Q_m(t)] = \mathcal{S}(t)\phi'(t)Q_m''(t) + (\phi'(t))^2\mathcal{S}(t)Q_m'(t) + \mathcal{D}(t)Q_m(t) + \lambda(\phi'(t))^3Q_m(t),
\]

can be expanded in a linear combination (with constant coefficients) of the three families \{Q_m(x)\}, \{Q'_m(x)\} or \{Q''_m(x)\}, giving rise to three different recurrence relations for the connection coefficients. From them one can choose the minimal one (i.e. the shortest in order) which, in most cases, is attained when \{Q_m''(t)\} is chosen as expanding basis. Of course, the minimal order strongly depends upon the degree of \( \phi(t) \).

4.1.1. Examples

As illustration, we consider now several examples giving for each of them, the concrete connection problem, the minimal recurrence relation for the corresponding connection coefficients (together with the initial conditions), the expanding basis and, sometimes, the solution of this recurrence. In all examples only the case in which the families \{P_n(x)\} and \{Q_m(x)\} in (7) coincide is treated. Nevertheless, as described above, the algorithm can be used when both families are not the same.

1. Hermite polynomials: \( H_n(x) \).

1.1. Connection problem: \( H_n(t + y) = \sum_{m=0}^{n_i} C_m(n; y)H_m(t) \).

- Minimal recurrence relation (of order one)

\[
(2 - m + n)C_{m-2}(n; y) - y(m - 1)C_{m-1}(n; y) - 0, \quad (2 \leq m \leq n + 1)
\]

- Initial condition: \( C_n(n; y) = 1 \).
- Expanding basis: \{H_m''(t)\}.
- Solution: \( C_m(n; y) = \binom{n}{m}y^{n-m} \).
With a change of normalization (here monic polynomials are considered), this expression coincides with the one given in [8, Eq. 10.12 (41)].

1.2. Connection problem: \( H_n(at) = a^n \sum_{m=0}^n C_m(n; a)H_m(t) \).
- Minimal recurrence relation (of order two)

\[
2a^2(1-m+n)C_{m-1}(n; a) + m(1-a^2)(m+1)C_{m+1}(n; a) = 0,
\]
\((1 \leq m \leq n)\).
- Initial conditions: \( C_{n+1}(n; a) = 0 \) and \( C_n(n; a) = 1 \).
- Expanding basis: \( \{H'_m(t)\} \).
- Solution:

\[
C_m(n; a) = \begin{cases} 
(m+1)_{-m} \left( \frac{a^2-1}{2a^2} \right)^{\frac{m}{2}} & \text{if } n-m = 2k, \\
0 & \text{if } n-m = 2k+1.
\end{cases}
\]

Here, \((A)_h = A(A+1)(A+2) \cdots (A+n-1)\) denotes the Pochhammer symbol and \( n!! = 2^{n/2} \Gamma(n/2 + 1) \) is the double factorial for even argument.

1.3. Connection problem: \( H_n(t^2) = \sum_{m=0}^{2n} C_m(n)H_m(t) \).
- Minimal recurrence relation (of order eight)

\[
16(2n-m+5)C_{m-5}(n)
+ 16[m(11-2m+3n)-(6n+15)]C_{m-3}(n)
+ 4(m-1)[m(19-6m+6n)-6(n+3)]C_{m-1}(n)
+ 2m(m^2-1)(2n-4m+3)C_{m+1}(n)
+ m(1-m^2)(m+2)(m+3)C_{m+3}(n) = 0, \quad (5 \leq m \leq 2n+4).
\]
- Initial conditions: \( C_{2n+i}(n) = 0 \) \((i = 1, \ldots, 8)\) and \( C_{2n}(n) = 1 \).
- Expanding basis: \( \{H'_m(t)\} \).

Of course, in this case one has \( C_{2m+i}(n) = 0 \) \((m = 0, \ldots, n-1)\).

