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# On non-local and non-homogeneous elastic continua

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ABSTRACT

A thermodynamic framework endowed with the concept of non-locality residual is adopted to derive non-local models of integral-type for non-homogeneous linear elastic materials. Two expressions of the free energy are considered: the former yields a one-component non-local stress, the latter leads to a two-component local–non-local stress since the stress is expressed as the sum of the classical local stress and of a non-local component identically vanishing in the case of constant strains. The attenuation effects are accounted for by a symmetric space weight function which guarantees the constant strain requirement as well as the dual constant stress condition everywhere in the body. The non-local and non-homogeneous elastic structural boundary-value problem under quasi-static loads is addressed in a geometrically linear range. The complete set of variational formulations for the structural problem is then provided in a unitary framework. The solution uniqueness of the non-local structural model is proved and the non-local FEM is addressed starting from the non-local counterpart of the total potential energy. Numerical applications are provided with reference to a non-homogeneous bar in tension using the Fredholm integral equation and the non-local FEM. The solutions show no pathological features such as numerical instability and mesh sensitivity for degraded bar conditions.



## 1. Introduction

Non-local continua have been proposed in the literature with the purpose of accounting for long distance cohesive forces appearing in many materials such as concretes, ceramics, soils, rocks and to avoid some troubles appearing in classical continuum elasticity such as strain softening, size effects and stress singularities in crack tips.

Non-local theories introduce in classical material models the intrinsic length which is a material parameter accounting for non-local effects in the continuum. Accordingly an elastic material can transmit information regarding the behaviour of the material microstructure to neighbouring points within a certain distance. The contributions of Bažant and Cedolin (1991), Bažant and Planas (1998), Mühlhaus (1995), de Borst and van der Giessen (1998), Bažant and Jirásek (2002) and Aifantis (2003), among many others, can be referred to for more details.

The formulation of a constitutive theory for a non-local elastic model has been presented in Polizzotto et al. (2006). The approach and the ideas contributed in Polizzotto et al. (2006) are interesting and deserve a special attention. Starting from the cited work, in the present paper two non-local models for non-homogeneous elastostatics in isothermal conditions are proposed and are cast in a thermodynamic framework which constitutes an appropriate theoretical basis to develop a consistent phenomenological non-local constitutive model. The former is called one-component non-local model because the related stress is provided in a non-local form by means of a unique term, the latter is called two-component non-local model because the corresponding stress is expressed as the sum of the local stress and of a non-local contribution vanishing in the

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case of constant strains. It is worth emphasizing that the non-homogeneity of the model refers to the macroscopic scale by means of the spatial variation of the elastic stiffness.

The non-locality residual was introduced in the framework of non-local continuum theories by Edelen and Laws (1971), Eringen and Edelen (1972), Eringen (1972, 1987) in which almost all the state variables were treated as non-local. However such theories turn out to be quite unsuitable for modelling the strain-softening behaviour of real materials in which only a limited number of state variables such as the strain tensor in elasticity, a measure of plastic strain in plasticity or a damage variable in damage mechanics need to be treated as non-local.

Nowadays a single non-locality residual in the internal energy balance equation (first thermodynamics principle) for every non-locality source is considered. The residual meets the so-called insulation condition (Polizzotto and Borino, 1998) which guarantees that there are no interactions of the body with the exterior world at the constitutive level. This condition is imposed by the vanishing of the integral of the non-locality residual over the volume of the body.

The second thermodynamics principle (entropy production inequality) is assumed to hold pointwise (see e.g. Polizzotto, 2003) as in classical (local) thermodynamics in order to guarantee that the energy dissipation density is non-negative everywhere in the body for any irreversible deformation process. Deformation processes are qualified as reversible if the second principle is satisfied as an equality.

The energy dissipation density is assumed to exhibit a bilinear form in terms of local strain rates and related non-local stresses. Well-known procedures of classical thermodynamics can then be extended to non-local non-homogeneous elasticity. Accordingly the relevant state equations, i.e. stress–strain law and the expression of the residual, can be derived in terms of local and non-local quantities.

In particular the elastic energy and the stress tensor of the considered models can be expressed as the sum of the classical local term and of a non-local term.

In addition to the internal length and to a space variable elastic stiffness, the proposed models contain a parameter which controls the proportion of the non-local addition to the local part of the free energy and of the stress. In existing models (see e.g. Polizzotto et al., 2004, 2006) such a parameter is directly added to the non-local part of the free energy as a multiplicative term. In the proposed approach this parameter is added to the expression of the space weight function in a suitable form such that the normalizing condition involving the weight function (constant strain requirement) is unaffected by its value. As a consequence, once the expression of the free energy is defined, the parameter consistently appears as a multiplicative term of the non-local part of the elastic energy and of the stress.

Then the considered non-homogeneous non-local models and the non-local model for piecewise non-homogeneous bodies proposed in Marotti de Sciarra (2008) are comparatively analyzed. Moreover it is shown that the two-component non-local model coincides to the one proposed by Polizzotto et al. (2006).

It is worth noting that the two-component non-local model behaves as a local one under any uniform strain field since the elastic energy and the stress tensor coincide to their local counterparts and the non-local residual vanishes everywhere in the body. In this case the non-local model does not interchange energy at the microstructural level. On the contrary, the one-component non-local model and the non-local model for piecewise non-homogeneous bodies behave as a local one under any uniform strain field only if the elastic stiffness is constant. In the case of a non-homogeneous material the elastic strain energy and the non-local stress do not reduce to their local counterparts under any uniform strain field due to the space variability of the elastic stiffness (see Polizzotto et al., 2006). The residual is pointwise vanishing and, at the global level, the non-local elastic energy functional coincides to the local one.

The boundary-value problem associated with the considered non-local elastic models can be formulated in a unified framework. The complete set of non-local mixed variational principles is then provided. Variational formulations can then be specialized to a particular model by considering the relevant expression of the elastic or complementary energy functionals. The extension to non-local linear elasticity of the classical principles of the total potential energy, complementary energy, mixed Hu-Washizu and Hellinger-Reissner principles are provided. A discussion on uniqueness of the solution of the non-local structural model is then provided.

Starting from the non-local total potential energy, a non-local-type finite element method (NFEM) which encompasses the considered non-local models is proposed. The symmetric global stiffness matrix contains the non-locality features of the model and shows a larger band width than in the local-type FEM due to the long distance interelement influence. A homogeneous bar having a piecewise constant elastic modulus or a continuous variable Young modulus is solved by the recourse to the Fredholm equation and to the proposed NFEM for several load conditions. The solutions obtained following the outlined procedures are in a good agreement one another and no pathological behaviours such as numerical instability, mesh sensitivity and boundary effects are present.

The outline of the paper is the following. In Section 2, the non-local models are cast in the thermodynamic framework and the residual function appears as an additional state variable. The non-local stress–strain relations for a linearly elastic non-homogeneous body are then provided together with the related expressions of the residual. The non-local and non-homogeneous elastic structural model is then addressed in Section 3 where the complete set of variational formulations are provided. In Section 4 a NFEM is formulated and in Section 5 a one-dimensional bar in tension is considered from a computational point of view by using the Fredholm integral equation and the proposed NFEM. The paper is closed with three Appendices. The Appendices A and B are devoted to the explicit evaluation of the integrals pertaining, respectively, to the one-component and two-component non-local models in terms of the attenuation function. The third Appendix deals with the Fredholm integral equation of the second kind associated with the proposed non-local models.

## 2. Non-local model

A non-local elastic structural problem is defined on a regular bounded domain $\Omega$ of an Euclidean space and is referred to orthogonal axes with Cartesian co-ordinates $\boldsymbol{x}$ in its undeformed state. The classical linearized theory is considered so that configurations assumed by the structure are sufficiently close to a reference one. Strains $\boldsymbol{\varepsilon}$ and stresses $\boldsymbol{\sigma}$ belong to dual spaces $\Sigma$ and $\mathscr{S}$, respectively. The inner product $\langle\cdot,\cdot\rangle$ in the dual spaces has the mechanical meaning of the internal virtual work and, for the Cauchy model, it is given by:

$$\langle\boldsymbol{\sigma},\boldsymbol{\varepsilon}\rangle=\int_\Omega \boldsymbol{\sigma}(\boldsymbol{x})*\boldsymbol{\varepsilon}(\boldsymbol{x})\mathrm{d}\boldsymbol{x},$$

where $*$ denotes the scalar product between dual local quantities (simple or double index saturation operation between vectors or tensors) at a given point $\boldsymbol{x}$ of the body $\Omega$. A compact notation is adopted throughout the paper with bold-face letters associated with vectors and tensors. For convenience, non-local variables are denoted by a superposed bar.

Non-local strains due to long-range interactions arising in an elastic structure are provided by the following relation:

$$\bar{\boldsymbol{\varepsilon}}(\boldsymbol{x})=(\mathbf{R}\boldsymbol{\varepsilon})(\boldsymbol{x})=\int_\Omega W(\boldsymbol{x},\xi)\boldsymbol{\varepsilon}(\xi)\mathrm{d}\xi\quad\forall\boldsymbol{x}\in\Omega. \tag{1}$$

The linear regularization operator $\mathbf{R}:\Sigma\to\mathscr{D}$ transforms the local strains $\boldsymbol{\varepsilon}$ into the related non-local strain $\bar{\boldsymbol{\varepsilon}}$ since its value at the point $\boldsymbol{x}$ of the body $\Omega$ depends on the entire field $\boldsymbol{\varepsilon}$ (Eringen and Kim, 1974; Eringen et al., 1977; Polizzotto et al., 2004, 2006).

From a physical point of view the space weight function $W$ appearing in (1) describes the mutual long-range elastic interaction. It is non-negative, have its maximum for $\boldsymbol{x}=\xi$ and decreases monotonically and rapidly to zero approaching the boundary of the interaction zone defined by the influence distance $R$ which is a multiple of the internal length. Accordingly the regularization takes place if the distance $r$ between the source point $\xi$, at which a local variable is considered, and the point $\boldsymbol{x}$, where the non-local effect is considered, is such that $r(\boldsymbol{x},\xi)\leqslant R$.

Since a non-local behaviour is present for high space variation of the local strain $\varepsilon$, the regularization operator $\mathbf{R}$ coincides to the identity operator $\mathbf{I}$ for uniform fields $\boldsymbol{\varepsilon}$, that is $\mathbf{R}=\mathbf{I}$. As a consequence the weight function $W$ meets the normalizing condition:

$$\int_\Omega W(\boldsymbol{x},\xi)\mathrm{d}\xi=1 \tag{2}$$

for any $\boldsymbol{x}$ in $\Omega$.

The following symmetric expression of the weight function $W$ is considered in the present paper:

$$W(\boldsymbol{x},\xi)=\left[1-\alpha\frac{V(\boldsymbol{x})}{V_\infty}\right]\delta(\boldsymbol{x},\xi)+\frac{\alpha}{V_\infty}g(\boldsymbol{x},\xi), \tag{3}$$

where the symbol $\delta(\boldsymbol{x},\xi)$ denotes the Dirac delta distribution, the scalar function $g(\boldsymbol{x},\xi)$ is the attenuation (or influence) function depending on the material internal length scale $l$ and $V(\boldsymbol{x})$ is the representative volume given by:

$$V(\boldsymbol{x})=\int_\Omega g(\boldsymbol{x},\xi)\mathrm{d}\xi. \tag{4}$$

The value assumed by the representative volume $V(\boldsymbol{x})$ for an unbounded body is denoted by $V_\infty$ and $\alpha$ is an adimensional scalar parameter which is concerned with the attenuations effects. The relation (3) is similar to the one proposed in Borino et al. (2003) within the context of non-local damage. The first term appearing in (4) is a local one. Setting $\alpha=1$, it is effective for points $\boldsymbol{x}$ close to the boundary since, for points $\boldsymbol{x}$ far from the boundary, $V(\boldsymbol{x})$ tends to $V_\infty$ and the local term vanishes. The second term is the classical non-local term associated with an unbounded body. The parameter $\alpha$ is added to the local and non-local terms in order to control the proportion of the non-local addition since it will be shown in the sequel that $\alpha$ appears only in the non-local part of the free energy and of the stress.

The dual averaging $\mathbf{R}'$ associated with the symmetric weight function (3) preserves constant fields since the regularization operator is self-adjoint independent of the choice of the attenuation function $g$ and for any value of the parameter $\alpha$, i.e. $\mathbf{R}=\mathbf{R}'$ (see Marotti de Sciarra, 2008). On the contrary, the dual averaging arising from the non-symmetric weight functions typically adopted in the literature (see e.g. Jirásek and Rolshoven, 2003) does not fulfil the constant requirement.

The attenuation function $g(\boldsymbol{x},\xi)$ is a function of the Euclidean distance $\|\boldsymbol{x}-\xi\|$ or of the geodetical distance $r(\boldsymbol{x},\xi)\geqslant\|\boldsymbol{x}-\xi\|$ defined as the length of the shortest path joining $\boldsymbol{x}$ with $\xi$ without intersecting the boundary of the body, see Polizzotto (2001). The geodetical distance turns out to be useful if the domain $\Omega$ occupied by the body is not convex or an obstacle, such as a hole, is located between the points $\boldsymbol{x}$ and $\xi$.

Typical choices for the attenuation functions are the Gauss-like function:

$$g(\boldsymbol{x},\xi)=\frac{1}{l\sqrt{2\pi}}\exp(-\|\boldsymbol{x}-\xi\|^2/2l^2), \tag{5}$$

the bi-exponential function:

$$g(\boldsymbol{x},\xi) = \frac{1}{2l}\exp(-\|\boldsymbol{x}-\xi\|/l) \tag{6}$$

or the bell-shaped polynomial function:

$$g(\boldsymbol{x},\xi) = \begin{cases} \frac{15}{16R}(1-\|\boldsymbol{x}-\xi\|^2/R^2)^2 & \text{if } \|\boldsymbol{x}-\xi\| \leqslant R, \\ 0 & \text{if } \|\boldsymbol{x}-\xi\| > R. \end{cases} \tag{7}$$

The Gauss-like and the bi-exponential functions have an unbounded support whereas the bell-shaped polynomial function (7) has a bounded support since $g(\boldsymbol{x},\xi)$ vanishes for $\|\boldsymbol{x}-\xi\|$ greater than the influence distance $R$. It is to remark that the decay of the exponential attenuation functions (5) and (6) for increasing $\|\boldsymbol{x}-\xi\|/l$ is very fast. Hence, from a computational point of view, it is possible to assume that the attenuation functions (5) and (6) are vanishing if $\|\boldsymbol{x}-\xi\| > R$ where $R$ is a multiple of the internal length $l$. In the examples of Section 5, the ratio $R/l$ is set equal to 6.

In what follows, two non-local models are analyzed. The former is referred to as one-component non-local model since the free energy is expressed in terms of a quadratic non-local form. The latter is named two-component non-local model since the free energy is expressed as the sum of the standard local elastic energy and of a quadratic non-local contribution vanishing in the case of uniform strains.

According to a mechanical requirement, it is shown in the sequel that the considered non-local models tend to local elasticity if the material length scale $l$ tends to zero.

### 2.1. Thermodynamical aspects

Let us assume, for simplicity, that the absolute temperature $T$ is constant, i.e. isothermal conditions, and the density of mass is constant. The first law of thermodynamics (see e.g. Lemaitre and Chaboche, 1994) for isothermal processes and for a non-local behaviour can be formulated as follows:

$$\int_\Omega \dot{e}\,\mathrm{d}\boldsymbol{x} = \langle \bar{\boldsymbol{\sigma}}, \dot{\varepsilon}\rangle, \tag{8}$$

where $e$ is the internal energy density depending on strain $\boldsymbol{\varepsilon}$ and entropy $s$. The symbol $\bar{\boldsymbol{\sigma}}$ denotes the non-local stress whose expression will be identified in the sequel. A superimposed dot means time derivative. Dropping for simplicity the explicit dependence on the variable $\boldsymbol{x}$, the energy balance in (8) can be written pointwise in $\Omega$ in the form:

$$\dot{e} = \bar{\boldsymbol{\sigma}} * \dot{\boldsymbol{\varepsilon}} + P. \tag{9}$$

The non-locality residual function $P$ takes into account the energy exchanges between neighbour particles (see e.g. Edelen and Laws, 1971) and the residual $P$ fulfils the insulation condition:

$$\int_\Omega P\,\mathrm{d}\boldsymbol{x} = 0 \tag{10}$$

for any strain rate $\dot{\varepsilon}$ since the body is a thermodynamically isolated system with reference to energy exchanges due to non-locality.

The second principle of thermodynamics for isothermal processes, in the non-local context, is written in its local form $\dot{s}T \geqslant 0$ everywhere in $\Omega$ where $\dot{s}$ is the internal entropy production rate per unit volume (see Polizzotto, 2002). In fact if the second principle holds in the global form $\int_\Omega \dot{s}T\,\mathrm{d}\boldsymbol{x} \geqslant 0$, there would be a class of deformation mechanisms which are reversible at the global level, being zero the global form of the second principle, but the same deformation mechanisms turn out to be irreversible at the local level which is not physically acceptable.

Let $\phi(\boldsymbol{\varepsilon},T)$ be the Helmholtz free energy defined by means of the Legendre transform $\phi = e - sT$. Performing the time derivative of the Helmholtz free energy in connection with the second principle and being $\dot{T}=0$, since the temperature is assumed to be constant, the dissipation $D$ at a given point of the body follows from the relation (9) in the form:

$$D = \dot{s}T = \bar{\boldsymbol{\sigma}} * \dot{\boldsymbol{\varepsilon}} - \dot{\phi} + P \geqslant 0 \tag{11}$$

which represents the Clausius–Duhem inequality for isothermal processes. The presence of the non-locality residual function $P$ guarantees the non-negativeness of the dissipation and accounts for material non-locality. The body energy dissipation $\mathfrak{W}$ follows by integrating the relation (11) to get:

$$\mathfrak{W} = \int_\Omega \dot{s}T\,\mathrm{d}\boldsymbol{x} = \langle \bar{\sigma}, \dot{\boldsymbol{\varepsilon}}\rangle - \int_\Omega \dot{\phi}\,\mathrm{d}\boldsymbol{x} \geqslant 0. \tag{12}$$

The free energy function at a point $\boldsymbol{x}$ of the body $\Omega$ is defined, for each of the two non-local models, according to the following relations.

- One-component non-local model – the free energy is given in the form:

$$\phi(\boldsymbol{\varepsilon}(\boldsymbol{x})) = \frac{1}{2}(\mathbf{RER}\boldsymbol{\varepsilon})(\boldsymbol{x}) * \boldsymbol{\varepsilon}(\boldsymbol{x}) \tag{13}$$

where $\mathbf{E}$ denotes the elastic stiffness. The global free energy is the functional of the strain $\boldsymbol{\varepsilon}$ obtained by integrating the specific free energy (13) over the domain of the body and represents the elastic energy stored in the whole structure. It is provided by the following quadratic functional:

$$\Phi(\boldsymbol{\varepsilon}) = \int_\Omega \phi(\boldsymbol{\varepsilon}(\boldsymbol{x}))\mathrm{d}\boldsymbol{x} = \frac{1}{2}\langle \mathbf{RER}\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}\rangle = \frac{1}{2}\langle \mathbf{ER}\boldsymbol{\varepsilon}, \mathbf{R}\boldsymbol{\varepsilon}\rangle. \tag{14}$$

The last equality holds true since the regularization operator $\mathbf{R}$ is self-adjoint with respect to the scalar product in $L^2(\Omega)$. The complementary elastic energy is evaluated by means of the Fenchel's conjugate of the elastic energy and is given by the following convex quadratic functional:

$$\Phi^*(\bar{\boldsymbol{\sigma}}) = \frac{1}{2}\langle \bar{\boldsymbol{\sigma}}, (\mathbf{RER})^{-1}\bar{\boldsymbol{\sigma}}\rangle. \tag{15}$$

- Two-component non-local model – the two-component free energy is:

$$\phi(\boldsymbol{\varepsilon}(\boldsymbol{x})) = \frac{1}{2}\mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x}) * \boldsymbol{\varepsilon}(\boldsymbol{x}) + \frac{1}{2}\mathbf{E}(\boldsymbol{x})(\mathbf{A}\boldsymbol{\varepsilon})(\boldsymbol{x}) * (\mathbf{A}\boldsymbol{\varepsilon})(\boldsymbol{x}), \tag{16}$$

where $\mathbf{A} = \mathbf{R} - \mathbf{I}$ being $\mathbf{I}$ the identity operator in the strain space. Hence the elastic energy stored in the whole structure is provided by the quadratic functional:

$$\Phi(\boldsymbol{\varepsilon}) = \int_\Omega \phi(\boldsymbol{\varepsilon}(\boldsymbol{x}))\mathrm{d}\boldsymbol{x} = \frac{1}{2}\langle \mathbf{E}\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}\rangle + \frac{1}{2}\langle \mathbf{EA}\boldsymbol{\varepsilon}, \mathbf{A}\boldsymbol{\varepsilon}\rangle = \frac{1}{2}\langle(\mathbf{AEA} + \mathbf{E})\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}\rangle \tag{17}$$

since it is immediate to show that the operator $\mathbf{A}$ is self-adjoint with respect to the scalar product in $L^2(\Omega)$. The complementary elastic energy is the Fenchel's conjugate of the elastic energy and is given by:

$$\Phi^*(\bar{\boldsymbol{\sigma}}) = \frac{1}{2}\langle \bar{\boldsymbol{\sigma}}, (\mathbf{AEA} + \mathbf{E})^{-1}\bar{\boldsymbol{\sigma}}\rangle. \tag{18}$$

