NOTE

Drawings of $C_m \times C_n$ with One Disjoint Family II

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Received March 23, 1999

A long-standing conjecture states that the crossing number of the Cartesian product of cycles $C_m \times C_n$ is $(m-2)n$, for every $m, n$ satisfying $n \geq m \geq 3$. A crossing is proper if it occurs between edges in different principal cycles. In this paper drawings of $C_m \times C_n$ with the principal $n$-cycles pairwise disjoint or the principal $m$-cycles pairwise disjoint are analyzed, and it is proved that every such drawing has at least $(m-2)n$ proper crossings. As an application of this result, we prove that the crossing number of $C_m \times C_n$ is at least $(m-2)n/2$, for all integers $m, n$ such that $n \geq m \geq 4$. This is the best general lower bound known for the crossing number of $C_m \times C_n$.

1. INTRODUCTION

The crossing number $\text{cr}(G)$ of a graph $G$ is the minimum number of pairwise intersections of edges in a drawing of $G$ in the plane. A long-standing conjecture is that the crossing number of the Cartesian product $C_m \times C_n$ is at least $(m-2)n$, if $n \geq m \geq 3$ [5]. This has been proved only for $m \leq 6$ [1, 3, 4, 6–9], and for the special case $m = n = 7$ [2].

It is not difficult to exhibit drawings of $C_m \times C_n$ with exactly $(m-2)n$ crossings, which is thus an upper bound. General lower bounds are a lot more difficult to compute.

The Cartesian product $C_m \times C_n$ is a 4-regular graph with $mn$ vertices $v_{i,j}$, where $0 \leq i \leq m-1$ and $0 \leq j \leq n-1$. The vertices are labeled in such a way that the four vertices adjacent to $v_{i,j}$ are $v_{i-1,j}$, $v_{i+1,j}$, $v_{i,j+1}$, and $v_{i,j-1}$, where indices $i$ and $j$ are read modulo $m$ and $n$, respectively.

The edge set of $C_m \times C_n$ is naturally partitioned into $m$ edge sets of principal blue $n$-cycles and $n$ edge sets of principal red $m$-cycles. We label the blue cycles $(v_{i,j})$, $0 \leq j \leq n-1$, by $B_j$, $0 \leq i \leq m-1$, and the red cycles $(v_{i,j})$, $0 \leq i \leq m-1$, by $R_j$, $0 \leq j \leq n-1$.

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We remark that if \( m \neq n \), then the edges in \( C_m \times C_n \) and \( C_n \times C_m \) are colored in a different way.

A crossing in a drawing of \( C_m \times C_n \) is proper if no principal cycle contains both edges involved in the crossing. In Section 2 we give a short proof of the following statement.

**Theorem 1.** Let \( m, n \) be such that \( n \geq m \geq 3 \). Then every drawing of \( C_m \times C_n \) such that either the principal \( m \)-cycles are pairwise disjoint or the principal \( n \)-cycles are pairwise disjoint has at least \((m - 2) n\) proper crossings.

In [10], the following weaker version of Theorem 1 was proved: Every drawing of \( C_m \times C_n \) with either the principal \( m \)-cycles pairwise disjoint or the principal \( n \)-cycles pairwise disjoint has at least \((m - 2) n\) crossings. By Theorem 1, these \((m - 2) n\) crossings can be chosen to be proper. This refinement is crucial in the proof of the following result.

**Theorem 2.** If \( m, n \) are positive integers such that \( n \geq 4 \), then \( \text{cr}(C_m \times C_n) \geq (m - 2) n/2 \).

This clearly implies that if \( n \geq m \geq 4 \), then \( \text{cr}(C_m \times C_n) \) is at least half its conjectured value. Moreover, since \( C_m \times C_n \) and \( C_n \times C_m \) are isomorphic, this shows that if \( n > m \geq 4 \), then \( \text{cr}(C_m \times C_n) \) is strictly greater than half its conjectured value.

### 2. Proof of Theorem 1

In a drawing of \( C_m \times C_n \), a cycle \( C \) separates the cycles \( C' \) and \( C'' \) if \( C' \) and \( C'' \) are contained in different components of \( \mathbb{R}^2 \setminus C \).