2. Bessel polynomials: \( Y_n^{(a)}(x) \) \((a \neq k, k \geq 2)\).

2.1. Connection problem: \( Y_n^{(b)}(t+y) = \sum_{m=0}^n C_m(n; y)Y_m^{(b)}(t) \).
- Minimal recurrence relation (of order four)

\[
(b+2m-2)(b+2m+2)(m-n-2)(m+n+b-1)C_{m-2}(n; y)
+ (m-1)(b+2m-1)(b+2m+3)2[8(b+m+n)(m-n-1)
+ (b+2m+2)(b+2m-2)y]C_{m-1}(n; y) + (m-1)2(b+2m+1)
\times (b+2m-2)(b+2m+4)[24(m-n)(b+m+n+1)
\]
\[-8(b + 2m + 3)(b + 2m - 1)y + (b + 2m - 1)z(b + 2m + 2)y^2]C_m(n; y) + 4(m - 1)z(b + 2m + 3)y_2(b + 2m - 2)z \times [8(b + m + n + 2)(m + 1 - n) - (b + 2m)(b + 2m + 4)y]C_{m+1}(n; y) + 16(m - 1)z(b + 2m - 2)(m + 2 - n)(m + n + b + 3)C_{m+2}(n; y) = 0, (2 \leq m \leq n + 1),

\]

where \((A)_n\) denotes the already mentioned Pochhammer symbol.

- Initial conditions: \(C_{n+i}(n) = 0\) \((i = 1, 2, 3)\) and \(C_{n}(n) = 1\).
- Expanding basis: \(\{[Y^{(b)}(t)]'\}\).
- The solution can be written in the "almost" explicit form

\[
C_m(n; y) = \binom{n}{m} \sum_{k=0}^{m-1} (-1)^{n-m} \frac{(b + n + m + 1 - k)2k+1}{(b + 2m + 2)2n-2m-1} \alpha_{k,m}(-y)^{k+1},
\]

where \((A)_n\) denotes the Pochhammer symbol and \(\alpha_{k,m}\) are coefficients which do not depend on the parameters \(b, y\) and \(n\). For the sake of completeness in Table 2 some values of these \(\alpha_{k,m}\) numbers are given.

<table>
<thead>
<tr>
<th>(m) (\backslash) (k)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(n-1)</td>
<td>(1)</td>
<td>(8)</td>
<td>(1)</td>
<td>(672)</td>
<td>(480)</td>
<td>(48)</td>
<td>(1)</td>
</tr>
<tr>
<td>(n-2)</td>
<td>(1)</td>
<td>(8)</td>
<td>(1)</td>
<td>(672)</td>
<td>(480)</td>
<td>(48)</td>
<td>(1)</td>
</tr>
<tr>
<td>(n-3)</td>
<td>(72)</td>
<td>(24)</td>
<td>(1)</td>
<td>(53760)</td>
<td>(4320)</td>
<td>(120)</td>
<td>(1)</td>
</tr>
<tr>
<td>(n-4)</td>
<td>(768)</td>
<td>(480)</td>
<td>(48)</td>
<td>(53760)</td>
<td>(4320)</td>
<td>(120)</td>
<td>(1)</td>
</tr>
<tr>
<td>(n-5)</td>
<td>(9600)</td>
<td>(9600)</td>
<td>(1680)</td>
<td>(80)</td>
<td>(1)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(n-6)</td>
<td>(138240)</td>
<td>(201600)</td>
<td>(53760)</td>
<td>(4320)</td>
<td>(120)</td>
<td>(1)</td>
<td></td>
</tr>
<tr>
<td>(n-7)</td>
<td>(2257920)</td>
<td>(4515840)</td>
<td>(1693440)</td>
<td>(201600)</td>
<td>(9240)</td>
<td>(168)</td>
<td>(1)</td>
</tr>
</tbody>
</table>

\[2.2.\] Connection problem: \(Y_n^{(0)}(at) = a^n \sum_{m=0}^{n} C_m(n; a)Y_m(t)\).

- Minimal recurrence relation (of order two)

\[
a^2(1 + m)(1 + 2m)(3 + 2m)(m - n - 1)(m + n)C_{m-1}(n; a) + a(1 + 2m)(3 + 2m)(m(m + 1)(2 - a) - an(n + 1)]C_m(n; a) + a^2m(1 + m - n)(2 + m + n)C_{m+1}(n; a) = 0, (1 \leq m \leq n),
\]

- Initial conditions: \(C_{n+i}(n; a) = 0\) and \(C_{n}(n; a) = 1\).
- Expanding basis: \(\{[Y_m(t)]'_n\}\).
3. Chebyshev polynomials of first kind: $T_n(x)$.