In what follows, let $\phi$ denote the free energy of one of the above considered models. Expanding the inequality (12), it results:

$$\langle \bar{\boldsymbol{\sigma}}, \dot{\boldsymbol{\varepsilon}}\rangle - \langle d\phi(\boldsymbol{\varepsilon}), \dot{\boldsymbol{\varepsilon}}\rangle \geqslant 0, \tag{19}$$

where $d\phi(\boldsymbol{\varepsilon})$ denotes the derivative of the free energy with respect to the strain, so that it turns out to be:

$$\begin{cases} \langle \bar{\boldsymbol{\sigma}}, \dot{\boldsymbol{\varepsilon}}\rangle - \langle \mathbf{RER}\boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}}\rangle \geqslant 0 & \text{one-component model,} \\ \langle \bar{\boldsymbol{\sigma}}, \dot{\boldsymbol{\varepsilon}}\rangle - \langle \mathbf{AEA}\boldsymbol{\varepsilon} + \mathbf{E}\boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}}\rangle \geqslant 0 & \text{two-component model.} \end{cases}$$

The relation (19) must hold for any admissible deformation mechanism so that, following widely used arguments (see e.g. Lemaitre and Chaboche, 1994), the state law is obtained:

$$\bar{\boldsymbol{\sigma}}(\boldsymbol{x}) = d\phi(\boldsymbol{\varepsilon}). \tag{20}$$

The expressions of the non-local stress related to the considered models are then given by:

$$\begin{cases} \bar{\boldsymbol{\sigma}}(\boldsymbol{x}) = (\mathbf{RER}\boldsymbol{\varepsilon})(\boldsymbol{x}) & \text{one-component model,} \\ \bar{\boldsymbol{\sigma}}(\boldsymbol{x}) = (\mathbf{AEA}\boldsymbol{\varepsilon} + \mathbf{E}\boldsymbol{\varepsilon})(\boldsymbol{x}) & \text{two-component model.} \end{cases} \tag{21}$$

It is then apparent that the relation (19) becomes an equality. Moreover the inequality (11) holds as an equality too since, recalling (10), it can be viewed as the non-negative integrand of (19). Hence the dissipation is pointwise vanishing due to the presence of the non-local residual function according to the reversible nature of the model:

$$D = \bar{\boldsymbol{\sigma}} * \dot{\boldsymbol{\varepsilon}} - \dot{\phi} + P = 0.$$

Then the expression for the non-locality residual function at a given point of the body $\Omega$ is given by:

$$P = \dot{\phi} - \bar{\boldsymbol{\sigma}} * \dot{\boldsymbol{\varepsilon}}. \tag{22}$$

By using the free energy (13) or (16) in connection with the corresponding non-local stress reported in (21), the non-locality residual function $P$ pertaining to the two models is given by:

$$\begin{cases} P(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{\varepsilon}(\boldsymbol{x}) * (\mathbf{RER}\dot{\boldsymbol{\varepsilon}})(\boldsymbol{x}) - \frac{1}{2}(\mathbf{RER}\boldsymbol{\varepsilon})(\boldsymbol{x}) * \dot{\boldsymbol{\varepsilon}}(\boldsymbol{x}) & \text{one-component model,} \\ P(\boldsymbol{x}) = (\mathbf{EA}\boldsymbol{\varepsilon})(\boldsymbol{x}) * (\mathbf{A}\dot{\varepsilon})(\boldsymbol{x}) - (\mathbf{AEA}\varepsilon)(\boldsymbol{x}) * \dot{\boldsymbol{\varepsilon}}(\boldsymbol{x}) & \text{two-component model.} \end{cases} \tag{23}$$

Then the relation (20) represents the constitutive law for the non-local non-homogeneous elastic material endowed with the elastic energy $\phi$. It is worth noting that the non-homogeneity of these models is referred to the spatial variation of the elastic stiffness.

Accordingly the non-local elastic relation can be expressed in terms of the state variable fields $\bar{\boldsymbol{\sigma}}$ and $\boldsymbol{\varepsilon}$ in the following equivalent forms:

$$\bar{\boldsymbol{\sigma}} = d\Phi(\boldsymbol{\varepsilon}) \Longleftrightarrow \boldsymbol{\varepsilon} = d\Phi^*(\bar{\boldsymbol{\sigma}}) \Longleftrightarrow \Phi(\boldsymbol{\varepsilon}) + \Phi^*(\bar{\boldsymbol{\sigma}}) = \langle \bar{\boldsymbol{\sigma}}, \boldsymbol{\varepsilon} \rangle, \tag{24}$$

where the last equality represents Fenchel's relation.

In the following subsections the considered models are explicitly analyzed.

### 2.2. Elastic energy, stress and residual for the one-component non-local model

The elastic energy (13) for the one-component non-local model can be explicitly evaluated in order to make evident the contribution of non-locality. In fact it results:

$$\phi(\boldsymbol{\varepsilon}(\boldsymbol{x})) = \frac{1}{2} \int_\Omega \int_\Omega W(\xi, \boldsymbol{x}) \mathbf{E}(\xi) W(\xi, \boldsymbol{z}) \boldsymbol{\varepsilon}(\boldsymbol{z}) \mathrm{d}\boldsymbol{z} \, \mathrm{d}\xi * \boldsymbol{\varepsilon}(\boldsymbol{x}). \tag{25}$$

By inserting in the expression above the space weight function $W$ given by (3), after some rearrangements of the various terms and with the definitions:

$$\begin{aligned}
\boldsymbol{\Lambda}_1(\boldsymbol{x}, \boldsymbol{z}) &= \left[ \left(1 - \alpha \frac{V(\boldsymbol{x})}{V_\infty}\right) \mathbf{E}(\boldsymbol{x}) + \left(1 - \alpha \frac{V(\boldsymbol{z})}{V_\infty}\right) \mathbf{E}(\boldsymbol{z}) \right] g(\boldsymbol{x}, \boldsymbol{z}), \\
\boldsymbol{\Lambda}_2(\boldsymbol{x}, \boldsymbol{z}) &= \int_\Omega g(\xi, \boldsymbol{x}) \mathbf{E}(\xi) g(\xi, \boldsymbol{z}) \mathrm{d}\xi, \\
\mathbf{H}(\boldsymbol{x}, \boldsymbol{z}) &= \boldsymbol{\Lambda}_1(\boldsymbol{x}, \boldsymbol{z}) + \frac{\alpha}{V_\infty} \boldsymbol{\Lambda}_2(\boldsymbol{x}, \boldsymbol{z}),
\end{aligned} \tag{26}$$

a more synthetic expression (see Appendix A) can be given to the elastic energy $\phi(\varepsilon(\boldsymbol{x}))$ in the form:

$$\begin{aligned}
\phi(\boldsymbol{\varepsilon}(\boldsymbol{x})) &= \frac{1}{2} \left[ 1 - \alpha \frac{V(\boldsymbol{x})}{V_\infty} \right]^2 \mathbf{E}(\boldsymbol{x}) \boldsymbol{\varepsilon}(\boldsymbol{x}) * \boldsymbol{\varepsilon}(\boldsymbol{x}) + \frac{\alpha}{2V_\infty} \int_\Omega \mathbf{H}(\boldsymbol{x}, \boldsymbol{z}) \boldsymbol{\varepsilon}(\boldsymbol{z}) \mathrm{d}\boldsymbol{z} * \boldsymbol{\varepsilon}(\boldsymbol{x}) \\
&= \frac{1}{2} \mathbf{E}(\boldsymbol{x}) \boldsymbol{\varepsilon}(\boldsymbol{x}) * \boldsymbol{\varepsilon}(\boldsymbol{x}) + \frac{\alpha}{2V_\infty} \int_\Omega \mathbf{H}_1(\boldsymbol{x}, \boldsymbol{z}) \boldsymbol{\varepsilon}(\boldsymbol{z}) \mathrm{d}\boldsymbol{z} * \boldsymbol{\varepsilon}(\boldsymbol{x}),
\end{aligned} \tag{27}$$

where

$$\mathbf{H}_1(\boldsymbol{x}, \boldsymbol{z}) = \left[ \frac{\alpha}{V_\infty} V^2(\boldsymbol{z}) - 2V(\boldsymbol{z}) \right] \mathbf{E}(\boldsymbol{z}) \delta(\boldsymbol{x}, \boldsymbol{z}) + \mathbf{H}(\boldsymbol{x}, \boldsymbol{z}). \tag{28}$$

For a homogeneous one-dimensional bar of length $L = 100$ cm with a unitary elastic modulus $E$, the functions $\Lambda_1$, $\Lambda_2$ and $H$ are plotted in Fig. 1(a) and (b) in terms of $z$ for a fixed $x$ assuming that the attenuation function $g$ is the bi-exponential function (6). Different length scales $l = 1$, 2 and 6 cm are considered, the interaction distance is such that $R/l = 6$ and $\alpha = -1$. The functions $\Lambda_1(x, \cdot)$, $\Lambda_2(x, \cdot)$ and $H(x, \cdot)$ are reported in Fig. 1(a) with reference to a point $x \in \Omega$ far from the boundary. If the point $x$ belongs to the boundary layer, the shape of the functions remain the same whereas they are cut by the presence of the boundary, see Fig. 1(b).

According to (14), the elastic energy functional is obtained by performing the integral of (27) over $\Omega$ to get:

$$\Phi(\boldsymbol{\varepsilon}) = \frac{1}{2} \int_\Omega \mathbf{E}(\boldsymbol{x}) \boldsymbol{\varepsilon}(\boldsymbol{x}) * \boldsymbol{\varepsilon}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} + \frac{\alpha}{2V_\infty} \int_\Omega \int_\Omega \mathbf{H}_1(\boldsymbol{x}, \boldsymbol{z}) \boldsymbol{\varepsilon}(\boldsymbol{z}) \mathrm{d}\boldsymbol{z} * \boldsymbol{\varepsilon}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}. \tag{29}$$

The constitutive relation between strains $\boldsymbol{\varepsilon}(\boldsymbol{x})$ and stresses $\bar{\boldsymbol{\sigma}}(\boldsymbol{x})$ for the one-component non-local model is reported in $(21)_1$. It can be rewritten in terms of the attenuation function $g$ as follows:

$$\bar{\boldsymbol{\sigma}}(\boldsymbol{x}) = \left[ 1 - \alpha \frac{V(\boldsymbol{x})}{V_\infty} \right]^2 \mathbf{E}(\boldsymbol{x}) \boldsymbol{\varepsilon}(\boldsymbol{x}) + \frac{\alpha}{V_\infty} \int_\Omega \mathbf{H}(\boldsymbol{x}, \boldsymbol{z}) \boldsymbol{\varepsilon}(\boldsymbol{z}) \mathrm{d}\boldsymbol{z} = \mathbf{E}(\boldsymbol{x}) \boldsymbol{\varepsilon}(\boldsymbol{x}) + \frac{\alpha}{V_\infty} \int_\Omega \mathbf{H}_1(\boldsymbol{x}, \boldsymbol{z}) \boldsymbol{\varepsilon}(\boldsymbol{z}) \mathrm{d}\boldsymbol{z}. \tag{30}$$

From a mechanical point of view, the elastic energy (27) is given by the sum of the strain energy related to the local behaviour and the strain energy due to the non-local constitutive behaviour. Note that the non-local term is symmetric due to the symmetry of $\mathbf{H}$, see (26).

Since the function $\mathbf{H}_1$ is vanishing for points $\boldsymbol{z} \in \Omega$ outside the influence region associated with a given point $\boldsymbol{x} \in \Omega$, a non-local term is added to the local strain energy (27) depending on strains belonging to the influence region. The amplitude of the non-local addition is controlled by the parameter $\alpha$. A similar remark holds for the non-local stress (30).

The expression for the non-locality residual function $P$ at a point $\boldsymbol{x}$ of the body $\Omega$ is given by $(23)_1$ and can be synthetically expressed in terms of the attenuation function $g$ in the form:

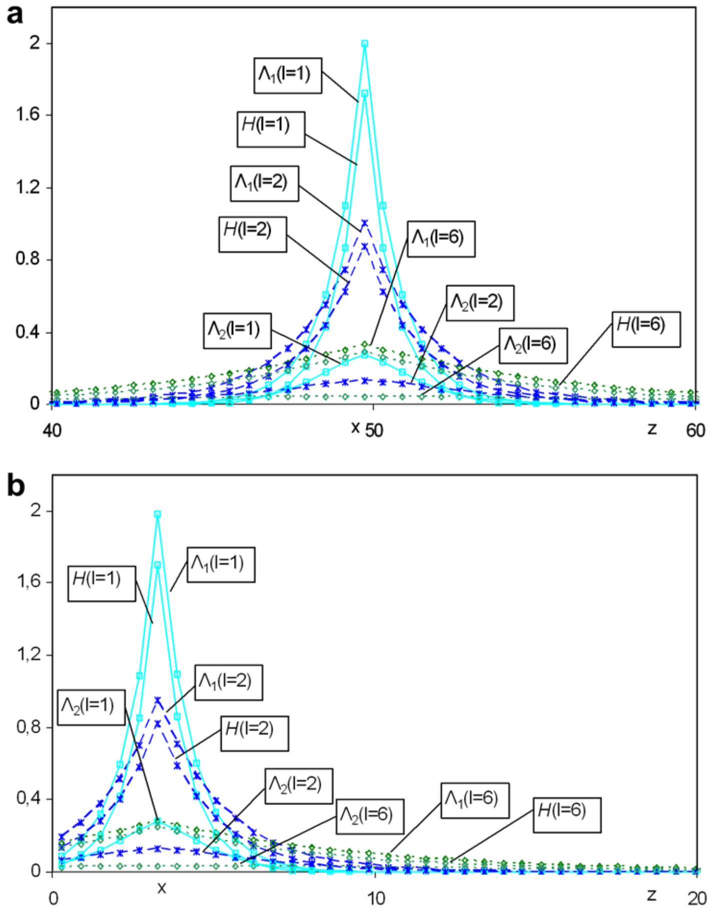


**Fig. 1.** Plots of the functions $\Lambda_1, \Lambda_2$ and $H$ in terms of $z$ for a fixed $x$ assuming a bi-exponential attenuation function and a unitary elastic modulus. Three different length scales $l$ are considered, namely $l = 1$ cm, $l = 2$ cm, $l = 6$ cm, the interaction distance $R$ is such that $R/l = 6$ and $\alpha = -1$: (a) the point $x$ is far from the boundary; (b) the point $x$ is in the boundary layer.

$$P(\mathbf{x}) = \frac{\alpha}{2V_\infty}\int_\Omega \mathbf{H}(\mathbf{x},\mathbf{z})\dot{\boldsymbol{\varepsilon}}(\mathbf{z})\mathrm{d}\mathbf{z} * \boldsymbol{\varepsilon}(\mathbf{x}) - \frac{\alpha}{2V_\infty}\int_\Omega \mathbf{H}(\mathbf{x},\mathbf{z})\boldsymbol{\varepsilon}(\mathbf{z})\mathrm{d}\mathbf{z} * \dot{\boldsymbol{\varepsilon}}(\mathbf{x}) \tag{31}$$

as provided in Appendix A.

The non-locality residual $P$ turns out to be a homogeneous function of the strain rate. Accordingly, for a given strain field $\boldsymbol{\varepsilon}$ corresponding to a prescribed configuration of the body, the residual $P$ can be rewritten in the following form:

$$P(\mathbf{x}) = \int_\Omega \mathbf{f}_1(\mathbf{x},\mathbf{z}) * \dot{\boldsymbol{\varepsilon}}(\mathbf{z})\mathrm{d}\mathbf{z} + \mathbf{F}_1(\mathbf{x}) * \dot{\boldsymbol{\varepsilon}}(\mathbf{x}), \tag{32}$$

where:

$$\begin{aligned} \mathbf{f}_1(\mathbf{x},\mathbf{z}) &= \frac{\alpha}{2V_\infty}\mathbf{H}(\mathbf{x},\mathbf{z})\boldsymbol{\varepsilon}(\mathbf{x}), \\ \mathbf{F}_1(\mathbf{x}) &= -\frac{\alpha}{2V_\infty}\int_\Omega \mathbf{H}(\mathbf{x},\mathbf{z})\boldsymbol{\varepsilon}(\mathbf{z})\mathrm{d}\mathbf{z} = -\int_\Omega \mathbf{f}_1(\mathbf{z},\mathbf{x})\mathrm{d}\mathbf{z}. \end{aligned} \tag{33}$$

The non-locality residual $P$ can then be evaluated by means of the functions $\mathbf{f}_1$ and $\mathbf{F}_1$ once the strain rate $\dot{\boldsymbol{\varepsilon}}$ is assigned. Plots regarding the functions $\mathbf{f}_1$ and $\mathbf{F}_1$, given by (33), are reported in the examples developed in Section 5 with reference to a non-homogeneous one-dimensional bar.

**Remark 1.** From a mechanical standpoint, if the internal length $l$ tends to vanishing, i.e. $l \to 0$, the non-local material must tend to a local behaviour. It is then useful to check whether the proposed model identifies to the local response for a vanishing internal length. Since the attenuation function $g(\mathbf{x}, \xi)$ tends to the Dirac distribution $\delta(\mathbf{x}, \xi)$ for a vanishing internal length $l$, it results $V(\mathbf{x}) = V_\infty \to 1$. Hence the following relations hold:

$$\begin{aligned}
&\boldsymbol{\Lambda}_1(\boldsymbol{x},\boldsymbol{z}) \to (1-\alpha)[\mathbf{E}(\boldsymbol{x})+\mathbf{E}(\boldsymbol{z})]\delta(\boldsymbol{x},\boldsymbol{z})\\
&\quad\int_\Omega \boldsymbol{\Lambda}_1(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})d\boldsymbol{z} \to 2(1-\alpha)\mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x})\\
&\quad\int_\Omega \boldsymbol{\Lambda}_2(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})d\boldsymbol{z} \to \mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x})\\
&\quad\int_\Omega \mathbf{H}(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})d\boldsymbol{z} \to (2-\alpha)\mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x})
\end{aligned}$$

so that

$$\int_\Omega \mathbf{H}_1(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z} \to (\alpha-2)\mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x})+(2-\alpha)\mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x})=\mathbf{0}$$

for any $\boldsymbol{x}\in\Omega$ and strain $\boldsymbol{\varepsilon}$. Accordingly, the non-local elastic energy (27) and the non-local stress (30) reduce to the related local terms since the non-local quantities vanish. Moreover the relation (31) shows that the residual $P(\boldsymbol{x})$ identically vanishes. Hence, the non-local model tends to the local model for the internal length $l$ tending to zero.

In the case of a homogeneous elastic stiffness, i.e. $\mathbf{E}(\boldsymbol{x})=\mathbf{E}$, the following simplifications hold true:

$$\begin{aligned}
&\boldsymbol{\Lambda}_1(\boldsymbol{x},\boldsymbol{z})=\mathbf{E}\left[\left(1-\alpha\frac{V(\boldsymbol{x})}{V_\infty}\right)+\left(1-\alpha\frac{V(\boldsymbol{z})}{V_\infty}\right)\right]g(\boldsymbol{x},\boldsymbol{z})=\mathbf{E}\lambda_1(\boldsymbol{x},\boldsymbol{z}),\\
&\boldsymbol{\Lambda}_2(\boldsymbol{x},\boldsymbol{z})=\mathbf{E}\int_\Omega g(\xi,\boldsymbol{x})g(\boldsymbol{z},\xi)\mathrm{d}\xi=\mathbf{E}\lambda_2(\boldsymbol{x},\boldsymbol{z}),\\
&\mathbf{H}(\boldsymbol{x},\boldsymbol{z})=\mathbf{E}\left[\lambda_1(\boldsymbol{x},\boldsymbol{z})+\frac{\alpha}{V_\infty}\lambda_2(\boldsymbol{x},\boldsymbol{z})\right]=\mathbf{E}h(\boldsymbol{x},\boldsymbol{z}),\\
&\mathbf{H}_1(\boldsymbol{x},\boldsymbol{z})=\mathbf{E}\left[\frac{\alpha}{V_\infty}V^2(\boldsymbol{z})-2V(\boldsymbol{z})\right]\delta(\boldsymbol{x},\boldsymbol{z})+\mathbf{H}(\boldsymbol{x},\boldsymbol{z})=\mathbf{E}h_1(\boldsymbol{x},\boldsymbol{z}).
\end{aligned}\tag{34}$$

The non-local elastic energy (27) is then provided in the form:

$$\phi(\boldsymbol{\varepsilon}(\boldsymbol{x}))=\frac{1}{2}\mathbf{E}\boldsymbol{\varepsilon}(\boldsymbol{x})*\boldsymbol{\varepsilon}(\boldsymbol{x})+\frac{\alpha}{2V_\infty}\mathbf{E}\int_\Omega h_1(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z}*\boldsymbol{\varepsilon}(\boldsymbol{x}).\tag{35}$$

Analogous simplifications hold for the non-local stress (30):

$$\bar{\boldsymbol{\sigma}}(\boldsymbol{x})=\mathbf{E}\boldsymbol{\varepsilon}(\boldsymbol{x})+\frac{\alpha}{V_\infty}\mathbf{E}\int_\Omega h_1(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z}\tag{36}$$

and for the non-locality residual function (31):

$$P(\boldsymbol{x})=\frac{\alpha}{2V_\infty}\mathbf{E}\int_\Omega h(\boldsymbol{x},\boldsymbol{z})\dot{\boldsymbol{\varepsilon}}(\boldsymbol{z})\mathrm{d}\boldsymbol{z}*\boldsymbol{\varepsilon}(\boldsymbol{x})-\frac{\alpha}{2V_\infty}\mathbf{E}\int_\Omega h(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z}*\dot{\boldsymbol{\varepsilon}}(\boldsymbol{x}).\tag{37}$$

### 2.3. Elastic energy, stress and residual for the two-component non-local model

The elastic energy $\phi(\boldsymbol{\varepsilon}(\boldsymbol{x}))$, the non-local stress $\bar{\boldsymbol{\sigma}}(\boldsymbol{x})$ and the non-locality residual function $P(\boldsymbol{x})$ for the two-component non-local model are, respectively, reported in (16), $(21)_2$ and $(23)_2$.

It is useful to compare the present two-component non-local model with the non-local and non-homogeneous model contributed in Polizzotto et al. (2006). With the present notations, the space weight function adopted in Polizzotto et al. (2006) can be written in the form $W_p(\boldsymbol{x},\xi)=-V(\boldsymbol{x})\delta(\boldsymbol{x},\xi)+g(\boldsymbol{x},\xi)$ and the related non-local strain is given as $\bar{\boldsymbol{\varepsilon}}_p(\boldsymbol{x})=(\mathbf{R}_p\boldsymbol{\varepsilon})(\boldsymbol{x})$.

The non-local strain (1) can then be rewritten in terms of the regularization operator $\mathbf{R}_p$ in the form:

$$\bar{\boldsymbol{\varepsilon}}(\boldsymbol{x})=(\mathbf{R}\boldsymbol{\varepsilon})(\boldsymbol{x})=\boldsymbol{\varepsilon}(\boldsymbol{x})+\frac{\alpha}{V_\infty}(\mathbf{R}_p\boldsymbol{\varepsilon})(\boldsymbol{x})=\left[\left(\mathbf{I}+\frac{\alpha}{V_\infty}\mathbf{R}_p\right)\boldsymbol{\varepsilon}\right](\boldsymbol{x})$$

so that the regularization operator $\mathbf{R}$ is expressed in terms of $\mathbf{R}_p$ as $\mathbf{R}=\mathbf{I}+\alpha/V_\infty\mathbf{R}_p$. As a consequence, it turns out to be $\mathbf{A}=\mathbf{R}-\mathbf{I}=\alpha/V_\infty\mathbf{R}_p$ and the elastic energy (16) becomes:

$$\phi(\boldsymbol{\varepsilon}(\boldsymbol{x}))=\frac{1}{2}\mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x})*\boldsymbol{\varepsilon}(\boldsymbol{x})+\frac{\alpha^2}{2V_\infty^2}\mathbf{E}(\boldsymbol{x})(\mathbf{R}_p\boldsymbol{\varepsilon})(\boldsymbol{x})*(\mathbf{R}_p\boldsymbol{\varepsilon})(\boldsymbol{x}).\tag{38}$$

The above expression coincides to the elastic energy assumed in Polizzotto et al. (2006) provided that the parameter introduced in Polizzotto et al. (2006) (therein denoted as $\alpha$) is equal to $\alpha^2/V_\infty^2$. It is worth noting that the non-local regularization operator $\mathbf{R}_p$ acts in a similar way as the strain gradient in gradient-dependent materials. In fact constant strains belong to the kernel of the non-local regularization operator $\mathbf{R}_p$. As a consequence, the corresponding weight function $W_p$ meets a "zero" condition, that is $\int_\Omega W_p(\boldsymbol{x},\xi)\mathrm{d}\xi = 0$, instead of the usual normalizing condition (4) of the non-local integral-type models.

The approach proposed in the present paper has the advantage to cast the two-component non-local model in the same framework of the one-component non-local model so that a comparison between them can be straightforwardly carried out. Accordingly the main relations pertaining to the two-component non-local model are hereafter provided in the proposed framework.

The elastic energy (16) for the two-component non-local model at a point $\boldsymbol{x}$ of a non-homogeneous body can be explicitly evaluated in order to make evident the contribution of non-locality. In fact it results (see Appendix B):

$$\begin{aligned}\phi(\boldsymbol{\varepsilon}(\boldsymbol{x})) = {} & \frac{1}{2}\mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x}) * \boldsymbol{\varepsilon}(\boldsymbol{x}) + \alpha^2\frac{V^2(\boldsymbol{x})}{2V_\infty^2}\mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x}) * \boldsymbol{\varepsilon}(\boldsymbol{x}) - \alpha^2\frac{V(\boldsymbol{x})}{V_\infty^2}\int_\Omega g(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z} * \mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x}) \\ & + \frac{\alpha^2}{2V_\infty^2}\mathbf{E}(\boldsymbol{x})\int_\Omega\int_\Omega g(\boldsymbol{x},\xi)g(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\varepsilon}(\xi) * \boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\xi\,\mathrm{d}\boldsymbol{z}.\end{aligned} \tag{39}$$

The elastic energy functional pertaining to the body $\Omega$ is then obtained by performing the integral of (39) over $\Omega$ according to (17). After some rearrangements of the various terms, the following expression for the elastic energy (39) is obtained:

$$\Phi(\boldsymbol{\varepsilon}) = \frac{1}{2}\int_\Omega \mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x}) * \boldsymbol{\varepsilon}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} + \frac{\alpha^2}{2V_\infty^2}\int_\Omega\int_\Omega \mathbf{J}_1(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z} * \boldsymbol{\varepsilon}(\boldsymbol{x})\mathrm{d}\boldsymbol{\varepsilon}, \tag{40}$$

where

$$\mathbf{J}_1(\boldsymbol{x},\boldsymbol{z}) = V^2(\boldsymbol{z})\mathbf{E}(\boldsymbol{z})\delta(\boldsymbol{x},\boldsymbol{z}) + \mathbf{J}(\boldsymbol{x},\boldsymbol{z}) \tag{41}$$

with the definitions:

$$\begin{aligned}&\boldsymbol{\Gamma}_1(\boldsymbol{x},\boldsymbol{z}) = [V(\boldsymbol{x})\mathbf{E}(\boldsymbol{x}) + V(\boldsymbol{z})\mathbf{E}(\boldsymbol{z})]g(\boldsymbol{x},\boldsymbol{z}),\\ &\boldsymbol{\Gamma}_2(\boldsymbol{x},\boldsymbol{z}) = \int_\Omega g(\xi,\boldsymbol{x})\mathbf{E}(\xi)g(\xi,\boldsymbol{z})\mathrm{d}\xi,\\ &\mathbf{J}(\boldsymbol{x},\boldsymbol{z}) = \boldsymbol{\Gamma}_2(\boldsymbol{x},\boldsymbol{z}) - \boldsymbol{\Gamma}_1(\boldsymbol{x},\boldsymbol{z}).\end{aligned} \tag{42}$$

The constitutive relation for the two-component non-local model is reported in $(21)_2$ and can be rewritten in the form:

$$\bar{\boldsymbol{\sigma}}(\boldsymbol{x}) = (\mathbf{REA}\boldsymbol{\varepsilon})(\boldsymbol{x}) - (\mathbf{EA}\boldsymbol{\varepsilon})(\boldsymbol{x}) + \mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x}) = (\mathbf{AER}\boldsymbol{\varepsilon})(\boldsymbol{x}) - (\mathbf{AE}\boldsymbol{\varepsilon})(\boldsymbol{x}) + \mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x}). \tag{43}$$

The relation above is evaluated in Appendix B so that a more synthetic expression, which will be used in the sequel for computations, can be given to the non-local stress in the form:

$$\begin{aligned}\bar{\boldsymbol{\sigma}}(\boldsymbol{x}) &= \left[1 + \alpha^2\frac{V^2(\boldsymbol{x})}{V_\infty^2}\right]\mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x}) - \frac{\alpha^2}{V_\infty^2}\int_\Omega \boldsymbol{\Gamma}_1(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z} + \frac{\alpha^2}{V_\infty^2}\int_\Omega \boldsymbol{\Gamma}_2(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z}\\ &= \left[1 + \alpha^2\frac{V^2(\boldsymbol{x})}{V_\infty^2}\right]\mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x}) + \frac{\alpha^2}{V_\infty^2}\int_\Omega \mathbf{J}(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z} = \mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x}) + \frac{\alpha^2}{V_\infty^2}\int_\Omega \mathbf{J}_1(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z}.\end{aligned} \tag{44}$$

It is immediate to show that the relations among the functions $\boldsymbol{\Lambda}_1, \boldsymbol{\Lambda}_2, \mathbf{H}, \mathbf{H}_1$ related to the one-component non-local model and the corresponding functions $\boldsymbol{\Gamma}_1, \boldsymbol{\Gamma}_2, \mathbf{J}, \mathbf{J}_1$ pertaining to the two-component non-local model are given by:

$$\begin{aligned}&\boldsymbol{\Lambda}_1(\boldsymbol{x},\boldsymbol{z}) = [\mathbf{E}(\boldsymbol{x}) + \mathbf{E}(\boldsymbol{z})]g(\boldsymbol{x},\boldsymbol{z}) - \frac{\alpha}{V_\infty}\boldsymbol{\Gamma}_1(\boldsymbol{x},\boldsymbol{z})\\ &\boldsymbol{\Lambda}_2(\boldsymbol{x},\boldsymbol{z}) = \boldsymbol{\Gamma}_2(\boldsymbol{x},\boldsymbol{z})\\ &\mathbf{H}(\boldsymbol{x},\boldsymbol{z}) = [\mathbf{E}(\boldsymbol{x}) + \mathbf{E}(\boldsymbol{z})]g(\boldsymbol{x},\boldsymbol{z}) + \frac{\alpha}{V_\infty}\mathbf{J}(\boldsymbol{x},\boldsymbol{z})\\ &\mathbf{H}_1(\boldsymbol{x},\boldsymbol{z}) = [\mathbf{E}(\boldsymbol{x}) + \mathbf{E}(\boldsymbol{z})]g(\boldsymbol{x},\boldsymbol{z}) - 2V(\boldsymbol{z})\mathbf{E}(\boldsymbol{z})\delta(\boldsymbol{x},\boldsymbol{z}) + \frac{\alpha}{V_\infty}\mathbf{J}_1(\boldsymbol{x},\boldsymbol{z}).\end{aligned} \tag{45}$$

For the homogeneous one-dimensional bar considered in Fig. 1, the functions $\Gamma_1, \Gamma_2$ and $J$ are plotted in Fig. 2(a) and (b) in terms of $z$ for a fixed $x$ assuming that the attenuation function $g$ is the bi-exponential function (6) and $\alpha = -1$. Moreover, the functions $-1/V_\infty \int_\Omega H(x,z)\varepsilon(z)\mathrm{d}z$, pertaining to the one-component model, and $1/V_\infty^2 \int_\Omega J(x,z)\varepsilon(z)\mathrm{d}z$, referred to the two-component model, are reported in Fig. 2(c) for the reported step strain function. A comparison shows the similarity of the shape of the two functions.

Analogously to the one-component non-local model, the elastic energy (39) for the non-homogeneous body is given by the sum of the strain energy related to the local behaviour and the strain energy due to the non-local constitutive behaviour. The non-local terms depend on strains belonging to the influence region and the amplitude of the non-local addition is con-

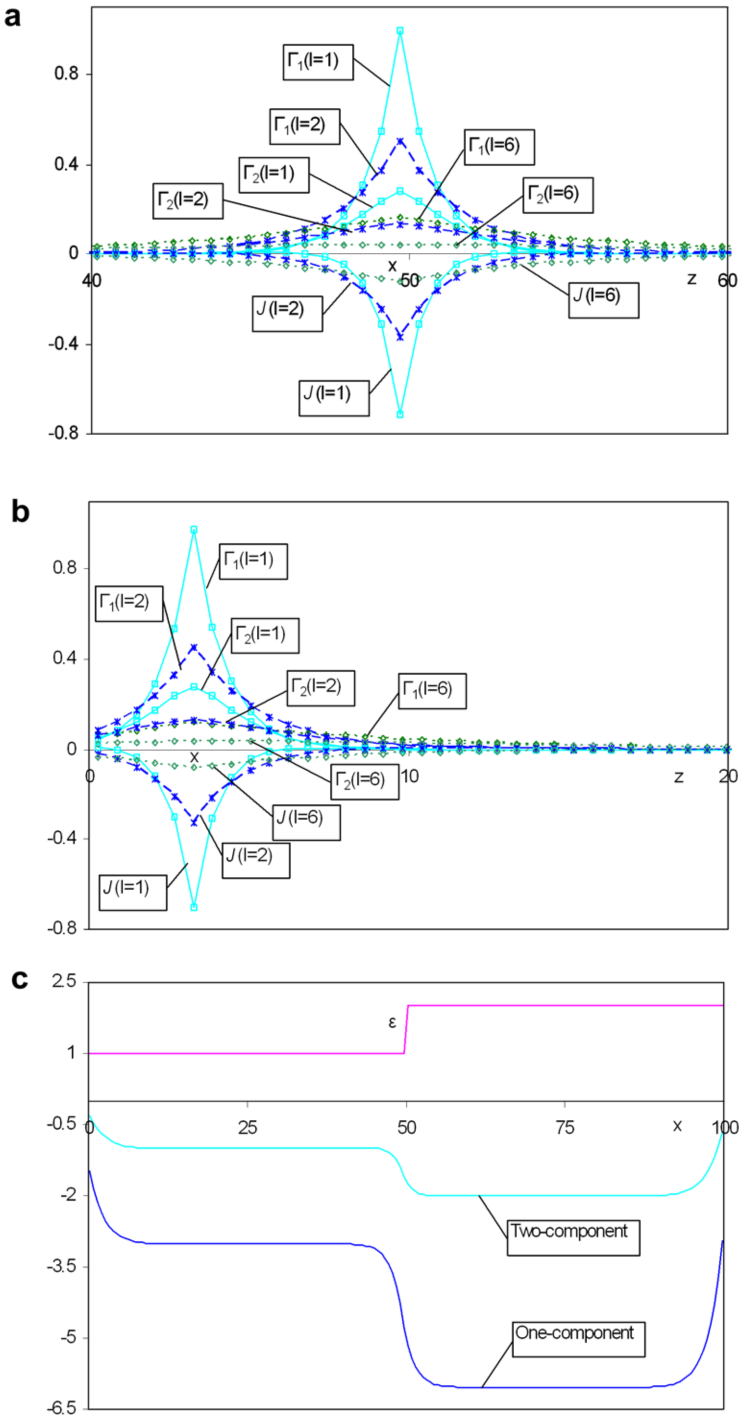


**Fig. 2.** Plots of the functions $\Gamma_1, \Gamma_2$ and $J$ in terms of $z$ for a fixed $x$ assuming a bi-exponential attenuation function and a unitary elastic modulus. The length scales are $l = 1$ cm, $l = 2$ cm, $l = 6$ cm, the interaction distance $R$ is such that $R/l = 6$ and $\alpha = -1$: (a) the point $x$ is far from the boundary; (b) the point $x$ is in the boundary layer; (c) plot of the functions $-1/V_\infty \int_\Omega H(x,z)\varepsilon(z)dz$ for the one-component model and $1/V_\infty^2 \int_\Omega J(x,z)\varepsilon(z)dz$ for the two-component model assuming $l = 2$ cm and the reported step function $\varepsilon$.

trolled by the parameter $\alpha$. Note that the elastic energy functional (40) is symmetric due to the symmetry of $J$, see (42). Analogous observations hold for the non-local stress (44).

For a given state of the body characterized by a strain $\boldsymbol{\varepsilon}(\boldsymbol{x})$, the residual $P$ at a point $\boldsymbol{x}$ of the body $\Omega$ is reported in $(23)_2$ and can be explicitly expressed in terms of the attenuation function $g$ in the form:

$$
\begin{aligned}
P(\boldsymbol{x}) = & -\alpha^2 \frac{V(\boldsymbol{x})}{V_\infty^2}\mathbf{E}(\boldsymbol{x})\int_\Omega g(\boldsymbol{x},\xi)\dot{\boldsymbol{\varepsilon}}(\xi)\mathrm{d}\xi * \boldsymbol{\varepsilon}(\boldsymbol{x}) \\
& + \frac{\alpha^2}{V_\infty^2}\mathbf{E}(\boldsymbol{x})\int_\Omega\int_\Omega g(\boldsymbol{x},\xi)g(\boldsymbol{x},\boldsymbol{z})\dot{\boldsymbol{\varepsilon}}(\xi) * \boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\xi\,\mathrm{d}\boldsymbol{z} \\
& + \frac{\alpha^2}{V_\infty^2}\int_\Omega V(\xi)g(\xi,\boldsymbol{x})\mathbf{E}(\xi)\boldsymbol{\varepsilon}(\xi)\mathrm{d}\xi * \dot{\varepsilon}(\boldsymbol{x}) \\
& - \frac{\alpha^2}{V_\infty^2}\int_\Omega\int_\Omega g(\xi,\boldsymbol{x})\mathbf{E}(\xi)g(\xi,\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z}\,\mathrm{d}\xi * \dot{\boldsymbol{\varepsilon}}(\boldsymbol{x})
\end{aligned}
\tag{46}
$$

as proved in Appendix B.

The non-locality residual $P$ turns out to be a homogeneous function of the strain rate. For a given strain field $\boldsymbol{\varepsilon}$, the residual $P$ can then be rewritten in a similar form to the one pertaining to the one-component non-local model:

$$
P(\boldsymbol{x}) = \int_\Omega \mathbf{f}_2(\boldsymbol{x},\xi) * \dot{\boldsymbol{\varepsilon}}(\xi)\mathrm{d}\xi + \mathbf{F}_2(\boldsymbol{x}) * \dot{\boldsymbol{\varepsilon}}(\boldsymbol{x}), \tag{47}
$$

where the functions $f_2$ and $F_2$ are hereafter reported:

$$
\begin{aligned}
\mathbf{f}_2(\boldsymbol{x},\xi) &= -\alpha^2\frac{V(\boldsymbol{x})}{V_\infty^2}\mathbf{E}(\boldsymbol{x})g(\boldsymbol{x},\xi)\boldsymbol{\varepsilon}(\boldsymbol{x}) + \frac{\alpha^2}{V_\infty^2}\mathbf{E}(\boldsymbol{x})g(\boldsymbol{x},\xi)\int_\Omega g(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z} \\
\mathbf{F}_2(\boldsymbol{x}) &= \frac{\alpha^2}{V_\infty^2}\int_\Omega V(\xi)g(\xi,\boldsymbol{x})\mathbf{E}(\xi)\boldsymbol{\varepsilon}(\xi)\mathrm{d}\xi - \frac{\alpha^2}{V_\infty^2}\int_\Omega g(\xi,\boldsymbol{x})\mathbf{E}(\xi)\left[\int_\Omega g(\xi,\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z}\right]\mathrm{d}\xi = -\int_\Omega \mathbf{f}_2(\xi,\boldsymbol{x})\mathrm{d}\xi.
\end{aligned}
\tag{48}
$$

The non-locality residual $P$ can then be evaluated by means of the functions $\mathbf{f}_2$ and $\mathbf{F}_2$ once the strain rate $\dot{\boldsymbol{\varepsilon}}$ is assigned. Plots regarding the functions $\mathbf{f}_2$ and $\mathbf{F}_2$, given by (48), are reported in the examples developed in Section 5.

The non-local stress $\bar{\boldsymbol{\sigma}}$ and the elastic energy functional $\Phi$ pertaining to the considered non-local models of non-homogeneous elasticity can be collected in a unified form as reported in Table 1.

A reasoning analogous to the one reported in Remark 1 can be followed for the two-component non-local model in order to show that the non-local material tends to a local behaviour if the internal length tends to vanishing.

### 2.4. A non-local model for piecewise non-homogeneous media

A non-local model for piecewise non-homogeneous bodies has been provided in Marotti de Sciarra (2008). The main relations are hereafter briefly summarized and cast in the present framework in order to make reasonably self-contained the paper.

The domain $\Omega$ occupied by the piecewise non-homogeneous body is partitioned in $\mathcal{N}$ homogeneous subdomains $\Omega_i \subseteq \Omega$ fulfilling the conditions $\cup_{i=1}^{\mathcal{N}}\overline{\Omega}_i = \overline{\Omega}$ and $\Omega_i \cap \Omega_j = \emptyset$ for any $i \neq j$. Accordingly the boundaries between subdomains of the piecewise non-homogeneous body constitute discontinuity surfaces for the elastic stiffness. The elastic stiffness associated with a homogeneous subdomain $\Omega_i$ is denoted by $\mathbf{E}_i$.