3.1. Connection problem: $T_n(at+b) = a^n \sum_{m=0}^{n} C_m(n;a,b) T_m(t)$. 
- Minimal recurrence relation (of order four)

$$
16a^2(1 + m)(2 - m + n)(-2 + m + n)C_{m-2}(n;a,b) \\
+ 16ab(1 - m)(1 + m)(-3 + 2m)C_{m-1}(n;a,b) \\
+ 8m[2(a^2 + b^2) - 1] + (2 - a^2 - 2b^2)m^2 - a^2n^2]C_m(n;a,b) \\
+ 4ab(1 - m)(1 + m)(3 + 2m)C_{m+1}(n;a,b) \\
+ a^2(-1 + m)(-2 - m + n)(2 + m + n)C_{m+2}(n;a,b) = 0,
$$

$$(2 \leq m \leq n+1).$$
- Initial conditions: $C_{n+i}(n;a,b) = 0$ ($i = 1, 2, 3$) and $C_n(n;a,b) = 1$.
- Expanding basis: \{T_m(t)\}.

In [28, p. 175] a more complicated recurrence relation (mixing both indices $n$ and $m$) for these coefficients is obtained by a completely different method.

4.2. The connection problem: $(ax + b)^{k}P_n(x) = a^{k}\sum_{m=0}^{n+k} C_m(n)Q_m(x)$

To use our algorithm for solving recurrently the connection problem (8) a similar procedure to the one described in the previous section can be applied. Notice that, being $P_n(x)$ classical and hence solution of (11), the polynomial $(ax + b)^k P_n(x)$ satisfies the differential equation $\mathcal{N}_{2,n}[(ax + b)^k P_n(x)] = 0$, where

$$
\mathcal{N}_{2,n} := (ax + b)^2 \sigma(x)\frac{D^2}{\frac{d}{dx}} + [(ax + b)^2 \tau(x) - 2ka(ax + b)\sigma(x)]D \\
+ [k(k + 1)ax + b)^2 \tau(x) + \lambda_n(ax + b)^2]f.
$$

Then, applying this $\mathcal{N}_{2,n}$ operator to both sides of (8), the recurrence relation comes out by expanding the resulting quantity $\mathcal{N}_{2,n}[Q_n(x)]$ on one of the three available basis (i.e. \{Q_m(x)\}, \{Q'_m(x)\} or \{Q''_m(x)\}). In this case, the minimal recurrence for the connection coefficients is obtained (in most cases) by using the \{Q'_m(x)\} basis.

As an illustrative example, let us consider the connection problem

$$
x^{k}H_n(x) = \sum_{m=0}^{n+k} C_m(n,k)H_m(x)
$$

for Hermite polynomials, $H_n(x)$. In this case the minimal recurrence relation is of order four and reads as follows

$$
4(3 + k - m + n)C_{m-3}(n;k) \\
+ 2(-1 + 2k + k^2 + 3m - 2m^2 - n + 2mn)C_{m-1}(n;k) \\
+ m(1 + m)(-1 - k - m + n)C_{m+1}(n;k) = 0 \quad (3 \leq m \leq n + k + 2)
$$
with the initial conditions \( C_{n+k+i}(n;k) = 0 \) \( (i = 1, 2, 3) \) and \( C_{n+k}(n;k) = 1 \). Obviously, one has \( C_{n+k-i}(n;k) = 0, \) \( i = 1, \ldots, \lfloor (n+k+1)/2 \rfloor \), being \( [A] \) the integer part of \( A \). For \( n = 0 \), it can be easily checked that the coefficients \( C_m(0;k) \) given by (27) exactly coincide with the numerical values computed in [1, Table 22.12].