The space weight function has the expression (3) and the elastic energy is assumed in the following form:

$$
\begin{aligned}
\phi(\boldsymbol{\varepsilon}(\boldsymbol{x})) &= \frac{1}{2}(\mathbf{RE}\varepsilon)(\boldsymbol{x}) * \boldsymbol{\varepsilon}(\boldsymbol{x}) = \frac{1}{2}\left[\sum_{j=1}^{\mathcal{N}}\mathbf{E}_j\int_{\Omega_j}W(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z}\right] * \boldsymbol{\varepsilon}(\boldsymbol{x}) \\
&= \frac{1}{2}\left[1 - \alpha\frac{V(\boldsymbol{x})}{V_\infty}\right]\mathbf{E}_i\boldsymbol{\varepsilon}(\boldsymbol{x}) * \boldsymbol{\varepsilon}(\boldsymbol{x}) + \frac{\alpha}{2V_\infty}\sum_{j=1}^{\mathcal{N}}\mathbf{E}_j\int_{\Omega_j}g(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z} * \boldsymbol{\varepsilon}(\boldsymbol{x}),
\end{aligned}
\tag{49}
$$

where $\boldsymbol{x} \in \Omega_i$ and $\boldsymbol{z} \in \Omega_j$. After some algebra, the elastic energy can be rewritten as the sum of the local elastic energy and of a non-local contribution:

$$
\phi(\boldsymbol{\varepsilon}(\boldsymbol{x})) = \frac{1}{2}\mathbf{E}_i\boldsymbol{\varepsilon}(\boldsymbol{x}) * \boldsymbol{\varepsilon}(\boldsymbol{x}) + \frac{\alpha}{2V_\infty}\sum_{j=1}^{\mathcal{N}}\int_{\Omega_j}\mathbf{L}(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z} * \boldsymbol{\varepsilon}(\boldsymbol{x}), \tag{50}
$$

**Table 1**
Unified expressions of the non-local stress and of the non-local elastic energy functional associated with the considered non-homogeneous formulations

| Non-local stress $\bar{\boldsymbol{\sigma}}(\boldsymbol{x})$ | $\mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x}) + \int_\Omega \mathbf{Z}(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z}$ |
|---|---|
| Elastic energy functional $\Phi(\varepsilon)$ | $\frac{1}{2}\int_\Omega \mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x}) * \boldsymbol{\varepsilon}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} + \frac{1}{2}\int_\Omega\int_\Omega \mathbf{Z}(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z} * \boldsymbol{\varepsilon}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$ |
| | $\mathbf{Z}(\boldsymbol{x},\boldsymbol{z})$ |
| Non-local model for piecewise non-homogeneous bodies | $\frac{\alpha}{V_\infty}\mathbf{L}(\boldsymbol{x},\boldsymbol{z}) = -\alpha\frac{V(\boldsymbol{x})}{V_\infty}\mathbf{E}_i\delta(\boldsymbol{x},\boldsymbol{z}) + \frac{\alpha}{V_\infty}\mathbf{E}_j g(\boldsymbol{x},\boldsymbol{z})$ where $\boldsymbol{x} \in \Omega_i, \quad \boldsymbol{z} \in \Omega_j$ |
| One-component non-local model | $\frac{\alpha}{V_\infty}\mathbf{H}_1(\boldsymbol{x},\boldsymbol{z}) = \frac{\alpha}{V_\infty}[\mathbf{E}(\boldsymbol{x}) + \mathbf{E}(\boldsymbol{z})]g(\boldsymbol{x},\boldsymbol{z}) - 2\alpha\frac{V(\boldsymbol{z})}{V_\infty}\mathbf{E}(\boldsymbol{z})\delta(\boldsymbol{x},\boldsymbol{z}) + \frac{\alpha^2}{V_\infty^2}\mathbf{J}_1(\boldsymbol{x},\boldsymbol{z})$ |
| Two-component non-local model | $\frac{\alpha^2}{V_\infty^2}\mathbf{J}_1(\boldsymbol{x},\boldsymbol{z})$ |

where $\mathbf{L}(\boldsymbol{x},\boldsymbol{z}) = -V(\boldsymbol{x})\mathbf{E}_i\delta(\boldsymbol{x},\boldsymbol{z}) + \mathbf{E}_j g(\boldsymbol{x},\boldsymbol{z})$ with $\boldsymbol{x}\in\Omega_i, \boldsymbol{z}\in\Omega_j$ and $\mathbf{E}_i(\mathbf{E}_j)$ denotes the elastic stiffness pertaining to the homogeneous subdomains $\Omega_i(\Omega_j)$ with $i,j = 1,\ldots,\mathcal{N}$.

The elastic energy functional is then given by:

$$\Phi(\boldsymbol{\varepsilon}) = \frac{1}{2}\sum_{i=1}^{\mathcal{N}}\mathbf{E}_i\int_{\Omega_i}\boldsymbol{\varepsilon}(\boldsymbol{x})*\boldsymbol{\varepsilon}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} + \frac{\alpha}{2V_\infty}\sum_{i,j=1}^{\mathcal{N}}\int_{\Omega_i}\int_{\Omega_j}\mathbf{L}(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z}*\boldsymbol{\varepsilon}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \tag{51}$$

with $\boldsymbol{x}\in\Omega_i$ and $\boldsymbol{z}\in\Omega_j$. The stress–strain relation is provided by the relation:

$$\bar{\boldsymbol{\sigma}}(\boldsymbol{x}) = (\mathbf{RE}\boldsymbol{\varepsilon})(\boldsymbol{x}) = \left[1-\alpha\frac{V(\boldsymbol{x})}{V_\infty}\right]\mathbf{E}_i\boldsymbol{\varepsilon}(\boldsymbol{x}) + \frac{\alpha}{V_\infty}\sum_{j=1}^{\mathcal{N}}\mathbf{E}_j\int_{\Omega_j}g(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z} = \mathbf{E}_i\boldsymbol{\varepsilon}(\boldsymbol{x}) + \frac{\alpha}{V_\infty}\sum_{j=1}^{\mathcal{N}}\int_{\Omega_j}\mathbf{L}(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z} \tag{52}$$

with $\boldsymbol{x}\in\Omega_i$ and $\boldsymbol{z}\in\Omega_j$.

The non-locality residual function is then given by:

$$\begin{aligned}P(\boldsymbol{x}) &= \frac{1}{2}(\mathbf{RE}\dot{\varepsilon})(\boldsymbol{x})*\boldsymbol{\varepsilon}(\boldsymbol{x}) - \frac{1}{2}(\mathbf{RE}\boldsymbol{\varepsilon})(\boldsymbol{x})*\dot{\boldsymbol{\varepsilon}}(\boldsymbol{x}) = \frac{1}{2}\left[1-\alpha\frac{V(\boldsymbol{x})}{V_\infty}\right]\mathbf{E}_i\dot{\boldsymbol{\varepsilon}}(\boldsymbol{x})*\boldsymbol{\varepsilon}(\boldsymbol{x})\\
&\quad+\frac{\alpha}{2V_\infty}\sum_{j=1}^{\mathcal{N}}\mathbf{E}_j\int_{\Omega_j}g(\boldsymbol{x},\boldsymbol{z})\dot{\boldsymbol{\varepsilon}}(\boldsymbol{z})\mathrm{d}\boldsymbol{z}*\boldsymbol{\varepsilon}(\boldsymbol{x}) - \frac{1}{2}\left[1-\alpha\frac{V(\boldsymbol{x})}{V_\infty}\right]\mathbf{E}_i\boldsymbol{\varepsilon}(\boldsymbol{x})*\dot{\boldsymbol{\varepsilon}}(\boldsymbol{x})\\
&\quad-\frac{\alpha}{2V_\infty}\sum_{j=1}^{\mathcal{N}}\mathbf{E}_j\int_{\Omega_j}g(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z}*\dot{\boldsymbol{\varepsilon}}(\boldsymbol{x}) = \frac{\alpha}{2V_\infty}\sum_{j=1}^{\mathcal{N}}\mathbf{E}_j\int_{\Omega_j}g(\boldsymbol{x},\boldsymbol{z})\dot{\boldsymbol{\varepsilon}}(\boldsymbol{z})\mathrm{d}\boldsymbol{z}*\boldsymbol{\varepsilon}(\boldsymbol{x})\\
&\quad-\frac{\alpha}{2V_\infty}\sum_{j=1}^{\mathcal{N}}\mathbf{E}_j\int_{\Omega_j}g(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z}*\dot{\boldsymbol{\varepsilon}}(\boldsymbol{x})\end{aligned} \tag{53}$$

with $\boldsymbol{x}\in\Omega_i$ and $\boldsymbol{z}\in\Omega_j$. The non-local elastic energy functional (51) and the non-local stress (52) pertaining to the piecewise non-homogeneous model are cast in the unified framework as reported in Table 1.

### 2.5. A comparison between the non-local models

Let us now analyze the behaviour of the non-local models in the case of homogeneous or non-homogeneous elasticity subject to uniform or non-uniform strains $\boldsymbol{\varepsilon}(\boldsymbol{x})$ and strain rates $\dot{\boldsymbol{\varepsilon}}(\boldsymbol{x})$ in $\Omega$. The results are summarized in Table 2 where the labels $u$ and $n$, respectively, stand for uniform and non-uniform in $\Omega$ with reference to the elastic stiffness $\mathbf{E}$, strain $\boldsymbol{\varepsilon}$ and strain rate $\dot{\boldsymbol{\varepsilon}}$. For conciseness, the dependence of $\mathbf{E}$ and $\boldsymbol{\varepsilon}$ on the variables $\boldsymbol{x}$ and $\xi$ is written as a subscript in Table 2.

- Two-component non-local model

Let us consider a uniform strain $\boldsymbol{\varepsilon}$ in a non-homogeneous body $\Omega$. The elastic energy $\phi(\boldsymbol{\varepsilon}(\boldsymbol{x}))$, the elastic energy functional $\Phi(\boldsymbol{\varepsilon})$ and the non-local stress $\bar{\boldsymbol{\sigma}}(\boldsymbol{x})$, respectively, given by (16), (17) and $(21)_2$, reduce to their local counterparts (see Table 2) since $\mathbf{A}\boldsymbol{\varepsilon} = (\mathbf{R}-\mathbf{I})\boldsymbol{\varepsilon} = \mathbf{0}$ being $\bar{\boldsymbol{\varepsilon}} = \mathbf{R}\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}$. The same results can be obtained by considering the explicit expressions of $\phi, \Phi$ and $\bar{\boldsymbol{\sigma}}$ in terms of the attenuation function $g$, respectively, given in (39), (40) and (44).

Trivially, if the elastic stiffness is constant (homogeneous body), analogous results hold with $\mathbf{E}(\boldsymbol{x}) = \mathbf{E}$.

Similar arguments lead to the result that the non-locality residual function $P$, see the relations $(23)_2$ and (46) in terms of the attenuation function $g$, vanishes in the case of homogeneous or non-homogeneous elasticity for any uniform strain rate $\dot{\boldsymbol{\varepsilon}}$ in $\Omega$ which is the locality recovery condition (Polizzotto et al., 2006).

Let us consider a non-uniform strain field $\boldsymbol{\varepsilon}$ in a non-homogeneous body $\Omega$. The elastic energy $\phi(\boldsymbol{\varepsilon}(\boldsymbol{x}))$, the elastic energy functional $\Phi(\boldsymbol{\varepsilon})$ and the non-local stress $\bar{\boldsymbol{\sigma}}(\boldsymbol{x})$ have the explicit expressions (39), (40) and (44). The non-locality residual $P$ given by (46) does not vanish independently of the elastic stiffness $\mathbf{E}(\boldsymbol{x})$ for any strain rate $\dot{\boldsymbol{\varepsilon}}$. In particular, assuming a uniform strain rate in $\Omega$, it results:

$$\begin{aligned}P(\boldsymbol{x}) &= -\alpha^2\frac{V^2(\boldsymbol{x})}{V_\infty^2}\mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x})*\dot{\boldsymbol{\varepsilon}} + \alpha^2\frac{V(\boldsymbol{x})}{V_\infty^2}\mathbf{E}(\boldsymbol{x})\int_\Omega g(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z}*\dot{\boldsymbol{\varepsilon}}\\
&\quad+\frac{\alpha^2}{V_\infty^2}\int_\Omega V(\xi)g(\xi,\boldsymbol{x})\mathbf{E}(\xi)\boldsymbol{\varepsilon}(\xi)\mathrm{d}\xi*\dot{\boldsymbol{\varepsilon}} - \frac{\alpha^2}{V_\infty^2}\int_\Omega\int_\Omega g(\xi,\boldsymbol{x})\mathbf{E}(\xi)g(\xi,\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z}\,\mathrm{d}\xi*\dot{\boldsymbol{\varepsilon}}.\end{aligned} \tag{54}$$

For a homogeneous body, analogous results hold with $\mathbf{E}(\boldsymbol{x}) = \mathbf{E}$. For sake of conciseness the specialization of the constitutive relations are omitted and the results are reported in Table 2 in which the following function is used:

$$j_1(\boldsymbol{x},\boldsymbol{z}) = V^2(\boldsymbol{z})\delta(\boldsymbol{x},\boldsymbol{z}) + j(\boldsymbol{x},\boldsymbol{z}),$$

where:

$$j(\boldsymbol{x},\boldsymbol{z}) = \lambda_2(\boldsymbol{x},\boldsymbol{z}) - \gamma_1(\boldsymbol{x},\boldsymbol{z}),\quad \gamma_1(\boldsymbol{x},\boldsymbol{z}) = [V(\boldsymbol{x}) + V(\boldsymbol{z})]g(\boldsymbol{x},\boldsymbol{z})$$

**Table 2**
Comparison among the nonlocal model for piecewise non-homogeneous bodies and for the one-component and two-component nonlocal models (respectively denoted by the subscripts 0, 1 and 2)

| $\mathbf{E}(\boldsymbol{x})$ <br> $\boldsymbol{\varepsilon}(\boldsymbol{x})$ | $u$ <br> $u$ | $u$ <br> $n$ | | $n$ <br> $u$ | $n$ <br> $n$ | |
|---|---|---|---|---|---|---|
| $\phi_1(\boldsymbol{\varepsilon_x})$ | $\frac{1}{2}\mathbf{E}\boldsymbol{\varepsilon} * \boldsymbol{\varepsilon}$ | $\frac{1}{2}\mathbf{E}\boldsymbol{\varepsilon_x} * \boldsymbol{\varepsilon_x} + \frac{\alpha\mathbf{E}}{2V_\infty}(h_1\boldsymbol{\varepsilon})_{\boldsymbol{x}} * \boldsymbol{\varepsilon_x}$ | | $\frac{1}{2}\mathbf{E}_{\boldsymbol{x}}\boldsymbol{\varepsilon} * \boldsymbol{\varepsilon} - \frac{\alpha}{2V_\infty}V_{\boldsymbol{x}}\mathbf{E}_{\boldsymbol{x}}\boldsymbol{\varepsilon} * \boldsymbol{\varepsilon} + \frac{\alpha}{2V_\infty}\int_\Omega g(\boldsymbol{x},\xi)\mathbf{E}_\xi d\xi\boldsymbol{\varepsilon} * \boldsymbol{\varepsilon}$ | $\frac{1}{2}\mathbf{E}_{\boldsymbol{x}}\boldsymbol{\varepsilon_x} * \boldsymbol{\varepsilon_x} + \frac{\alpha}{2V_\infty}(\mathbf{H}_1\boldsymbol{\varepsilon})_{\boldsymbol{x}} * \boldsymbol{\varepsilon_x}$ | |
| $\Phi_1(\boldsymbol{\varepsilon})$ | $\frac{1}{2}\mathbf{E}\Omega\boldsymbol{\varepsilon} * \boldsymbol{\varepsilon}$ | $\frac{1}{2}\mathbf{E}\langle\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}\rangle + \frac{\alpha\mathbf{E}}{2V_\infty}\langle h_1\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}\rangle$ | | $\frac{1}{2}\langle\mathbf{E}\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}\rangle$ | $\frac{1}{2}\langle\mathbf{E}\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}\rangle + \frac{\alpha}{2V_\infty}\langle\mathbf{H}_1\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}\rangle$ | |
| $\bar{\boldsymbol{\sigma}}_1(\boldsymbol{x})$ | $\mathbf{E}\boldsymbol{\varepsilon}$ | $\mathbf{E}\boldsymbol{\varepsilon_x} + \frac{\alpha\mathbf{E}}{V_\infty}(h_1\boldsymbol{\varepsilon})_{\boldsymbol{x}}$ | | $\mathbf{E}_{\boldsymbol{x}}\boldsymbol{\varepsilon} - \alpha\frac{V_{\boldsymbol{x}}}{V_\infty}\mathbf{E}_{\boldsymbol{x}}\boldsymbol{\varepsilon} + \frac{\alpha}{V_\infty}\int_\Omega g(\boldsymbol{x},\xi)\mathbf{E}_\xi d\xi\boldsymbol{\varepsilon}$ | $\mathbf{E}_{\boldsymbol{x}}\boldsymbol{\varepsilon_x} + \frac{\alpha}{V_\infty}(\mathbf{H}_1\boldsymbol{\varepsilon})_{\boldsymbol{x}}$ | |
| $\phi_2(\boldsymbol{\varepsilon_x})$ | $\frac{1}{2}\mathbf{E}\boldsymbol{\varepsilon} * \boldsymbol{\varepsilon}$ | (39) with $\mathbf{E}_{\boldsymbol{x}}$=$\mathbf{E}$ | | $\frac{1}{2}\mathbf{E}_{\boldsymbol{x}}\boldsymbol{\varepsilon} * \boldsymbol{\varepsilon}$ | (39) | |
| $\Phi_2(\boldsymbol{\varepsilon})$ | $\frac{1}{2}\mathbf{E}\Omega\boldsymbol{\varepsilon} * \boldsymbol{\varepsilon}$ | $\frac{1}{2}\mathbf{E}\langle\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}\rangle + \frac{\alpha^2\mathbf{E}}{2V_\infty^2}\langle j_1\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}\rangle$ | | $\frac{1}{2}\langle\mathbf{E}\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}\rangle$ | $\frac{1}{2}\langle\mathbf{E}\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}\rangle + \frac{\alpha^2}{2V_\infty^2}\langle\mathbf{J}_1\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}\rangle$ | |
| $\bar{\boldsymbol{\sigma}}_2(\boldsymbol{x})$ | $\mathbf{E}\boldsymbol{\varepsilon}$ | $\mathbf{E}\boldsymbol{\varepsilon_x} + \frac{\alpha^2\mathbf{E}}{V_\infty^2}(j_1\boldsymbol{\varepsilon})_{\boldsymbol{x}}$ | | $\mathbf{E}_{\boldsymbol{x}}\boldsymbol{\varepsilon}$ | $\mathbf{E}_{\boldsymbol{x}}\boldsymbol{\varepsilon_x} + \frac{\alpha^2}{V_\infty^2}(\mathbf{J}_1\boldsymbol{\varepsilon})_{\boldsymbol{x}}$ | |
| $\phi_o(\boldsymbol{\varepsilon_x})$ | $\frac{1}{2}\mathbf{E}\boldsymbol{\varepsilon} * \boldsymbol{\varepsilon}$ | $\frac{1}{2}\mathbf{E}\boldsymbol{\varepsilon_x} * \boldsymbol{\varepsilon_x} + \frac{\alpha\mathbf{E}}{2V_\infty}(l\boldsymbol{\varepsilon})_{\boldsymbol{x}} * \boldsymbol{\varepsilon_x}$ | | $\frac{1}{2}\mathbf{E}_i\boldsymbol{\varepsilon} * \boldsymbol{\varepsilon} + \frac{\alpha}{2V_\infty}\sum_{j=1}^{\mathcal{N}}\int_{\Omega_j}\mathbf{L}(\boldsymbol{x},\boldsymbol{z})d\boldsymbol{z}\boldsymbol{\varepsilon} * \boldsymbol{\varepsilon}$ | $\frac{1}{2}\mathbf{E}_i\boldsymbol{\varepsilon_x} * \boldsymbol{\varepsilon_x} + \frac{\alpha}{2V_\infty}(\mathbf{L}\boldsymbol{\varepsilon})_{\boldsymbol{x}} * \boldsymbol{\varepsilon_x}$ | |
| $\Phi_o(\boldsymbol{\varepsilon})$ | $\frac{1}{2}\mathbf{E}\Omega\boldsymbol{\varepsilon} * \boldsymbol{\varepsilon}$ | $\frac{1}{2}\mathbf{E}\sum_i\langle\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}\rangle_i + \frac{\alpha\mathbf{E}}{2V_\infty}\sum_i\langle l\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}\rangle_i$ | | $\frac{1}{2}\sum_{i=1}^{\mathcal{N}}\mathbf{E}_i\Omega_i\boldsymbol{\varepsilon} * \boldsymbol{\varepsilon}$ | $\frac{1}{2}\sum_i\langle\mathbf{E}_i\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}\rangle_i + \frac{\alpha}{2V_\infty}\sum_i\langle\mathbf{L}\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}\rangle_i$ | |
| $\bar{\boldsymbol{\sigma}}_o(\boldsymbol{x})$ | $\mathbf{E}\boldsymbol{\varepsilon}$ | $\mathbf{E}\boldsymbol{\varepsilon_x} + \frac{\alpha\mathbf{E}}{V_\infty}(l\boldsymbol{\varepsilon})_{\boldsymbol{x}}$ | | $\mathbf{E}_i\boldsymbol{\varepsilon} + \frac{\alpha}{V_\infty}\sum_{j=1}^{\mathcal{N}}\int_{\Omega_j}\mathbf{L}(\boldsymbol{x},\boldsymbol{z})d\boldsymbol{z}\boldsymbol{\varepsilon}$ | $\mathbf{E}_i\boldsymbol{\varepsilon_x} + \frac{\alpha}{V_\infty}(\mathbf{L}\boldsymbol{\varepsilon})_{\boldsymbol{x}}$ | |
| $\dot{\boldsymbol{\varepsilon}}(\boldsymbol{x})$ | $u$ | $u$ | $n$ | $u$ | $u$ | $n$ |
| $P_1(\boldsymbol{x})$ | 0 | (60) | (37) | 0 | (58) | (31) |
| $P_2(\boldsymbol{x})$ | 0 | (54) with $\mathbf{E}_{\boldsymbol{x}}$=$\mathbf{E}$ | (46) with $\mathbf{E}_{\boldsymbol{x}}$=$\mathbf{E}$ | 0 | (54) | (46) |
| $P_o(\boldsymbol{x})$ | 0 | (65) with $\mathbf{E}_j$=$\mathbf{E}$ | (53) with $\mathbf{E}_j$=$\mathbf{E}$ | 0 | (65) | (53) |

and $\lambda_2$ is defined according to $(34)_2$.

- One-component non-local model

Let us consider a uniform strain field $\boldsymbol{\varepsilon}$ in a non-homogeneous body $\Omega$. The relation (13) providing the elastic energy becomes:

$$\phi(\boldsymbol{\varepsilon}(\boldsymbol{x})) = \frac{1}{2}(\mathbf{RE})(\boldsymbol{x})\boldsymbol{\varepsilon} * \boldsymbol{\varepsilon} = \frac{1}{2}\mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon} * \boldsymbol{\varepsilon} - \frac{\alpha}{2V_\infty}\left[V(\boldsymbol{x})\mathbf{E}(\boldsymbol{x}) - \int_\Omega g(\boldsymbol{x},\xi)\mathbf{E}(\xi)\mathrm{d}\xi\right]\boldsymbol{\varepsilon} * \boldsymbol{\varepsilon}. \tag{55}$$

It is then immediate to show that the elastic energy functional (14) coincides to its local counterpart, i.e. $\Phi(\boldsymbol{\varepsilon}) = \frac{1}{2}\langle \mathbf{E}\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}\rangle$. The relation $(21)_1$ for the non-local stress $\bar{\boldsymbol{\sigma}}(\boldsymbol{x})$ yields:

$$\bar{\boldsymbol{\sigma}}(\boldsymbol{x}) = (\mathbf{RE})(\boldsymbol{x})\boldsymbol{\varepsilon} = \mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon} - \frac{\alpha}{V_\infty}\left[V(\boldsymbol{x})\mathbf{E}(\boldsymbol{x}) - \int_\Omega g(\boldsymbol{x},\xi)\mathbf{E}(\xi)\mathrm{d}\xi\right]\boldsymbol{\varepsilon}. \tag{56}$$

The non-locality residual $P$ vanishes for any uniform strain rate $\dot{\varepsilon}$. In fact, the relation $(23)_1$ yields:

$$P(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{\varepsilon} * (\mathbf{RE})(\boldsymbol{x})\dot{\boldsymbol{\varepsilon}} - \frac{1}{2}(\mathbf{RE})(\boldsymbol{x})\boldsymbol{\varepsilon} * \dot{\boldsymbol{\varepsilon}} = 0 \tag{57}$$

since $\bar{\boldsymbol{\varepsilon}} = \mathbf{R}\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}$, $\mathbf{R}\dot{\varepsilon} = (\mathbf{R}\boldsymbol{\varepsilon})^{\cdot} = \dot{\boldsymbol{\varepsilon}}$ and it can be easily proved that $\boldsymbol{\varepsilon} * (\mathbf{RE})(\boldsymbol{x})\dot{\boldsymbol{\varepsilon}} = (\mathbf{RE})(\boldsymbol{x})\boldsymbol{\varepsilon} * \dot{\boldsymbol{\varepsilon}}$.

The values of the elastic energy (55) and of the non-local stress (56) pertaining to a non-homogeneous body in a uniform strain state do not coincide to the relevant local counterparts due to the non-homogeneity of the material, i.e. to the space variation of the elastic stiffness. Further the elastic energy functional (pertaining to the whole body) coincides to its local counterpart.

It is worth noting that the insulation condition (10) is fulfilled and the dissipation $D$, given by (11), is pointwise vanishing according to the reversible nature of the model since $\dot{\phi} = \bar{\boldsymbol{\sigma}} * \dot{\boldsymbol{\varepsilon}}$.

As a consequence, the vanishing of the residual function everywhere in the body for any uniform strain (see the locality recovery condition in Polizzotto et al., 2006) ensures that there is no energy exchanges between neighbour particles. Nevertheless the presence of a spatial variation of the elastic stiffness yields a non-local expression for the free energy and for the stress.

In fact the non-local behaviour is due to the non-homogeneity of the material and is effective even if the strain is uniform in $\Omega$ and the non-locality residual is pointwise vanishing.

Hence the considered model follows the heuristic considerations advanced by Polizzotto et al. (2004, 2006) according to which the presence of non-homogeneity due to the space variation of the elastic stiffness $\mathbf{E}(\boldsymbol{x})$ contributes to the interactions between distant particles. In fact if a subdomain $\Omega_o \subset \Omega$ pertaining to an initially homogeneous non-local elastic body $\Omega$, has a completely deteriorated elastic modulus, no long distance interactions between points inside and outside $\Omega_o$ are allowed to occur. In any intermediate state of this degradation process, the interactions between particles inside and outside $\Omega_o$ are influenced by the non-homogeneity of the elastic modulus with respect to the initial non-local homogeneous situation. In Polizzotto et al. (2006) the particle interaction attenuation effects due to non-homogeneity of the elastic modulus are conventionally accounted for by means of a larger equivalent distance. Experimental data seems to be lacking on this issue and further analyses are necessary.

Let us consider a uniform strain $\boldsymbol{\varepsilon}$ in a homogeneous body $\Omega$, i.e. $\mathbf{E}(\boldsymbol{x}) = \mathbf{E}$. The elastic energy $\phi(\boldsymbol{\varepsilon}(\boldsymbol{x}))$, the elastic energy functional $\Phi(\boldsymbol{\varepsilon})$ and the non-local stress $\bar{\boldsymbol{\sigma}}(\boldsymbol{x})$ given by (13), (14) and $(21)_1$ reduce to their local counterparts (see Table 2) being $\mathbf{R}\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}$ and $\mathbf{RER}\boldsymbol{\varepsilon} = \mathbf{E}\boldsymbol{\varepsilon}$.

Similar arguments show that the non-locality residual function $P$, given by $(23)_1$, vanishes for any uniform strain rate $\dot{\boldsymbol{\varepsilon}}$ by the symmetry of the elastic stiffness $\mathbf{E}$ and the condition $\mathbf{R}\dot{\boldsymbol{\varepsilon}} = (\mathbf{R}\boldsymbol{\varepsilon})^{\cdot} = \dot{\boldsymbol{\varepsilon}}$.

Let us consider a non-uniform strain field $\boldsymbol{\varepsilon}$ in a non-homogeneous body $\Omega$. The elastic energy $\phi(\boldsymbol{\varepsilon}(\boldsymbol{x}))$, the elastic energy functional $\Phi(\boldsymbol{\varepsilon})$ and the non-local stress $\bar{\boldsymbol{\sigma}}(\boldsymbol{x})$ have the expressions (27), (29) and (30). The non-locality residual $P$ is given by (31) and does not vanish, independently of the elastic stiffness $\mathbf{E}(\boldsymbol{x})$, for any strain rate $\dot{\boldsymbol{\varepsilon}}$. In particular, assuming a uniform strain rate, it results:

$$P(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{\varepsilon}(\boldsymbol{x}) * (\mathbf{RE})(\boldsymbol{x})\dot{\varepsilon} - \frac{1}{2}(\mathbf{RER}\boldsymbol{\varepsilon})(\boldsymbol{x}) * \dot{\varepsilon} \tag{58}$$

and explicitly it results:

$$P(\boldsymbol{x}) = \alpha\frac{V(\boldsymbol{x})}{2V_\infty}\left[1 - \alpha\frac{V(\boldsymbol{x})}{V_\infty}\right]\mathbf{E}(\boldsymbol{x})\dot{\boldsymbol{\varepsilon}} * \boldsymbol{\varepsilon}(\boldsymbol{x}) + \frac{\alpha}{2V_\infty}\int_\Omega \mathbf{E}(\boldsymbol{z})g(\boldsymbol{x},\boldsymbol{z})\mathrm{d}\boldsymbol{z}\dot{\boldsymbol{\varepsilon}} * \boldsymbol{\varepsilon}(\boldsymbol{x}) - \frac{\alpha}{2V_\infty}\int_\Omega \mathbf{H}(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z} * \dot{\boldsymbol{\varepsilon}}. \tag{59}$$

For a homogeneous body, analogous results hold with $\mathbf{E}(\boldsymbol{x}) = \mathbf{E}$. The non-local elastic energy, stress and residual are, respectively, given by (35)–(37). In the case of a uniform strain rate $\dot{\boldsymbol{\varepsilon}}$, the expression (37) becomes:

$$P(\boldsymbol{x}) = \alpha\frac{V(\boldsymbol{x})}{2V_\infty}\left[2 - \alpha\frac{V(\boldsymbol{x})}{V_\infty}\right]\mathbf{E}\dot{\varepsilon} * \boldsymbol{\varepsilon}(\boldsymbol{x}) - \frac{\alpha}{2V_\infty}\mathbf{E}\int_\Omega h(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z} * \dot{\boldsymbol{\varepsilon}}. \tag{60}$$

Clearly the same result is obtained starting from the expression (59) by setting $\mathbf{E}(\boldsymbol{x}) = \mathbf{E}$.

- Piecewise non-homogeneous model

Let us consider a uniform strain field $\boldsymbol{\varepsilon}$ in a non-local model for a piecewise non-homogeneous body $\Omega$. The relation (50) can be easily rewritten by setting $\boldsymbol{\varepsilon}(\boldsymbol{x}) = \boldsymbol{\varepsilon}$ in the form:

$$\phi(\boldsymbol{\varepsilon}(\boldsymbol{x})) = \frac{1}{2}\mathbf{E}_i\boldsymbol{\varepsilon} * \boldsymbol{\varepsilon} + \frac{\alpha}{2V_\infty}\sum_{j=1}^{\mathcal{N}}\int_{\Omega_j}\mathbf{L}(\boldsymbol{x},\boldsymbol{z})\mathrm{d}\boldsymbol{z}\boldsymbol{\varepsilon} * \boldsymbol{\varepsilon} \tag{61}$$

with $\boldsymbol{x} \in \Omega_i$ and $\boldsymbol{z} \in \Omega_j$. The elastic energy functional (51) coincides to its local counterpart:

$$\Phi(\boldsymbol{\varepsilon}) = \frac{1}{2}\sum_{i=1}^{\mathcal{N}}\mathbf{E}_i\Omega_i\boldsymbol{\varepsilon} * \boldsymbol{\varepsilon}, \tag{62}$$

being $\Omega_i$ the measure of the $i$th homogeneous subdomain, since it results:

$$\sum_{i,j=1}^{\mathcal{N}}\int_{\Omega_i}\int_{\Omega_j}\mathbf{L}(\boldsymbol{x},\boldsymbol{z})\mathrm{d}\boldsymbol{z}\,\mathrm{d}\boldsymbol{x} = \mathbf{0}$$

with $\boldsymbol{x} \in \Omega_i$ and $\boldsymbol{z} \in \Omega_j$. The non-local stress (52) becomes:

$$\bar{\boldsymbol{\sigma}}(\boldsymbol{x}) = \mathbf{E}_i\boldsymbol{\varepsilon} + \frac{\alpha}{V_\infty}\sum_{j=1}^{\mathcal{N}}\int_{\Omega_j}\mathbf{L}(\boldsymbol{x},\boldsymbol{z})\mathrm{d}\boldsymbol{z}\boldsymbol{\varepsilon} \tag{63}$$

and the non-locality residual (53) vanishes for any uniform strain rate being:

$$P(\boldsymbol{x}) = \frac{\alpha}{2V_\infty}\sum_{j=1}^{\mathcal{N}}\mathbf{E}_j\int_{\Omega_j}g(\boldsymbol{x},\xi)\mathrm{d}\xi\dot{\boldsymbol{\varepsilon}} * \boldsymbol{\varepsilon} - \frac{\alpha}{2V_\infty}\sum_{j=1}^{\mathcal{N}}\mathbf{E}_j\int_{\Omega_j}g(\boldsymbol{x},\xi)\mathrm{d}\xi\boldsymbol{\varepsilon} * \dot{\boldsymbol{\varepsilon}} = 0 \tag{64}$$

with $\boldsymbol{x} \in \Omega_i$ and $\boldsymbol{z} \in \Omega_j$.

It is worth noting that, similarly to the one-component non-local model, the elastic energy and the non-local stress do not reduce to the relevant local counterparts. The non-locality residual function vanishes for any uniform strain rate, the insulation condition is fulfilled and the dissipation is pointwise vanishing according to the reversible nature of the model. Moreover the global elastic energy (62) coincides to its classical (local) counterparts.

Accordingly the considered non-local model for piecewise non-homogeneous bodies follows the considerations advanced in Polizzotto et al. (2006) upon the possibility that a non-homogeneous elastic stiffness contributes to interactions between distant particles.

Let us consider a uniform strain field $\boldsymbol{\varepsilon}$ in a homogeneous body $\Omega$. The elastic energy (50), the elastic energy functional (51) and the non-local stress (52) reduce to their local counterparts. The residual $P$ vanishes for any uniform strain rate $\dot{\boldsymbol{\varepsilon}}$.

Let us consider a non-uniform strain field $\boldsymbol{\varepsilon}$ in a non-homogeneous body $\Omega$. The elastic energy $\phi(\boldsymbol{\varepsilon}(\boldsymbol{x}))$, the elastic energy functional $\Phi(\boldsymbol{\varepsilon})$ and the non-local stress $\bar{\boldsymbol{\sigma}}(\boldsymbol{x})$, respectively, follow from the relations (50)–(52). The non-locality residual $P$ does not vanish for any strain rate $\dot{\boldsymbol{\varepsilon}}$. In particular, assuming a uniform strain rate, it results:

$$P(\boldsymbol{x}) = \frac{\alpha}{2V_\infty}\sum_{j=1}^{\mathcal{N}}\mathbf{E}_j\int_{\Omega_j}g(\boldsymbol{x},\xi)\mathrm{d}\xi\dot{\boldsymbol{\varepsilon}} * \boldsymbol{\varepsilon}(\boldsymbol{x}) - \frac{\alpha}{2V_\infty}\sum_{j=1}^{\mathcal{N}}\mathbf{E}_j\int_{\Omega_j}g(\boldsymbol{x},\xi)\boldsymbol{\varepsilon}(\xi)\mathrm{d}\xi * \dot{\boldsymbol{\varepsilon}}. \tag{65}$$

For a homogeneous body, analogous results hold with $\mathbf{E}(\boldsymbol{x}) = \mathbf{E}$ where the function $l(\boldsymbol{x},\boldsymbol{z}) = -V(\boldsymbol{x})\delta(\boldsymbol{x},\boldsymbol{z}) + g(\boldsymbol{x},\boldsymbol{z})$ is introduced in Table 2.

## 3. The non-local elastostatic problem

The analysis of linear structural models makes reference to the following pairs of dual Hilbert spaces: the kinematic-force pair $\mathscr{U},\mathscr{F}$ and the strain–stress pair $\Sigma,\mathscr{S}$. The kinematic operator $\mathbf{B} \in \mathrm{Lin}\{\mathscr{U},\Sigma\}$ gives the linearized strain due to a prescribed displacement field and the dual equilibrium operator $\mathbf{B}' \in \mathrm{Lin}\{\mathscr{S},\mathscr{F}\}$ provides the force system in equilibrium with a given stress field. Stress and strain spaces may be identified with a pivot Hilbert space (square integrable fields). The duality pairing in $\mathscr{F} \times \mathscr{U}$ is denoted by $\langle\cdot,\cdot\rangle$ having the physical meaning of external virtual work. For avoiding proliferation of symbols, the internal and external virtual works are denoted by the same symbol. Linear boundary constraints define a linear subspace $\mathscr{L}_o \subset \mathscr{U}$ of conforming displacement fields $\mathbf{v}$ (see Marotti de Sciarra, 2008; Romano, 2002; Showalter, 1997).

The relations governing the non-local and non-homogeneous elastic structural problem for a given load history $\ell(t)$ are given in the form:

$$\begin{cases} \mathbf{B}'\bar{\boldsymbol{\sigma}} = \ell + \mathbf{r} & \text{equilibrium} \\ \mathbf{B}(\mathbf{v} + \mathbf{w}) = \boldsymbol{\varepsilon} & \text{compatibility} \\ \bar{\boldsymbol{\sigma}} = d\Phi(\boldsymbol{\varepsilon}) & \text{constitutive relation} \\ \mathbf{v} \in \partial\Upsilon^*(\mathbf{r}) & \text{external relation.} \end{cases} \tag{66}$$

The load functional $\ell = \{\mathbf{t}, \mathbf{b}\} \in \mathscr{F}$ collects tractions $\mathbf{t}$ and body forces $\mathbf{b}$. Reactions of the external constraints are denoted by $\mathbf{r}$ and $\mathbf{w} \in \mathscr{U}$ represents a displacement field which fulfils the non-homogeneous boundary conditions.

The structural model encompasses the one-component and two-component non-local models and the non-local model for piecewise non-homogeneous bodies since the elastic energy in $(66)_3$ can assume the expressions provided in (29), (40), (51) as reported in Table 1.

The external relation between reactions $\mathbf{r} \in \mathscr{F}$ and displacements $\mathbf{v} \in \mathscr{U}$ is expressed in terms of two conjugate concave functionals $\Upsilon : \mathscr{U} \to \Re \cup \{-\infty\}$ and $\Upsilon^* : \mathscr{F} \to \Re \cup \{-\infty\}$ by means of the following equivalent relations:

$$\mathbf{r} \in \partial\Upsilon(\mathbf{v}) \Longleftrightarrow \mathbf{v} \in \partial\Upsilon^*(\mathbf{r}) \Longleftrightarrow \Upsilon(\mathbf{v}) + \Upsilon^*(\mathbf{r}) = \langle \mathbf{r}, \mathbf{v} \rangle \tag{67}$$

where the symbol $\partial$ denotes the superdifferential of concave functionals (Rockafellar, 1970).

Different expressions can be given to the concave functionals $\Upsilon$ and $\Upsilon^*$ depending on the particular type of external constrains. In the case of external unilateral constraints, the set of conforming displacements is given by the convex cone $\mathscr{C}_o$. The subspace of external reactions is the positive polar cone $\mathscr{C}_o^+$ of $\mathscr{C}_o$ defined as the set of reactions $\mathbf{r} \in \mathscr{F}$ such that the external virtual work is non-negative, i.e. $\langle \mathbf{r}, \mathbf{v} \rangle \geqslant 0$, for any conforming displacement $\mathbf{v} \in \mathscr{C}_o$. Then the functionals $\Upsilon$ and $\Upsilon^*$ are defined in the form:

$$\Upsilon(\mathbf{v}) = \sqcap_{\mathscr{C}_o}(\mathbf{v}) = \begin{cases} 0 & \text{if } \mathbf{v} \in \mathscr{C}_o \\ -\infty & \text{otherwise} \end{cases} \quad \Upsilon^*(\mathbf{r}) = \sqcap_{\mathscr{C}_o^+}(\mathbf{r}) = \begin{cases} 0 & \text{if } \mathbf{r} \in \mathscr{C}_o^+ = R \\ -\infty & \text{otherwise.} \end{cases}$$

The external constraint relation $(66)_4$ then yields $\mathbf{v} \in \mathscr{C}_o$ and $\mathbf{r} \in R = \mathscr{C}_o^+$, i.e. $\langle \mathbf{r}, \mathbf{v} \rangle \geqslant 0$ for any conforming displacement $\mathbf{v} \in \mathscr{C}_o$.

In the sequel, external frictionless bilateral constraints with non-homogeneous boundary conditions are considered in the examples. Hence the admissible set of displacements is the subspace $\mathscr{L} = \mathbf{w} + \mathscr{L}_o$ where $\mathscr{L}_o$ collects conforming displacements which satisfy the homogeneous boundary conditions. The subspace of the external constraint reactions $R$ is the orthogonal complement of $\mathscr{L}_o$, that is $R = \mathscr{L}_o^\perp$. Then the functional $\Upsilon$ turns out to be the indicator of $\mathscr{L}_o$ defined in the form:

$$\Upsilon(\mathbf{v}) = \sqcap_{\mathscr{L}_o}(\mathbf{v} - \mathbf{w}) = \begin{cases} 0 & \text{if } \mathbf{v} - \mathbf{w} \in \mathscr{L}_o \\ -\infty & \text{otherwise} \end{cases} \tag{68}$$

and its conjugate $\Upsilon^*$ is given by:

$$\Upsilon^*(\mathbf{r}) = \langle \mathbf{r}, \mathbf{w} \rangle + \sqcap_{\mathscr{L}_o^\perp}(\mathbf{r}) = \langle \mathbf{r}, \mathbf{w} \rangle + \begin{cases} 0 & \text{if } \mathbf{r} \in \mathscr{L}_o^\perp = R \\ -\infty & \text{otherwise.} \end{cases} \tag{69}$$

Accordingly the external relation $(66)_4$ are equivalent to state $\mathbf{v} \in \mathbf{w} + \mathscr{L}_o$ and $\mathbf{r} \in R = \mathscr{L}_o^\perp$, i.e. $\langle \mathbf{r}, \mathbf{v} \rangle = 0$ for any conforming displacement $\mathbf{v} \in \mathscr{L}_o$.

The non-local model (66) turns out to be coincident to the non-local model for piecewise non-homogeneous bodies proposed in Marotti de Sciarra (2008) if the non-local elastic energy functional appearing in $(66)_3$ assumes the expression (51). From a mechanical point of view the one-component and the two-component non-local models can describe a fully non-homogeneous continuum while the non-local model provided in Marotti de Sciarra (2008) is limited to a piecewise non-homogeneous medium.

A comparison among the considered non-homogeneous models is provided in Section 5 with reference to a non-homogeneous one-dimensional bar.

It is possible to follow the same procedure shown in Marotti de Sciarra (2008) to prove that the non-local problem at hand admits variational formulations. The complete set of non-local variational formulations containing all the possible combinations of the state variables is provided by the ten functionals reported in Table 3. By enforcing the fulfilment of the relations (66), the state variables appearing in the mixed variational formulations of Table 3 can be alternatively eliminated and the variational formulations can be obtained from one another. All the functionals attain the same value at a solution point of the non-local and non-homogeneous elastic problem.

The potentials $P_2, P_1, H_1$ and $R_2$ turn out to be the non-local counterparts of the total potential energy, complementary energy and mixed principles of Hu-Washizu and Hellinger-Reissner in classical local elasticity (Washizu, 1982) in the case of convex external constraints.