### 4.3. Recurrence relations for integrals

As pointed out in the introduction (see Eq. (6)), it is interesting to notice here that the \( C_m(n;k) \)-coefficients in (26) can be written as \( C_m(n;k) = I(k,n,m)/d_m^2 \) where

\[
I(k,n,m) := \int_{-\infty}^{\infty} x^k H_m(x) H_n(x) \exp\{-x^2\} \, dx, \quad d_m^2 = I(0,m,m).
\]

For monic Hermite polynomials one has \( d_m^2 = m! \sqrt{\pi}/2^m \) and so, Eq. (27) gives, in fact, the following backward recurrence relation for the integrals \( I(k,n,m) \):

\[
\begin{align*}
(n - m - k)I(k,n,m) &+ \frac{1}{2}[(k + 1)^2 + n(2m - 3) - 2m^2 + 7m - 7]I(k,n,m - 2) \\
&+ \frac{1}{4}(m - 2)(m - 3)(k + n - m + 4)I(k,n,m - 4) = 0,
\end{align*}
\]

which only involves the index \( m \). Now the initial conditions are \( I(k,n,n+k+i) = 0 \) \( (i = 1, 2, 3) \) and \( I(k,n,n+k) = d_{k+n}^2 \).

This statement can be generalized to any classical orthogonal family. Let \( \{P_n(x)\} \) be a family of monic classical orthogonal polynomials with respect to the weight \( \rho(x) \), solution of the Eq. (10), in some interval \( I \subseteq \mathbb{R} \). From the connection problem

\[
x^k P_n(x) = \sum_{m=0}^{n+k} C_{m}(n,k) P_m(x),
\]

we deduce obviously:

\[
I(k,n,m) := \int_I x^k P_m(x) P_n(x) \rho(x) \, dx = C_{m}(n,k) d_m^2,
\]

where \( d_m^2 = I(0,m,m) \) is the normalization constant.

Then, from the second order differential equation satisfied by \( x^k P_n(x) \) and using the \( \{P_n(x)\} \) basis, our algorithm generates a minimal recurrence relation in \( m \) for the connection coefficients \( C_{m}(n,k) \) or the integrals \( I(k,n,m) \) of order five. This formula is, in fact, a recurrence relation for the integrals \( I(k,n,m) \) which only involves the index \( m \), and so, it is quite different from the usual recurrence in both indices \( k \) and \( m \), constructed from the three-term recurrence relation (12) satisfied by the polynomials \( \{P_n(x)\} \).
5. Extensions: generalized linearization problem

The algorithms considered in Sections 3 and 4 can be easily extended to solve recurrently the connection problem

\[ M_k(x) = \sum_{m=0}^{k} C_m(n; k) Q_m(x) \]  

being \( M_k(x) \) a polynomial of degree \( k \) solution of a \( r \)-th order linear differential equation (with polynomial coefficients) and \( \{ Q_m(x) \} \) any classical family.

In particular, if \( A(x) = \sum R_j(x) \) being \( \{ e(x) \} \) and \( \{ R_j(x) \} \) a so-called classical, a fourth order differential operator \( R_{4, i, j} \) can be constructed \([17, 18]\) such that

\[ R_{4, i, j}[e(x) R_j(x)] = 0. \]

Then, connection problem (28) becomes the so-called “generalized linearization problem” (see, e.g., \([20]\)) and can be handled recurrently with our algorithm.

Many interesting situations appear also when the operator killing the polynomial \( M_k(x) \) in (28) is of first order.

For example, if \( M_k(x) = (ax + b)^k \), the operator \( R_{1,k} \) such that \( R_{1,k}[(ax + b)^k] = 0 \) is, obviously:

\[ R_{1,1} = (ax + b)D - kaI \quad (D = d/dx). \]

It has been checked that the corresponding connection coefficients satisfy in all classical cases a minimal recurrence relation of second order, which appears when \( \{ Q'_m(x) \} \) is the expanding basis.

More generally, if \( M_k(x) \) is an arbitrary monic polynomial of degree \( k \), the expansion

\[ (M_k(x))' = \sum_{m=0}^{i+k} C_m(n; i, k) Q_m(x), \]

is solved with the operator \( R_{1,k,i} \equiv M_k(x)D - iM'_k(x)I \) and the minimal recurrence relation for \( C_m(n; i, k) \) is of order \( 2k \) in terms of the \( \{ Q'_m(x) \} \) basis.