**Table 3**
The non-local elastic functionals with non-homogeneous boundary conditions

| |
|---|
| $M(\mathbf{v}, \bar{\boldsymbol{\sigma}}, \boldsymbol{\varepsilon}, \mathbf{r}) = \Phi(\boldsymbol{\varepsilon}) + \Upsilon^*(\mathbf{r}) + \langle \bar{\boldsymbol{\sigma}}, \mathbf{B}(\mathbf{v} + \mathbf{w}) - \boldsymbol{\varepsilon} \rangle - \langle \ell + \mathbf{r}, \mathbf{v} \rangle$ |
| $H_1(\mathbf{v}, \bar{\boldsymbol{\sigma}}, \boldsymbol{\varepsilon}) = \Phi(\boldsymbol{\varepsilon}) - \Upsilon(\mathbf{v}) + \langle \bar{\boldsymbol{\sigma}}, \mathbf{B}(\mathbf{v} + \mathbf{w}) - \boldsymbol{\varepsilon} \rangle - \langle \ell, \mathbf{v} \rangle$ |
| $H_2(\mathbf{v}, \bar{\boldsymbol{\sigma}}, \mathbf{r}) = -\Phi^*(\bar{\boldsymbol{\sigma}}) + \Upsilon^*(\mathbf{r}) + \langle \bar{\boldsymbol{\sigma}}, \mathbf{B}(\mathbf{v} + \mathbf{w}) \rangle - \langle \ell + \mathbf{r}, \mathbf{v} \rangle$ |
| $R_1(\bar{\boldsymbol{\sigma}}, \boldsymbol{\varepsilon}) = \Phi(\boldsymbol{\varepsilon}) + \Upsilon^*(\mathbf{B}'\bar{\boldsymbol{\sigma}} - \ell) - \langle \bar{\boldsymbol{\sigma}}, \boldsymbol{\varepsilon} \rangle + \langle \bar{\boldsymbol{\sigma}}, \mathbf{B}\mathbf{w} \rangle$ |
| $R_2(\mathbf{v}, \bar{\boldsymbol{\sigma}}) = -\Phi^*(\bar{\boldsymbol{\sigma}}) - \Upsilon(\mathbf{v}) + \langle \bar{\boldsymbol{\sigma}}, \mathbf{B}(\mathbf{v} + \mathbf{w}) \rangle - \langle \ell, \mathbf{v} \rangle$ |
| $R_3(\mathbf{v}, \mathbf{r}) = \Phi(\mathbf{B}(\mathbf{v} + \mathbf{w})) + \Upsilon^*(\mathbf{r}) - \langle \ell + \mathbf{r}, \mathbf{v} \rangle$ |
| $P_1(\bar{\boldsymbol{\sigma}}) = -\Phi^*(\bar{\boldsymbol{\sigma}}) + \Upsilon^*(\mathbf{B}'\bar{\boldsymbol{\sigma}} - \ell) + \langle \bar{\boldsymbol{\sigma}}, \mathbf{B}\mathbf{w} \rangle$ |
| $P_2(\mathbf{v}) = \Phi(\mathbf{B}(\mathbf{v} + \mathbf{w})) - \Upsilon(\mathbf{v}) - \langle \ell, \mathbf{v} \rangle$ |
| $P_3(\boldsymbol{\varepsilon}) = \Phi(\boldsymbol{\varepsilon}) - (\Upsilon^* \circ \mathbf{B}')^*(\boldsymbol{\varepsilon})$ |
| $P_4(\mathbf{r}) = -(\Phi \circ \mathbf{B})^*(\ell + \mathbf{r}) + \Upsilon^*(\mathbf{r}) + \langle \mathbf{r}, \mathbf{w} \rangle$ |

The solution uniqueness of the boundary-value problem (66) for non-homogeneous non-local elastic materials requires the strictly convexity of the non-local elastic energy. In fact, if the non-local elastic energy functional $\Phi$ is strictly convex, the total potential energy $P_2$ turns out to be strictly convex so that the solution displacement is an absolute minimum for $P_2$ and the non-local elastic structural problem (66) admits a unique solution (if any).

### 3.1. Total potential energy for non-homogeneous non-local elasticity

From a computational standpoint, the non-local total potential energy $P_2$ provides a useful tool to derive the finite element method for non-homogeneous non-local elasticity as discussed in Section 4. The expression of the non-local total potential energy $P_2$ for the Cauchy model in the case of external frictionless bilateral constraints is given by:

$$P_2(\mathbf{v}) = \Phi(\mathbf{B}(\mathbf{v}+\mathbf{w})) - \langle \ell, \mathbf{v} \rangle,$$

where $\mathbf{v} \in \mathscr{L}_o$ is a conforming displacement. The unified form of the elastic energy functional $\Phi$ is reported in Table 1 so that it follows:

$$\begin{aligned} P_2(\mathbf{v}) = &\frac{1}{2}\int_\Omega \mathbf{E}(\boldsymbol{x})\mathbf{B}[\mathbf{v}(\boldsymbol{x})+\mathbf{w}(\boldsymbol{x})] * \mathbf{B}[\mathbf{v}(\boldsymbol{x})+\mathbf{w}(\boldsymbol{x})]\mathrm{d}\boldsymbol{x} + \frac{1}{2}\int_\Omega\int_\Omega \mathbf{Z}(\boldsymbol{x},\boldsymbol{z})\mathbf{B}[\mathbf{v}(\boldsymbol{z})+\mathbf{w}(\boldsymbol{z})] * \mathbf{B}[\mathbf{v}(\boldsymbol{x})+\mathbf{w}(\boldsymbol{x})]\mathrm{d}\boldsymbol{z}\,\mathrm{d}\boldsymbol{x} \\ &- \int_\Omega \mathbf{b}(\boldsymbol{x}) * \mathbf{v}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} - \int_{\partial\Omega} \mathbf{t}(\boldsymbol{x}) * \mathbf{v}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \end{aligned} \tag{70}$$

with $\mathbf{v} \in \mathscr{L}_o$ The explicit expressions of $\Phi$ are given by (29), (40) and (51) depending on the considered model.

## 4. A non-local finite element discretization

The non-local elastic problem can be numerically solved by a non-local finite element method (NFEM) starting from the total potential energy $P_2$. A NFEM requires to build up a non-local stiffness matrix which reflects the non-locality features of the structural problem (Polizzotto et al., 2006). Such a non-local stiffness matrix contains the contributions from all the elements of the mesh which lie within the influence distance from the considered element. Accordingly the non-local stiffness matrix turns out to be banded with a band width larger than in the standard FEM.

The domain $\Omega$ occupied by the non-homogeneous body is partitioned in subdomains $\Omega_e \subseteq \Omega$, with $e = 1, \ldots \mathscr{N}$, fulfilling the conditions $\cup_{e=1}^{\mathscr{N}} \overline{\Omega}_e = \overline{\Omega}$ and $\Omega_e \cap \Omega_j = \emptyset$ for any $e \neq j$.

Adopting a conforming finite element discretization, the unknown displacement field $\mathbf{v}(\boldsymbol{x})$ is given, for each element, in the interpolated form $\mathbf{v}_h^e(\boldsymbol{x}) = \mathbf{N}_e(\boldsymbol{x})\mathbf{q}_e$ with $\boldsymbol{x} \in \Omega_e$ where $\mathbf{q}_e$ is the vector collecting the nodal displacement of the $e$-th finite element and $\mathbf{N}_e(\boldsymbol{x})$ is the chosen shape-function matrix.

A conforming displacement field $\mathbf{v}_h = \{\mathbf{v}_h^1, \mathbf{v}_h^2, \ldots, \mathbf{v}_h^{\mathscr{N}}\}$ satisfies the interelement continuity conditions and the homogeneous boundary conditions so that the rigid-body displacements are ruled out.

The displacement parameters $\mathbf{q}_e$ can be expressed in terms of nodal parameters $\mathbf{q}$ by means of the standard assembly operator $\mathscr{A}_e$ according to the parametric expression $\mathbf{q}_e = \mathscr{A}_e\mathbf{q}$. The interpolated counterpart $P_{2h}$ of the non-local total potential energy $P_2$ can be obtained by adding up the contributions of each non-assembly element and imposing the conforming requirement to the interpolating displacement to get:

$$P_{2h}(\mathbf{v}_h) = \Phi[\mathbf{B}(\mathbf{v}_h + \mathbf{w}_h)] - \langle \ell, \mathbf{v}_h \rangle,$$

with $\mathbf{v}_h \in \mathscr{L}_o$. Accordingly the total potential energy $P_{2h}$ can be rewritten as follows:

$$\begin{aligned} P_{2h}(\mathbf{v}_h) = &\frac{1}{2}\sum_{e=1}^{\mathscr{N}} \int_{\Omega_e} \mathbf{E}(\boldsymbol{x})\mathbf{B}(\mathbf{v}_h^e + \mathbf{w}_h^e)(\boldsymbol{x}) * \mathbf{B}(\mathbf{v}_h^e + \mathbf{w}_h^e)(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \\ &+ \frac{1}{2}\sum_{e=1}^{\mathscr{N}}\sum_{m=1}^{\mathscr{N}} \int_{\Omega_e}\int_{\Omega_m} \mathbf{Z}(\boldsymbol{x},\boldsymbol{z})\mathbf{B}(\mathbf{v}_h^m + \mathbf{w}_h^m)(\boldsymbol{z}) * \mathbf{B}(\mathbf{v}_h^e + \mathbf{w}_h^e)(\boldsymbol{x})\mathrm{d}\boldsymbol{z}\,\mathrm{d}\boldsymbol{x} \\ &- \sum_{e=1}^{\mathscr{N}} \int_{\Omega_e} \mathbf{b}(\boldsymbol{x}) * \mathbf{v}_h^e(\boldsymbol{x})\mathrm{d}\boldsymbol{x} - \sum_{e=1}^{\mathscr{N}} \int_{S_e} \mathbf{t}(\boldsymbol{x}) * \mathbf{v}_h^e(\boldsymbol{x})\mathrm{d}\boldsymbol{x}, \end{aligned}$$

where $S_e = \partial\Omega \cap \partial\Omega_e$ and $Z(\boldsymbol{x}, \boldsymbol{z})$ is reported in Table 1.

The matrix form of the discrete problem is obtained by imposing the stationarity of $P_{2h}$ with respect to $\mathbf{v}_h$ and is given by:

$$\sum\nolimits_{e=1}^{\mathscr{N}} \mathscr{A}_e^T \mathbf{K}_{ee} \mathscr{A}_e \mathbf{q} + \sum\nolimits_{e=1}^{\mathscr{N}} \sum\nolimits_{m=1}^{\mathscr{N}} \mathscr{A}_e^T \mathbf{K}_{em}^{NL} \mathscr{A}_m \mathbf{q} = \sum\nolimits_{e=1}^{\mathscr{N}} \mathscr{A}_e^T \mathbf{f}_e \tag{71}$$

in which the component submatrices and subvectors are defined in the form:

$$\mathbf{K}_{ee} = \mathbf{K}_{ee}^L + \mathbf{K}_{ee}^{NL}, \quad \mathbf{f}_e = \mathbf{f}_e^L + \mathbf{f}_e^{NL}.$$

**Table 4**
The functions $K_1$ and $\mathbf{K}_2$ for the considered non-local formulations

| | Non-local model for piecewise non-homogeneous bodies | One-component non-local model | Two-component non-local model |
|---|---|---|---|
| $K_1(\boldsymbol{x})$ | $-\alpha\frac{V(\boldsymbol{x})}{V_\infty}$ | $\alpha^2\frac{V^2(\boldsymbol{x})}{V_\infty^2} - 2\alpha\frac{V(\boldsymbol{x})}{V_\infty}$ | $\alpha^2\frac{V^2(\boldsymbol{x})}{V_\infty^2}$ |
| $\mathbf{K}_2(\boldsymbol{x},\boldsymbol{z})$ | $\frac{\alpha}{V_\infty}g(\boldsymbol{x},\boldsymbol{z})\mathbf{E}_m$ | $\frac{\alpha}{V_\infty}\mathbf{H}(\boldsymbol{x},\boldsymbol{z}) = \frac{\alpha}{V_\infty}[\mathbf{E}(\boldsymbol{x})+\mathbf{E}(\boldsymbol{z})]g(\boldsymbol{x},\boldsymbol{z}) + \frac{\alpha^2}{V_\infty^2}\mathbf{J}(\boldsymbol{x},\boldsymbol{z})$ | $\frac{\alpha^2}{V_\infty^2}\mathbf{J}(\boldsymbol{x},\boldsymbol{z})$ |

The stiffness matrices are:

$$\begin{aligned}
\mathbf{K}^L_{ee} &= \int_{\Omega_e}(\mathbf{BN}_e)^T(\boldsymbol{x})\mathbf{E}(\boldsymbol{x})(\mathbf{BN}_e)(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\\
\mathbf{K}^{NL}_{ee} &= \int_{\Omega_e}(\mathbf{BN}_e)^T(\boldsymbol{x})K_1(\boldsymbol{x})\mathbf{E}(\boldsymbol{x})(\mathbf{BN}_e)(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\\
\mathbf{K}^{NL}_{em} &= \int_{\Omega_e}\int_{\Omega_m}(\mathbf{BN}_e)^T(\boldsymbol{x})\mathbf{K}_2(\boldsymbol{x},\boldsymbol{z})(\mathbf{BN}_m)(\boldsymbol{z})\mathrm{d}\boldsymbol{z}\,\mathrm{d}\boldsymbol{x}
\end{aligned} \tag{72}$$

and the force vectors are:

$$\begin{aligned}
\mathbf{f}^L_e &= \int_{\Omega_e}\mathbf{N}^T_e(\boldsymbol{x})\mathbf{b}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} + \int_{S_e}\mathbf{N}^T_e(\boldsymbol{x})\mathbf{t}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} - \int_{\Omega_e}(\mathbf{BN}_e)^T(\boldsymbol{x})\mathbf{E}(\boldsymbol{x})(\mathbf{BN}_e)(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\mathbf{w}_e\\
\mathbf{f}^{NL}_e &= -\int_{\Omega_e}(\mathbf{BN}_e)^T(\boldsymbol{x})K_1(\boldsymbol{x})\mathbf{E}(\boldsymbol{x})(\mathbf{BN}_e)(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\mathbf{w}_e - \int_{\Omega_e}\int_{\Omega_m}(\mathbf{BN}_e)^T(\boldsymbol{x})\mathbf{K}_2(\boldsymbol{x},\boldsymbol{z})(\mathbf{BN}_m)(\boldsymbol{z})\mathrm{d}\boldsymbol{z}\,\mathrm{d}\boldsymbol{x}\mathbf{w}_e,
\end{aligned} \tag{73}$$

where $K_1$ and $\mathbf{K}_2$ are defined in Table 4.

The integrations appearing in (72) are performed elementwise so that $\mathbf{K}^L_{ee}$ turns out to be the standard stiffness matrix while $\mathbf{K}^{NL}_{ee}$ and $\mathbf{K}^{NL}_{em}$ given by $(72)_{2\text{-}3}$ turn out to be the non-local symmetric stiffness matrices reflecting the non-locality of the model. The band width of the matrix $\mathbf{K}^{NL}_{em}$ is larger than in the standard stiffness matrix since the elements of $\mathbf{K}^{NL}_{em}$ vanish if the related elements are too far with respect to the influence distance. The solving linear equation system follows from (71) and is given by:

$$\mathbf{Kq} = (\mathbf{K}^L + \mathbf{K}^{NL})\mathbf{q} = \mathbf{f},$$

where the global stiffness matrix $\mathbf{K}$ is symmetric and positive definite.

In the case of a local elastic behaviour, the non-local terms disappear and the solving equation system reduces to the standard local FEM given by $\mathbf{K}^L\mathbf{q} = \mathbf{f}^L$.

## 5. Examples

Let us consider an elastic bar in tension having a unit cross-section and a length $L$. It is clamped at the end $x = 0$ and is subjected to a given displacement $w$ at the other end $x = L$.

The characteristics of the one-dimensional bar are length $L = 100$ cm, elastic modulus $E_o = 21 \times 10^4$ MPa, internal length $l = 2$ cm, influence distance $R = 12$ cm and parameter $\alpha = -1$. The displacement $w = 0.2$ cm is imposed at the end $x = L$. The

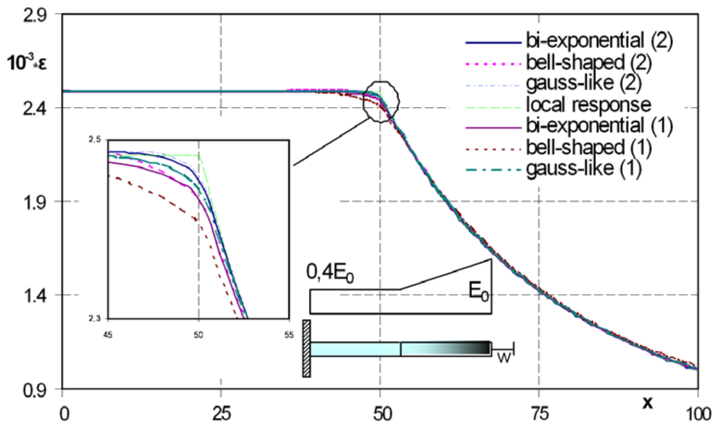


**Fig. 3.** Strain plots of a non-homogeneous bar with a piecewise continuous Young modulus having a constant value $E(x) = 0.4E_o$ for $0 \leqslant x \leqslant L/2$ and a linearly increasing Young modulus $E(x)$ within $0.4E_o$ and $E_o$ for $L/2 \leqslant x \leqslant L$ subjected to an imposed displacement at the end $x = L$. The symmetric function $W$ is expressed in terms of different attenuation functions $g$, the labels (1) and (2) are referred to the one-component and two-component non-local models, respectively.

parameter $\alpha$ in the two-component non-local model always appears as a square so that positive or negative values play the same role.

The stress $\bar{\sigma} = \sigma$ in the considered bar $\Omega = [0, L]$ is then constant for the equilibrium requirement and the problem is solved by means of the Fredholm integral equation. In fact the relations (30), (44) and (52) can be specialized to the present one-dimensional case in the following unique integral equation:

$$\sigma = [1 + K_1(x)]E(x)\varepsilon(x) + \int_\Omega K_2(x,z)\varepsilon(z)\mathrm{d}z \quad \Omega = [0, L], \tag{74}$$

where $K_1(x)$ and $K_2(x,z)$ are reported in Table 4.

The relation (74) can be transformed into a Fredholm integral equation of the second kind as shown in Appendix C and the strain $\varepsilon(x)$ can be obtained from the solution of Fredholm equation (80) according to the expression (83) of Appendix C.

In the case of a homogeneous material, the solution of the integral equation (74) yields $\varepsilon(x) = w/L = 2 * 10^{-3}$ and $\sigma = E_o w/L = 420$ MPa by $(78)_3$ (see Appendix C) since the solution of the Fredholm equation (77) is provided by $\theta(x) = 1$. Accordingly, the local values of strains and stresses for an homogeneous continuum are recovered independently of the internal length.

A non-homogeneous bar with a piecewise continuous Young modulus having a constant value $E(x) = 0.4E_o$ for $0 \leqslant x \leqslant L/2$ and a linearly increasing Young modulus within $0.4E_o$ and $E_o$ for $L/2 \leqslant x \leqslant L$ has been analyzed by means of the considered one-component and two-component non-local models. The strain response $\varepsilon$ is provided in Fig. 3. Three different attenuation functions $g$, given by the Gauss-like function (5), the bi-exponential function (6) and the bell-shaped polynomial function (7), are employed in the expression of the weight function $W$ of the type (3). For comparison the strain plot analytically derived in the case of a local behaviour is reported. The stress $\sigma$ evaluated for the local behaviour is 208.5 MPa. The one-component and two-component non-local models provide the following constant stresses (the subscripts 1 and 2 are referred to the one-component and two-component models, respectively): $\sigma_1 = 205.3$ MPa, $\sigma_2 = 208.6$ MPa using the bi-exponential function, $\sigma_1 = \sigma_2 = 208.6$ MPa using the bell-shaped polynomial function and $\sigma_1 = \sigma_2 = 208.6$ MPa using the Gauss-like function.

The close-up at the section $x = 50$ shows that the two-component model with the Gauss-like or bi-exponential functions provides similar results and the related curves are the closest to the knee of the local response. The curve associated with the one-component model with the bell-shaped polynomial function is the most distant one from the local response and it shows a higher slope on the left side of the middle section than the other curves. On the right side all the curves tend to coincide going away from the middle section.

In the case of a non-homogeneous bar with a piecewise Young modulus $E(x) = 0.1E_o$ for $0 \leqslant x \leqslant L/2$ and $E(x) = E_o$ for $L/2 \leqslant x \leqslant L$, the strain response $\varepsilon$ is reported in Fig. 4. For comparison purposes, the space weight function $W$ is defined in terms of the attenuation functions (5)–(7) and the one-component and two-component non-local models and the non-local model for piecewise non-homogeneous bodies are adopted. It is apparent the presence of the non-local response in a narrow layer around the middle section of the bar in which the strain $\varepsilon$ smoothly varies depending on the considered non-local models and the chosen attenuation function.

The comparison shows that the one-component model and the non-local model for piecewise non-homogeneous bodies provide the best fit of the constant strain. Moreover, the bell-shaped function presents a lower slope around the middle section of the bar than the one corresponding to the other two attenuation functions independent of the considered non-local model. The local stress value is $\sigma = 76.3$ MPa. The evaluated constant stresses pertaining to the non-local models vary from $\sigma = 75.2$ MPa using the bi-exponential function in the one-component non-local model to $\sigma = 79.1$ MPa using the bell-shaped function in the two-component non-local model.

It is worth noting that the maximum strain gap in the left part of the bar is provided by the constant strain ($\varepsilon = 3.765 * 10^{-3}$), obtained using the two-component model with the bell-shaped attenuation function, and the constant

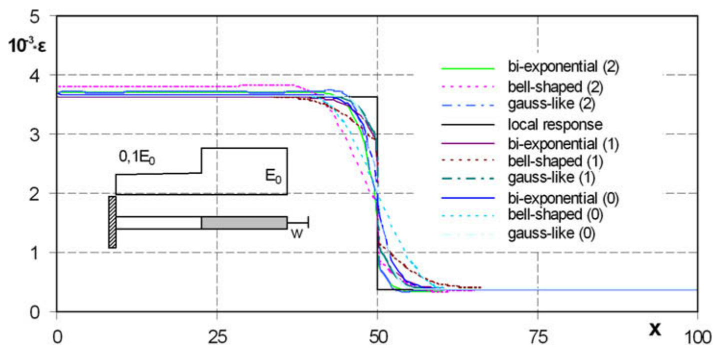


**Fig. 4.** Strain plots of a piecewise non-homogeneous bar in tension with $E(x) = 0.1E_o$ for $0 \leqslant x \leqslant L/2$ and $E(x) = E_o$ for $L/2 \leqslant x \leqslant L$ subjected to an imposed displacement at the end $x = L$. The symmetric function $W$ is expressed in terms of different attenuation functions $g$. The labels (0), (1) and (2) are referred to the non-local model for piecewise non-homogeneous bodies and to the one-component and two-component non-local models.

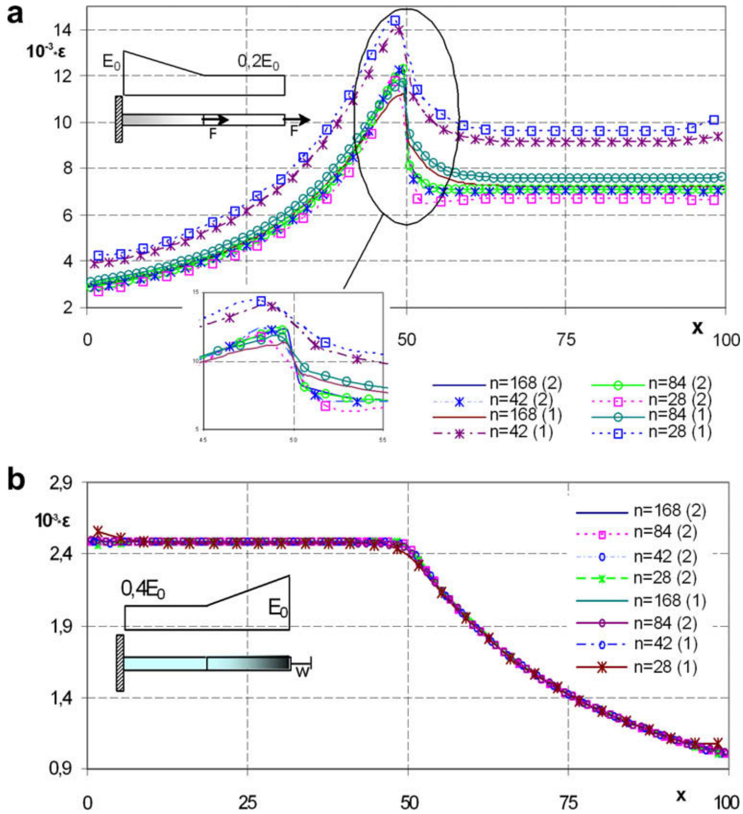


**Fig. 5.** Profile of the strain for different finite element discretizations, different load conditions and different non-homogeneous bars: (a) Applied forces $F = 30$ kN at $x = L/2$ and $x = L$ on a non-homogeneous piecewise continuous bar having a linearly decreasing elastic modulus $E(x)$ within $E_o$ and $0.2E_o$ for $0 \leqslant x \leqslant L/2$ and a constant value $E(x) = 0.2E_o$ for $L/2 \leqslant x \leqslant L$; (b) Imposed displacement $w = 0.2$ cm on a non-homogeneous piecewise bar having a constant elastic modulus $E(x) = 0.4E_o$ for $0 \leqslant x \leqslant L/2$ and a linearly increasing elastic modulus $E(x)$ within $0.4E_o$ and $E_o$ for $L/2 \leqslant x \leqslant L$.

local value of strain ($\varepsilon = 3.636 * 10^{-3}$). Such a gap is less than 4%. The gap between the strains obtained by using other attenuation functions or non-local models and the constant local strain is even less.

Further, the non-homogeneous elastic bar in tension is solved by adopting the proposed NFEM and the bi-exponential attenuation function. The problem is discretized with a different number of elements, all of equal size, namely $n = 28, n = 42, n = 84$ and $n = 168$.

The elastic bar in tension is loaded by two external forces of magnitude $F = 30$ kN applied at the sections $x = L/2$ and $x = L$. The strain plot is reported in Fig. 5(a). The non-homogeneous bar has a linearly decreasing Young modulus $E(x)$ within $E_o$ and $0.2E_o$ for $0 \leqslant x \leqslant L/2$ and a constant Young modulus $E(x) = 0.2E_o$ for $L/2 \leqslant x \leqslant L$. The curves related to the FE subdivision with $n = 28, n = 42$ and $n = 84$ are constructed from straight segments connecting individual points that are quite far apart since they are evaluated using coarse meshes in order to show the effectiveness of the models. The picture shows that the strain plots tend to the one obtained by the FE solution with a refined mesh of $n = 168$ elements for both the one- and two-component models. It is worth noting that the strains associated with the FE two-component model tends to the strain curve obtained with $n = 168$, for increasing $n$, from below, i.e. at a fixed point $x$ strains grow up if the mesh is refined so that the model tends to relax. On the contrary, the strain curves provided by the FE one-component model show the opposite behaviour with increasing $n$, i.e. at a fixed point $x$ strains decrease if the mesh is refined so that the model tends to stiff.

The strain plot in Fig. 5(b) is referred to the elastic bar in tension loaded by the displacement $w = 0.2$ cm at the end section $x = L$. The non-homogeneous bar has a constant Young modulus $E(x) = 0.4E_o$ for $0 \leqslant x \leqslant L/2$ and a linearly increasing Young modulus $E(x)$ within $0.4E_o$ and $E_o$ for $L/2 \leqslant x \leqslant L$. The stress assumes the values 222.6 MPa ($n$ = 28), 211.5 MPa ($n$ = 42), 208.7 MPa ($n$ = 84) and 208.6 MPa ($n$ = 168).

On comparing the strain results, no mesh dependence or boundary effects are pointed out by the considered non-local model. The strain plot reported in Fig. 5(a) shows that a minor approximation is exhibited by the one-component non-local model for the coarser meshes $n$ = 42 and $n$ = 28.

The plots of the two-dimensional function $f_1$ and of the one-dimensional function $F_1$, see (33), pertaining to the one-component non-local model is reported in Fig. 6 with reference to the bar analyzed in Fig. 5(a).

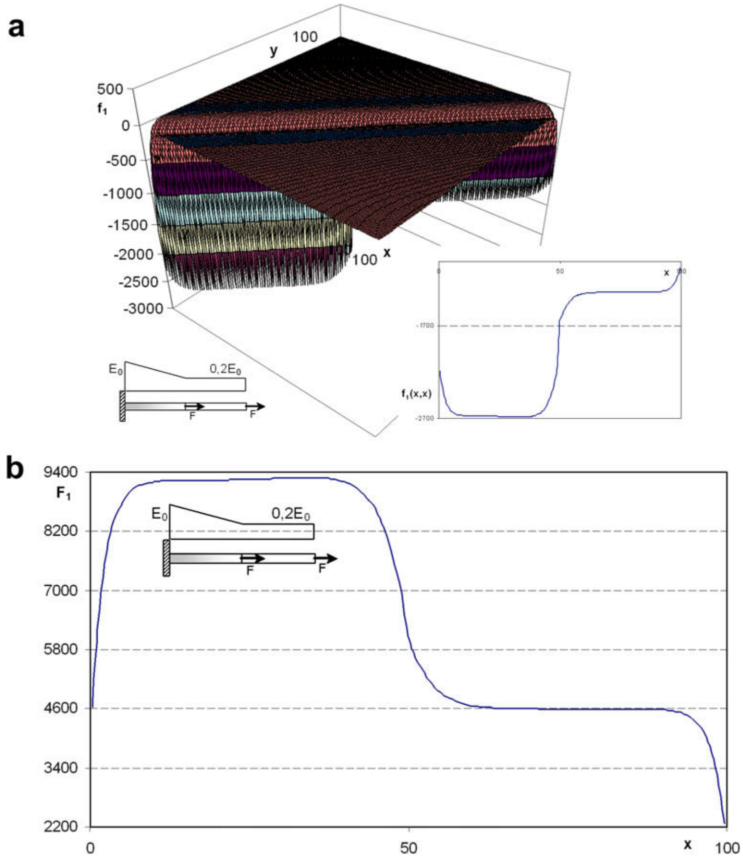


**Fig. 6.** Plots of the functions $f_1$ and $F_1$ providing the non-locality residual $P$ for the one-component non-local model related to the non-homogeneous bar analyzed in Fig. 5(a): (a) the two-dimensional function $f_1$; (b) the one-dimensional function $F_1$.

Similarly, the functions $f_2$ and $F_2$, see (48), related to the non-locality residual (47) are reported in Fig. 7(a) and (b) with reference to the two-component non-local model. In the second half of the bar beyond the boundary layer of the middle section, the strain is constant so that the functions $f_2$ and $F_2$ identically vanish according to the relations (48).

The non-locality residual $P$, associated with the one-component and two-component non-local models, is evaluated in Fig. 7(c) for the non-homogeneous bar analyzed in Fig. 5(a) by considering a uniform strain rate $\dot{\varepsilon}(x) = 0.004$. According to the results provided in Table 2, the residual $P$ in the first half of the bar is non-vanishing. On the contrary, beyond the boundary layer of the middle section, the non-locality residuals $P$ are vanishing being the strain $\varepsilon$ uniform.

Figs. 6 and 7 show that the non-locality residual component functions $f_1$ and $F_1$, pertaining to the one-component non-local model, and the non-locality residual functions $f_2$ and $F_2$, related to the two-component non-local model, appears to be quite different. Nevertheless the non-locality residual functions $P$ pertaining to the two models have a similar shape so that the energy exchanges between neighbour particles occur in a similar manner. Accordingly the considered examples show that the macroscopic behaviour of the non-local and non-homogeneous models turns out to be similar.

A non-homogeneous bar with a constant Young modulus $E(x) = 0.4E_o$ for $0 \leqslant x \leqslant L/2$ and a linearly increasing Young modulus $E(x)$ within $0.4E_o$ and $E_o$ for $L/2 \leqslant x \leqslant L$ is considered. The bar is subjected to a constant strain $\varepsilon = 0.001$ and the related stress plot is reported in Fig. 8. According to the results reported in Table 2, the non-local stress associated with the one-component non-local model coincides to the local one where the elastic modulus $E(x)$ is constant. Approaching the boundary layer of the middle section of the bar, the non-local stress does not coincide to the local one due to the non-homogeneity of the bar.

## 6. Closure

A contribution in the framework of non-local constitutive models for non-homogeneous elastic materials is addressed. Different stress–strain laws follow from suitable definitions of the free energy in terms of local and non-local strains.

A thermodynamic analysis is developed in order to consistently derive the non-local model. The two-component model satisfies the condition that the elastic energy and the stress coincide to their local counterparts whenever the strain field is

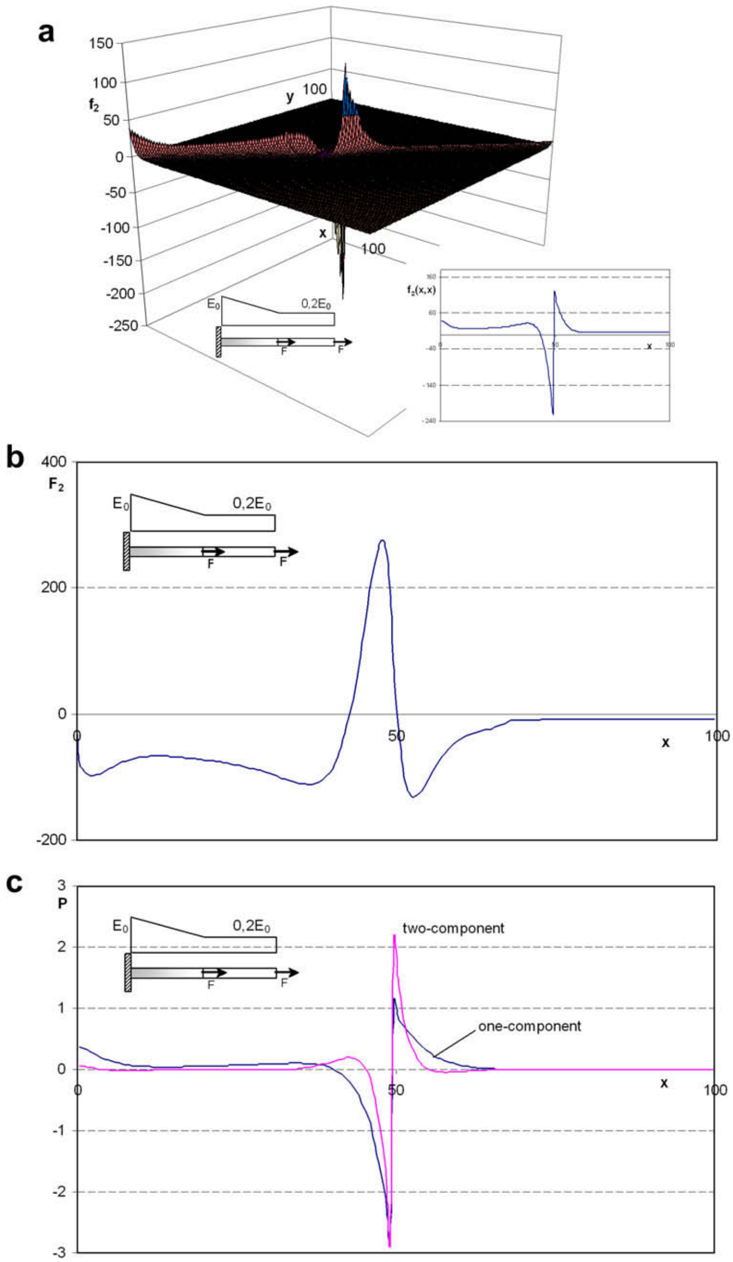


**Fig. 7.** Plots of the functions $f_2$ and $F_2$ providing the non-locality residual $P$ for the two-component non-local model related to the non-homogeneous bar analyzed in Fig. 5(a): (a) the two-dimensional function $f_2$; (b) the one-dimensional function $F_2$; (c) the non-locality residual $P$ associated with the one-component and two-component non-local models is evaluated for a uniform strain rate $\dot{\varepsilon}(x) = 0.004$.

uniform in the body. The elastic energy and the stress pertaining to the one-component model and to the non-local model for piecewise non-homogeneous bodies do not reduce to the local one under uniform strain fields in order to account for non-locality effects due to the non-homogeneity of the material.

The complete family of variational principles with different combinations of the state variables is provided in a unified framework. Uniqueness of the solution of the non-local elastic problem is also discussed.

The extension to the non-local elasticity of total potential energy, complementary energy, mixed Hu-Washizu and Hellinger-Reissner principles of classical (local) elasticity are provided.

A non-homogeneous bar under different load conditions is addressed. The numerical solution is obtained by the recourse to the Fredholm integral equation and the NFEM showing no pathological behaviours such as mesh dependence, numerical instability or boundary effects. Extensions of the present model to materials with a non-homogeneous internal length and with voids or holes are of practical interest and are the subject of ongoing researches.

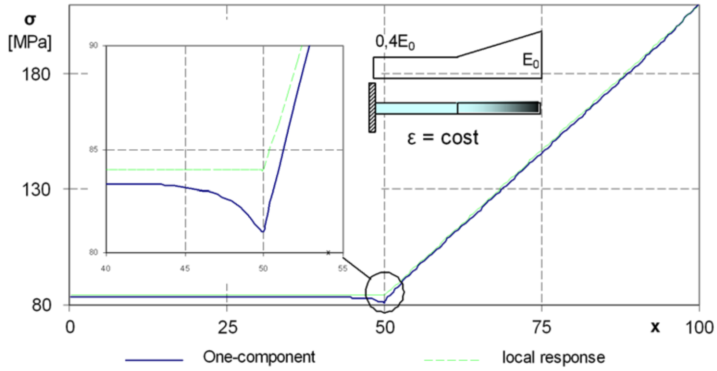


**Fig. 8.** Plot of stress for a non-homogeneous bar with a constant Young modulus $E(x) = 0.4E_o$ for $0 \leqslant x \leqslant L/2$ and a linearly increasing Young modulus $E(x)$ within $0.4E_o$ and $E_o$ for $L/2 \leqslant x \leqslant L$ subjected to a constant strain $\varepsilon = 0.001$.

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## Appendix A. One-component non-local model

- The integrals appearing in (25) can be transformed in the following form:

$$\begin{aligned}\int_\Omega\int_\Omega W(\xi,\boldsymbol{x})\mathbf{E}(\xi)W(\xi,\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z}\,\mathrm{d}\xi &= \int_\Omega\left[\left(1-\alpha\frac{V(\xi)}{V_\infty}\right)\delta(\xi,\boldsymbol{x})+\frac{\alpha}{V_\infty}g(\xi,\boldsymbol{x})\right]\mathbf{E}(\xi)\\ &\times\left\{\left[1-\alpha\frac{V(\xi)}{V_\infty}\right]\boldsymbol{\varepsilon}(\xi)+\frac{\alpha}{V_\infty}\int_\Omega g(\xi,\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z}\right\}\mathrm{d}\xi = \left[1-\alpha\frac{V(\boldsymbol{x})}{V_\infty}\right]^2\mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x})+\left[1-\alpha\frac{V(\boldsymbol{x})}{V_\infty}\right]\frac{\alpha}{V_\infty}\mathbf{E}(\boldsymbol{x})\int_\Omega g(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z}\\ &+\frac{\alpha}{V_\infty}\int_\Omega\left[1-\alpha\frac{V(\xi)}{V_\infty}\right]g(\xi,\boldsymbol{x})\mathbf{E}(\xi)\boldsymbol{\varepsilon}(\xi)\mathrm{d}\xi+\frac{\alpha^2}{V_\infty^2}\int_\Omega\int_\Omega g(\xi,\boldsymbol{x})\mathbf{E}(\xi)g(\xi,\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z}\,\mathrm{d}\xi.\end{aligned} \tag{75}$$

Using the definitions (26), the integrals in (75) become:

$$\int_\Omega\int_\Omega W(\xi,\boldsymbol{x})\mathbf{E}(\xi)W(\xi,\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z}\,\mathrm{d}\xi = \left[1-\alpha\frac{V(\boldsymbol{x})}{V_\infty}\right]^2\mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x})+\frac{\alpha}{V_\infty}\int_\Omega\boldsymbol{\Lambda}_1(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z}+\frac{\alpha^2}{V_\infty^2}\int_\Omega\boldsymbol{\Lambda}_2(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z}.$$

- The residual $P$ at the point $\boldsymbol{x}$ is given by $(23)_1$. Noting that:

$$(\mathbf{RER}\dot{\boldsymbol{\varepsilon}})(\boldsymbol{x}) = \int_\Omega\int_\Omega W(\xi,\boldsymbol{x})\mathbf{E}(\xi)W(\xi,\boldsymbol{z})\dot{\boldsymbol{\varepsilon}}(\boldsymbol{z})\mathrm{d}\boldsymbol{z}\,\mathrm{d}\xi,$$

and recalling the expression (75), it results:

$$\begin{aligned}P(\boldsymbol{x}) &= \frac{1}{2}\boldsymbol{\varepsilon}(\boldsymbol{x})*\left[1-\alpha\frac{V(\boldsymbol{x})}{V_\infty}\right]\frac{\alpha}{V_\infty}\mathbf{E}(\boldsymbol{x})\int_\Omega g(\boldsymbol{x},\boldsymbol{z})\dot{\boldsymbol{\varepsilon}}(\boldsymbol{z})\mathrm{d}\boldsymbol{z}+\frac{1}{2}\boldsymbol{\varepsilon}(\boldsymbol{x})*\frac{\alpha}{V_\infty}\int_\Omega\left[1-\alpha\frac{V(\boldsymbol{z})}{V_\infty}\right]g(\boldsymbol{z},\boldsymbol{x})\mathbf{E}(\boldsymbol{z})\dot{\boldsymbol{\varepsilon}}(\boldsymbol{z})\mathrm{d}\boldsymbol{z}\\ &+\frac{1}{2}\boldsymbol{\varepsilon}(\boldsymbol{x})*\frac{\alpha^2}{V_\infty^2}\int_\Omega\int_\Omega g(\xi,\boldsymbol{x})\mathbf{E}(\xi)g(\xi,\boldsymbol{z})\dot{\boldsymbol{\varepsilon}}(\boldsymbol{z})\mathrm{d}\xi\,\mathrm{d}\boldsymbol{z}-\frac{1}{2}\dot{\boldsymbol{\varepsilon}}(\boldsymbol{x})*\left[1-\alpha\frac{V(\boldsymbol{x})}{V_\infty}\right]\frac{\alpha}{V_\infty}\mathbf{E}(\boldsymbol{x})\int_\Omega g(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z}\\ &-\frac{1}{2}\dot{\boldsymbol{\varepsilon}}(\boldsymbol{x})*\frac{\alpha}{V_\infty}\int_\Omega\left[1-\alpha\frac{V(\boldsymbol{z})}{V_\infty}\right]g(\boldsymbol{x},\boldsymbol{z})\mathbf{E}(\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z}-\frac{1}{2}\dot{\boldsymbol{\varepsilon}}(\boldsymbol{x})*\frac{\alpha^2}{V_\infty^2}\int_\Omega\int_\Omega g(\xi,\boldsymbol{x})\mathbf{E}(\xi)g(\xi,\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\xi\,\mathrm{d}\boldsymbol{z}.\end{aligned}$$