The most general situation for which the degree of the polynomial coefficients in \( R_1 \) does not depend on the exponents \( i \) in the latter example corresponds to

\[ \prod_{j=1}^{s} (M_j(x))^k = \sum_m C_m(n; k_j, j) Q_m(x). \]

In this case, the operator \( R_{1,j,k} \), easily computed by the logarithmic derivative, is

\[ R_{1,j,k} = \prod_{j=1}^{s} M_j(x)D - \left( \sum_{j=1}^{s} k_j M'_j(x) \prod_{k=1, k\neq j}^{s} M_k(x) \right) I. \]

So, this kind of connection problems could be also managed with our method, giving minimal recurrences when choosing \( \{ Q'_m(x) \} \) as expanding basis.
Remark 6. The search for the minimal recurrence relations (see Eqs. (20) and (24)) satisfied by connection coefficients in the classical continuous case developed here, can be also considered for the classical discrete orthogonal polynomials of Hahn, Kravchuk, Meixner and Charlier, already approached in previous work [21, 22]. Results for connection problems inside these discrete families are now in progress [5].

Appendix: data for monic classical orthogonal polynomials

With the notation $\tau_\mu(x) = \tau(x) + \mu \sigma'(x)$ the expressions of the coefficients appearing in Eqs. (12)–(14) in terms of the polynomials $\sigma$ and $\tau$ characterizing the Pearson weight equation (see Remark 3) are as follows.

- Coefficients of the three-term recurrence relation (12).

$$B_n = \frac{1}{\tau'_n \tau_{n-1}} \left[ \sigma'' \tau_{n+1-n}(0) - \tau' \tau_{2n}(0) \right], \quad (n \geq 0) \quad (B_{-1} = 0),$$

$$C_n = \frac{n \tau'_n}{2(\tau'_{n-1})^2} \left\{ 2 \tau' \left[ \sigma'(0) \tau_{n-1}(0) - \sigma(0) \tau' \right] - \sigma'' \left[ \tau_{n-1}(0) \tau_{1-n}(0) + 4(n-1)\sigma(0)\tau'_{n-1} \right] \right\}, \quad (n \geq 0) \quad (C_{-1} = 0).$$

- Coefficients of the structure relation (13).

$$\alpha_n = \frac{n}{2} \sigma'',$$

$$\beta_n = n \tau'_n \left( \frac{2 \sigma'(0) \tau'_{n-\frac{1}{2}} \tau'_n - \sigma'' [\tau_n(0) \tau'_{n-\frac{1}{2}} + \tau(0) \tau'_n]}{2 \tau'_{n-1} \tau'_{n-\frac{1}{2}} \tau'_n} \right),$$

$$\gamma_n = n \tau'_n \tau'_{n-\frac{1}{2}} \left( \frac{2 \sigma(0) (\tau'_{n-1})^2 + \tau_n(0) [-2 \sigma'(0) \tau'_{n-\frac{1}{2}} + \tau(0) \sigma'']}{2 (\tau'_{n-1})^2 \tau'_{n-\frac{1}{2}} \tau'_{n-\frac{1}{2}}} \right).$$

- Coefficients of the derivative representation (14).

$$F_n = \frac{\tau(0) \sigma'' - \sigma'(0) \tau'}{\tau'_{n-1} \tau'_n},$$

$$G_n = \frac{\tau'_{n-\frac{1}{2}} - \tau'_{n+\frac{1}{2}}}{2 \tau'_{n-1} \tau'_{n-\frac{1}{2}} (\tau_n' - \frac{1}{2})^2} \left\{ \tau_n(0) [2 \sigma'(0) \tau' - \tau_{1-n}(0) \sigma''] - 2 \sigma(0) (\tau'_{n-1})^2 \right\}.$$
Tables 3 and 4 contain the specific values of these coefficients collected for each monic classical orthogonal family.