Accordingly, recalling the definitions (26), it follows:

$$\begin{aligned}P(\boldsymbol{x}) &= \frac{1}{2}\boldsymbol{\varepsilon}(\boldsymbol{x})*\frac{\alpha}{V_\infty}\int_\Omega\boldsymbol{\Lambda}_1(\boldsymbol{x},\boldsymbol{z})\dot{\boldsymbol{\varepsilon}}(\boldsymbol{z})\mathrm{d}\boldsymbol{z}+\frac{1}{2}\boldsymbol{\varepsilon}(\boldsymbol{x})*\frac{\alpha^2}{V_\infty^2}\int_\Omega\boldsymbol{\Lambda}_2(\boldsymbol{x},\boldsymbol{z})\dot{\boldsymbol{\varepsilon}}(\boldsymbol{z})\mathrm{d}\boldsymbol{z}\\ &-\frac{1}{2}\dot{\boldsymbol{\varepsilon}}(\boldsymbol{x})*\frac{\alpha}{V_\infty}\int_\Omega\boldsymbol{\Lambda}_1(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z}-\frac{1}{2}\dot{\boldsymbol{\varepsilon}}(\boldsymbol{x})*\frac{\alpha^2}{V_\infty^2}\int_\Omega\boldsymbol{\Lambda}_2(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z}.\end{aligned}$$

## Appendix B. Two-component non-local model

- The elastic energy appearing in the relation (16) can be given in the form:

$$\begin{aligned}\phi(\boldsymbol{\varepsilon}(\boldsymbol{x})) &= \frac{1}{2}\mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x}) * \boldsymbol{\varepsilon}(\boldsymbol{x}) + \frac{1}{2}\mathbf{E}(\boldsymbol{x})[(\mathbf{R}-\mathbf{I})\boldsymbol{\varepsilon}](\boldsymbol{x}) * [(\mathbf{R}-\mathbf{I})\boldsymbol{\varepsilon}](\boldsymbol{x}) \\ &= \mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x}) * \boldsymbol{\varepsilon}(\boldsymbol{x}) + \frac{1}{2}\mathbf{E}(\boldsymbol{x})\int_\Omega W(\boldsymbol{x},\xi)\boldsymbol{\varepsilon}(\xi)\mathrm{d}\xi * \int_\Omega W(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z} - \mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x}) * \int_\Omega W(\boldsymbol{x},\xi)\boldsymbol{\varepsilon}(\xi)\mathrm{d}\xi.\end{aligned}$$

Recalling the expression of the space weight function $W$, it results:

$$\begin{aligned}\phi(\boldsymbol{\varepsilon}(\boldsymbol{x})) &= \mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x}) * \boldsymbol{\varepsilon}(\boldsymbol{x}) + \frac{1}{2}\left[1-\alpha\frac{V(\boldsymbol{x})}{V_\infty}\right]^2\mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x}) * \boldsymbol{\varepsilon}(\boldsymbol{x}) \\ &\quad + \left[1-\alpha\frac{V(\boldsymbol{x})}{V_\infty}\right]\frac{\alpha}{V_\infty}\mathbf{E}(\boldsymbol{x})\int_\Omega g(\boldsymbol{x},\xi)\boldsymbol{\varepsilon}(\xi)\mathrm{d}\xi * \boldsymbol{\varepsilon}(\boldsymbol{x}) \\ &\quad + \frac{\alpha^2}{2V_\infty^2}\mathbf{E}(\boldsymbol{x})\int_\Omega g(\boldsymbol{x},\xi)\boldsymbol{\varepsilon}(\xi)\mathrm{d}\xi * \int_\Omega g(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z} \\ &\quad - \mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x}) * \left[1-\alpha\frac{V(\boldsymbol{x})}{V_\infty}\right]\boldsymbol{\varepsilon}(\boldsymbol{x}) - \mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x}) * \frac{\alpha}{V_\infty}\int_\Omega g(\boldsymbol{x},\xi)\boldsymbol{\varepsilon}(\xi)\mathrm{d}\xi\end{aligned}$$

and rearranging the terms it turns out to be:

$$\phi(\boldsymbol{\varepsilon}(\boldsymbol{x})) = \frac{1}{2}\left[1+\alpha^2\frac{V^2(\boldsymbol{x})}{V_\infty^2}\right]\mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x}) * \boldsymbol{\varepsilon}(\boldsymbol{x}) - \alpha^2\frac{V(\boldsymbol{x})}{V_\infty^2}\int_\Omega g(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z} * \mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x}) + \frac{\alpha^2}{2V_\infty^2}\mathbf{E}(\boldsymbol{x})\int_\Omega\int_\Omega g(\boldsymbol{x},\xi)g(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\varepsilon}(\xi) * \boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\xi\mathrm{d}\boldsymbol{z}.$$

- Let us evaluate the non-local stress (43). Recalling the expressions (1) and (3) of the regularization operator **R** and of the weight function $W$, together with the expression of the operator **A**, the non-local stress (43) turns out to be:

$$\begin{aligned}\bar{\boldsymbol{\sigma}}(\boldsymbol{x}) &= (\mathbf{RER}\boldsymbol{\varepsilon})(\boldsymbol{x}) - (\mathbf{RE}\boldsymbol{\varepsilon})(\boldsymbol{x}) - \mathbf{E}(\boldsymbol{x})(\mathbf{R}\boldsymbol{\varepsilon})(\boldsymbol{x}) + 2\mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x}) \\ &= \int_\Omega\int_\Omega W(\xi,\boldsymbol{x})\mathbf{E}(\xi)W(\xi,\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z}\,\mathrm{d}\xi - \int_\Omega W(\boldsymbol{x},\xi)\mathbf{E}(\xi)\boldsymbol{\varepsilon}(\xi)\mathrm{d}\xi - \mathbf{E}(\boldsymbol{x})\int_\Omega W(\boldsymbol{x},\xi)\boldsymbol{\varepsilon}(\xi)d\xi + 2\mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x}).\end{aligned} \tag{76}$$

The first integral in (76) has been evaluated in terms of the attenuation function $g$ in (75). The second integral in (76) becomes:

$$\int_\Omega W(\boldsymbol{x},\xi)\mathbf{E}(\xi)\boldsymbol{\varepsilon}(\xi)\mathrm{d}\xi = \left[1-\alpha\frac{V(\boldsymbol{x})}{V_\infty}\right]\mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x}) + \frac{\alpha}{V_\infty}\int_\Omega g(\boldsymbol{x},\xi)\mathbf{E}(\xi)\boldsymbol{\varepsilon}(\xi)\mathrm{d}\xi$$

and the third integral in (76) turns out to be:

$$\int_\Omega W(\boldsymbol{x},\xi)\boldsymbol{\varepsilon}(\xi)\mathrm{d}\xi = \left[1-\alpha\frac{V(\boldsymbol{x})}{V_\infty}\right]\boldsymbol{\varepsilon}(\boldsymbol{x}) + \frac{\alpha}{V_\infty}\int_\Omega g(\boldsymbol{x},\xi)\boldsymbol{\varepsilon}(\xi)\mathrm{d}\xi.$$

Hence it results:

$$\begin{aligned}\bar{\boldsymbol{\sigma}}(\boldsymbol{x}) &= \left[1-\alpha\frac{V(\boldsymbol{x})}{V_\infty}\right]^2\mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x}) + \left[1-\alpha\frac{V(\boldsymbol{x})}{V_\infty}\right]\frac{\alpha}{V_\infty}\mathbf{E}(\boldsymbol{x})\int_\Omega g(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z} + \frac{\alpha}{V_\infty}\int_\Omega\left[1-\alpha\frac{V(\xi)}{V_\infty}\right]g(\xi,\boldsymbol{x})\mathbf{E}(\xi)\boldsymbol{\varepsilon}(\xi)\mathrm{d}\xi \\ &\quad + \frac{\alpha^2}{V_\infty^2}\int_\Omega\int_\Omega g(\xi,\boldsymbol{x})\mathbf{E}(\xi)g(\xi,\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z}\,\mathrm{d}\xi - \left[1-\alpha\frac{V(\boldsymbol{x})}{V_\infty}\right]\mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x}) - \frac{\alpha}{V_\infty}\int_\Omega g(\boldsymbol{x},\xi)\mathbf{E}(\xi)\boldsymbol{\varepsilon}(\xi)\mathrm{d}\xi \\ &\quad - \left[1-\alpha\frac{V(\boldsymbol{x})}{V_\infty}\right]\mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x}) - \frac{\alpha}{V_\infty}\mathbf{E}(\boldsymbol{x})\int_\Omega g(\boldsymbol{x},\xi)\boldsymbol{\varepsilon}(\xi)\mathrm{d}\xi.\end{aligned}$$

- The non-local residual function $P$ is reported in $(23)_2$ for the two-component non-local model and can be rewritten in the form:

$$P(\boldsymbol{x}) = (\mathbf{EA}\boldsymbol{\varepsilon})(\boldsymbol{x}) * (\mathbf{A}\dot{\boldsymbol{\varepsilon}})(\boldsymbol{x}) - (\mathbf{REA}\boldsymbol{\varepsilon})(\boldsymbol{x}) * \dot{\boldsymbol{\varepsilon}}(\boldsymbol{x}) + (\mathbf{EA}\boldsymbol{\varepsilon})(\boldsymbol{x}) * \dot{\boldsymbol{\varepsilon}}(\boldsymbol{x})$$

so that it follows:

$$\begin{aligned}P(\boldsymbol{x}) &= \mathbf{E}(\boldsymbol{x})[(\mathbf{R}\boldsymbol{\varepsilon})(\boldsymbol{x}) - \boldsymbol{\varepsilon}(\boldsymbol{x})] * [(\mathbf{R}\dot{\boldsymbol{\varepsilon}})(\boldsymbol{x}) - \dot{\boldsymbol{\varepsilon}}(\boldsymbol{x})] - [(\mathbf{RER}\boldsymbol{\varepsilon})(\boldsymbol{x}) - (\mathbf{RE}\boldsymbol{\varepsilon})(\boldsymbol{x})] * \dot{\boldsymbol{\varepsilon}}(\boldsymbol{x}) + \mathbf{E}(\boldsymbol{x})[(\mathbf{R}\boldsymbol{\varepsilon})(\boldsymbol{x}) - \boldsymbol{\varepsilon}(\boldsymbol{x})] * \dot{\boldsymbol{\varepsilon}}(\boldsymbol{x}) \\ &= \mathbf{E}(\boldsymbol{x})(\mathbf{R}\boldsymbol{\varepsilon})(\boldsymbol{x}) * (\mathbf{R}\dot{\boldsymbol{\varepsilon}})(\boldsymbol{x}) - \mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x}) * (\mathbf{R}\dot{\boldsymbol{\varepsilon}})(\boldsymbol{x}) - (\mathbf{RER}\boldsymbol{\varepsilon})(\boldsymbol{x}) * \dot{\boldsymbol{\varepsilon}}(\boldsymbol{x}) + (\mathbf{RE}\boldsymbol{\varepsilon})(\boldsymbol{x}) * \dot{\boldsymbol{\varepsilon}}(\boldsymbol{x}).\end{aligned}$$

Recalling the expression of the space weight function $W$, it results:

$$
\begin{aligned}
P(\boldsymbol{x}) = & \left[1-\alpha\frac{V(\boldsymbol{x})}{V_\infty}\right]\mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x}) * \left[1-\alpha\frac{V(\boldsymbol{x})}{V_\infty}\right]\dot{\boldsymbol{\varepsilon}}(\boldsymbol{x}) + \frac{\alpha}{V_\infty}\mathbf{E}(\boldsymbol{x})\int_\Omega g(\boldsymbol{x},\xi)\boldsymbol{\varepsilon}(\xi)\mathrm{d}\xi * \left[1-\alpha\frac{V(\boldsymbol{x})}{V_\infty}\right]\dot{\boldsymbol{\varepsilon}}(\boldsymbol{x}) \\
& + \left[1-\alpha\frac{V(\boldsymbol{x})}{V_\infty}\right]\mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x}) * \frac{\alpha}{V_\infty}\int_\Omega g(\boldsymbol{x},\xi)\dot{\boldsymbol{\varepsilon}}(\xi)\mathrm{d}\xi + \frac{\alpha}{V_\infty}\mathbf{E}(\boldsymbol{x})\int_\Omega g(\boldsymbol{x},\xi)\boldsymbol{\varepsilon}(\xi)\mathrm{d}\xi * \frac{\alpha}{V_\infty}\int_\Omega g(\boldsymbol{x},\xi)\dot{\boldsymbol{\varepsilon}}(\xi)\mathrm{d}\xi \\
& - \mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x}) * \left\{\left[1-\alpha\frac{V(\boldsymbol{x})}{V_\infty}\right]\dot{\boldsymbol{\varepsilon}}(\boldsymbol{x}) + \frac{\alpha}{V_\infty}\int_\Omega g(\boldsymbol{x},\xi)\dot{\boldsymbol{\varepsilon}}(\xi)\mathrm{d}\xi\right\} - \left[1-\alpha\frac{V(\boldsymbol{x})}{V_\infty}\right]^2\mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x}) * \dot{\boldsymbol{\varepsilon}}(\boldsymbol{x}) \\
& - \left[1-\alpha\frac{V(\boldsymbol{x})}{V_\infty}\right]\frac{\alpha}{V_\infty}\mathbf{E}(\boldsymbol{x})\int_\Omega g(\boldsymbol{x},\xi)\boldsymbol{\varepsilon}(\xi)\mathrm{d}\xi * \dot{\boldsymbol{\varepsilon}}(\boldsymbol{x}) - \frac{\alpha}{V_\infty}\int_\Omega\left[1-\alpha\frac{V(\xi)}{V_\infty}\right]g(\boldsymbol{x},\xi)\mathbf{E}(\xi)\boldsymbol{\varepsilon}(\xi)\mathrm{d}\xi * \dot{\boldsymbol{\varepsilon}}(\boldsymbol{x}) \\
& - \frac{\alpha^2}{V_\infty^2}\int_\Omega\int_\Omega g(\xi,\boldsymbol{x})\mathbf{E}(\xi)g(\xi,\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z}\,\mathrm{d}\xi * \dot{\boldsymbol{\varepsilon}}(\boldsymbol{x}) + \left[1-\alpha\frac{V(\boldsymbol{x})}{V_\infty}\right]\mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x}) * \dot{\boldsymbol{\varepsilon}}(\boldsymbol{x}) + \frac{\alpha}{V_\infty}\int_\Omega g(\boldsymbol{x},\xi)\mathbf{E}(\xi)\boldsymbol{\varepsilon}(\xi)\mathrm{d}\xi * \dot{\boldsymbol{\varepsilon}}(\boldsymbol{x}).
\end{aligned}
$$

As a consequence it turns out to be:

$$
\begin{aligned}
P(\boldsymbol{x}) = & \left[1-\alpha\frac{V(\boldsymbol{x})}{V_\infty}\right]\frac{\alpha}{V_\infty}\mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x}) * \int_\Omega g(\boldsymbol{x},\boldsymbol{z})\dot{\boldsymbol{\varepsilon}}(\boldsymbol{z})\mathrm{d}\boldsymbol{z} + \left[1-\alpha\frac{V(\boldsymbol{x})}{V_\infty}\right]\frac{\alpha}{V_\infty}\int_\Omega g(\boldsymbol{x},\xi)\boldsymbol{\varepsilon}(\xi)\mathrm{d}\xi * \mathbf{E}(\boldsymbol{x})\dot{\boldsymbol{\varepsilon}}(\boldsymbol{x}) \\
& + \frac{\alpha^2}{V_\infty^2}\mathbf{E}(\boldsymbol{x})\int_\Omega g(\boldsymbol{x},\xi)\boldsymbol{\varepsilon}(\xi)\mathrm{d}\xi * \int_\Omega g(\boldsymbol{x},\boldsymbol{z})\dot{\boldsymbol{\varepsilon}}(\boldsymbol{z})\mathrm{d}\boldsymbol{z} - \mathbf{E}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x}) * \frac{\alpha}{V_\infty}\int_\Omega g(\boldsymbol{x},\boldsymbol{z})\dot{\boldsymbol{\varepsilon}}(\boldsymbol{z})\mathrm{d}\boldsymbol{z} \\
& - \left[1-\alpha\frac{V(\boldsymbol{x})}{V_\infty}\right]\frac{\alpha}{V_\infty}\int_\Omega g(\boldsymbol{x},\xi)\boldsymbol{\varepsilon}(\xi)\mathrm{d}\xi * \mathbf{E}(\boldsymbol{x})\dot{\boldsymbol{\varepsilon}}(\boldsymbol{x}) - \frac{\alpha}{V_\infty}\int_\Omega\left[1-\alpha\frac{V(\xi)}{V_\infty}\right]g(\xi,\boldsymbol{x})\mathbf{E}(\xi)\boldsymbol{\varepsilon}(\xi)\mathrm{d}\xi * \dot{\boldsymbol{\varepsilon}}(\boldsymbol{x}) \\
& - \frac{\alpha^2}{V_\infty^2}\int_\Omega\int_\Omega g(\xi,\boldsymbol{x})\mathbf{E}(\xi)g(\xi,\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z}\,\mathrm{d}\xi * \dot{\boldsymbol{\varepsilon}}(\boldsymbol{x}) + \frac{\alpha}{V_\infty}\int_\Omega g(\boldsymbol{x},\xi)\mathbf{E}(\xi)\boldsymbol{\varepsilon}(\xi)\mathrm{d}\xi * \dot{\boldsymbol{\varepsilon}}(\boldsymbol{x}).
\end{aligned}
$$

Finally it results:

$$
\begin{aligned}
P(\boldsymbol{x}) = & -\alpha^2\frac{V(\boldsymbol{x})}{V_\infty^2}\mathbf{E}(\boldsymbol{x})\int_\Omega g(\boldsymbol{x},\xi)\dot{\boldsymbol{\varepsilon}}(\xi)\mathrm{d}\xi * \boldsymbol{\varepsilon}(\boldsymbol{x}) + \frac{\alpha^2}{V_\infty^2}\mathbf{E}(\boldsymbol{x})\int_\Omega\int_\Omega g(\boldsymbol{x},\xi)g(\boldsymbol{x},\boldsymbol{z})\dot{\boldsymbol{\varepsilon}}(\xi) * \boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\xi\,\mathrm{d}\boldsymbol{z} \\
& + \frac{\alpha^2}{V_\infty^2}\int_\Omega V(\xi)g(\xi,\boldsymbol{x})\mathbf{E}(\xi)\boldsymbol{\varepsilon}(\xi)\mathrm{d}\xi * \dot{\boldsymbol{\varepsilon}}(\boldsymbol{x}) - \frac{\alpha^2}{V_\infty^2}\int_\Omega\int_\Omega g(\xi,\boldsymbol{x})\mathbf{E}(\xi)g(\xi,\boldsymbol{z})\boldsymbol{\varepsilon}(\boldsymbol{z})\mathrm{d}\boldsymbol{z}\,\mathrm{d}\xi * \dot{\boldsymbol{\varepsilon}}(\boldsymbol{x}).
\end{aligned}
$$

## Appendix C. Fredholm integral equation

The relation (74) can be transformed into a Fredholm integral equation since it can be rewritten in the form:

$$
A(x)\theta(x) + \int_\Omega B(x,z)\theta(z)\mathrm{d}z = 1, \tag{77}
$$

where

$$
A(x) = [1+K_1(x)]\frac{E(x)}{E_o} \quad B(x,z) = \frac{K_2(x,z)}{E_o} \quad \theta(x) = \frac{E_o}{\sigma}\varepsilon(x). \tag{78}
$$

Setting:

$$
a(x) = \frac{1}{\sqrt{A(x)}} \quad B^*(x,z) = \frac{B(x,z)}{\sqrt{A(x)A(z)}} \quad \theta^*(x) = \theta(x)\sqrt{A(x)} \tag{79}
$$

the relation (77) becomes:

$$
\theta^*(x) + \int_\Omega B^*(x,z)\theta^*(z)\mathrm{d}z = a(x) \tag{80}
$$

which is a Fredholm integral equation of the second kind for the unknown function $\theta^*$ (Tricomi, 1985). The kernel $B^*(x,z)$ is symmetric and, if $a(x) = 0$, the integral equation is said to be homogeneous. The functions $\psi_i$ fulfilling the homogeneous equation

$$
\psi_i(x) = \mu_i\int_\Omega B^*(x,z)\psi_i(z)\mathrm{d}z \tag{81}
$$

are called the eigenfunctions and the scalars $\mu_i$ are the eigenvalues of $B^*$ in $\Omega$. The solution of the Fredholm integral equation of the second kind (80) is then given by:

$$
\theta^*(x) = a(x) - \sum_{i=0}^{\infty}\frac{\int_\Omega a(z)\psi_i(z)\mathrm{d}z}{\mu_i+1}\psi_i(z)
$$

if the term $(-1)$ is not an eigenvalue.

On the contrary, if $\mu_e = -1$ is an eigenvalue, the solution of the Fredholm equation (80) is provided in the form:

$$\theta^*(x) = a(x) + c\psi_e(x) + \mu_e \sum_{i=0(i\neq e)}^{\infty} \frac{\int_\Omega a(z)\psi_i(z)\mathrm{d}z}{\mu_i - \mu_e}\psi_i(z),$$

where $c$ is an undetermined constant, $\psi_e$ is the eigenfunction corresponding to $\mu_e$ and the function $a$ has to be orthogonal to the eigenfunction $\psi_e$, that is:

$$\int_\Omega a(z)\psi_e(z)\mathrm{d}z = 0. \tag{82}$$

Accordingly if the orthogonality condition (82) is not fulfilled, a solution of the equation (80) does not exist. Otherwise, if the orthogonality condition (82) is fulfilled, the solution is not fully determined.

Hence if the term $(-1)$ is not an eigenvalue, the solution exists and the Fredholm equation (80) provides the function $\theta^*(x)$. Noting that the equality $(79)_3$ yields $\theta(x) = \theta^*(x)/\sqrt{A(x)}$, the strain $\varepsilon(x)$ can be obtained from the relation $(78)_3$ in the form:

$$\varepsilon(x) = \frac{\theta(x)\sigma}{E_o} = \frac{\theta(x)}{\int_\Omega \theta(z)\mathrm{d}z} w \tag{83}$$

being

$$\int_\Omega \theta(z)\mathrm{d}z = \frac{E_o w}{\sigma}.$$

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