### Table 3
Data for monic classical Hermite and Jacobi orthogonal polynomials

<table>
<thead>
<tr>
<th></th>
<th>Hermite $H_n(x)$</th>
<th>Jacobi $P_n^{\alpha, \beta}(x)$ $(\alpha &gt; -1, \beta &gt; -1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma(x)$</td>
<td>1</td>
<td>$1 - x^2$</td>
</tr>
<tr>
<td>$\tau(x)$</td>
<td>$-2x$</td>
<td>$-(\alpha + \beta + 2)x + \beta - \alpha$</td>
</tr>
<tr>
<td>$\lambda_n$</td>
<td>$2n$</td>
<td>$n(n + \alpha + \beta + 1)$</td>
</tr>
<tr>
<td>$B_n$</td>
<td>0</td>
<td>$\frac{\beta^2 - \alpha^2}{(2n + \alpha + \beta)(2n + 2 + \alpha + \beta)}$</td>
</tr>
<tr>
<td>$C_n$</td>
<td>$\frac{n}{2}$</td>
<td>$\frac{4n(n + \alpha)(n + \alpha + \beta)(n + \beta)}{(2n + \alpha + \beta - 1)(2n + \alpha + \beta + 1)^2(2n + \alpha + \beta + 1)}$</td>
</tr>
<tr>
<td>$\gamma_n$</td>
<td>0</td>
<td>$\frac{2n(\alpha - \beta)(n + \alpha + \beta + 1)}{(2n + \alpha + \beta)(2n + 2 + \alpha + \beta)}$</td>
</tr>
<tr>
<td>$F_n$</td>
<td>0</td>
<td>$\frac{2(\alpha - \beta)}{(2n + \alpha + \beta)(2n + 2 + \alpha + \beta)}$</td>
</tr>
<tr>
<td>$G_n$</td>
<td>0</td>
<td>$\frac{-4n(n + \alpha)(n + \beta)}{(2n + \alpha + \beta - 1)(2n + \alpha + \beta + 1)^2(2n + \alpha + \beta + 1)}$</td>
</tr>
</tbody>
</table>

### Table 4
Data for monic classical Laguerre and Bessel orthogonal polynomials

<table>
<thead>
<tr>
<th></th>
<th>Laguerre $L_n^{\alpha}(x)$ $(\alpha &gt; -1)$</th>
<th>Bessel $\gamma_n^{(\alpha)}(x)$ $(\alpha \neq -n, n \geq 2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma(x)$</td>
<td>$x$</td>
<td>$x^2$</td>
</tr>
<tr>
<td>$\tau(x)$</td>
<td>$-x + \alpha + 1$</td>
<td>$(\alpha + 2)x + 2$</td>
</tr>
<tr>
<td>$\lambda_n$</td>
<td>$n$</td>
<td>$-n(n + \alpha + 1)$</td>
</tr>
<tr>
<td>$B_n$</td>
<td>$2n + \alpha + 1$</td>
<td>$\frac{-2x}{(2n + \alpha)(2n + 2 + \alpha)}$</td>
</tr>
<tr>
<td>$C_n$</td>
<td>$n(n + \alpha)$</td>
<td>$\frac{-4n(n + \alpha)}{(2n + \alpha - 1)(2n + \alpha)^2(2n + \alpha + 1)}$</td>
</tr>
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</table>
Table 4 (contd.)

<table>
<thead>
<tr>
<th></th>
<th>Laguerre ( L_n^{(2)}(x) ) (( x &gt; -1 ))</th>
<th>Bessel ( J_n^{(2)}(x) ) (( x \neq -n, n \geq 2 ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_n )</td>
<td>0</td>
<td>( n )</td>
</tr>
<tr>
<td>( \beta_n )</td>
<td>( n )</td>
<td>( \frac{-4n(n + 1 + x)}{(2n + x)(2n + 2 + x)} )</td>
</tr>
<tr>
<td>( \gamma_n )</td>
<td>( n(n + x) )</td>
<td>( \frac{4n(n + x)(n + x + 1)}{(2n + x - 1)(2n + x)^2(2n + x + 1)} )</td>
</tr>
<tr>
<td>( F_n )</td>
<td>1</td>
<td>( \frac{4}{(2n + x)(2n + 2 + x)} )</td>
</tr>
<tr>
<td>( G_n )</td>
<td>0</td>
<td>( \frac{4n}{(2n + x - 1)(2n + x)^2(2n + x + 1)} )</td>
</tr>
</tbody>
</table>

References