**Lemma 3.** Suppose that \( \min\{p, q\} \geq 3 \). Let \( \mathcal{D} \) be a drawing of \( C_p \times C_q \), such that the \( p \) \( q \)-cycles are pairwise disjoint and no \( p \)-cycle separates two \( p \)-cycles. Then \( \mathcal{D} \) has at least \((p - 2) q\) proper crossings.

**Proof.** This proof is divided in 3 steps.

**Step 1. Definition of force.** The red \( p \)-cycles are naturally cyclically ordered \( R_0, R_1, ..., R_{q-1} \), so that each of the blue \( q \)-cycles has an edge that joins a vertex of \( R_{k-1} \) with a vertex of \( R_k \), for \( k = 0, 1, ..., q-1 \), where the indices are read modulo \( q \).

Let \( H_k \) denote the subgraph of \( C_p \times C_q \) induced by the vertices in \( R_{k-1} \cup R_k \), for each \( k = 0, 1, ..., q-1 \). Thus, \( H_k \) has \( 2p \) red edges and \( p \) blue edges, every red edge is in two of the \( H_k \), and every blue edge is in exactly one of the \( H_k \).

The force \( f(H_k) \) of \( H_k \) is the total number of proper crossings of the following types:
(1) a crossing of a blue edge in \( H_k \) with an edge in \( R_k \cup R_{k+1} \);
(2) a crossing of a blue edge in \( H_{k+1} \) with an edge in \( H_k \); and
(3) a crossing of a blue edge in \( H_k \) with a blue edge in \( H_k \).

Remark. If \( q = 3 \), then a crossing of a blue edge in \( H_k \) with an edge of \( R_{k+1} \) is counted in \( f(H_k) \) as type (1) and in \( f(H_{k-1}) \) as type (2). In this case we note that, since every blue edge of \( H_k \) can cross the cycle \( R_{k+1} \) only an even number of times, we can let one such crossing contribute to \( f(H_k) \) and another such crossing contribute to \( f(H_{k-1}) \).

It is not difficult to check that no crossing counted in \( f(H_k) \) is counted in \( f(H_j) \) if \( j \neq k \). Therefore the number \( cr_p(\mathcal{D}) \) of proper crossings in \( \mathcal{D} \) is at least \( \sum_{0 \leq q \leq q-1} f(H_k) \).

Therefore, in order to complete the proof of Lemma 3, it suffices to show that, for each \( k = 0, 1, \ldots, q-1 \), \( f(H_k) \geq p - 2 \). By the symmetry it suffices to show that \( f(H_0) \geq p - 2 \).

Step 2. Counting the contributions to \( f(H_0) \) of crossings of types (1) and (3). Let \( \mathcal{B} \) denote the collection of blue edges in \( H_0 \). Let \( \mathcal{B}_{R_0 \cup R_1} \) consist of the edges in \( \mathcal{B} \) that intersect \( R_0 \cup R_1 \). Let \( \mathcal{B}_a \) be a maximal collection of pairwise disjoint edges in \( \mathcal{B} \backslash \mathcal{B}_{R_0 \cup R_1} \).

Clearly, the crossings of edges in \( \mathcal{B}_{R_0 \cup R_1} \) contribute to \( f(H_0) \) in at least \( |\mathcal{B}_{R_0 \cup R_1}| \). Since each edge in \( \mathcal{B} \backslash (\mathcal{B}_{R_0 \cup R_1} \cup \mathcal{B}_a) \) contributes at least one edge in \( \mathcal{B}_a \), it follows that the edges in \( \mathcal{B} \backslash (\mathcal{B}_{R_0 \cup R_1} \cup \mathcal{B}_a) \) contribute to \( f(H_0) \) in at least \( p - (|\mathcal{B}_{R_0 \cup R_1}| + |\mathcal{B}_a|) \). Therefore, there are at least \( |\mathcal{B}_{R_0 \cup R_1}| + p - (|\mathcal{B}_{R_0 \cup R_1}| + |\mathcal{B}_a|) = p - |\mathcal{B}_a| \) crossings of types (1) and (3). Thus, to finish the proof it suffices to show that there are at least \( |\mathcal{B}_a| - 2 \) crossings of type (2).

Step 3. There are at least \( |\mathcal{B}_a| - 2 \) crossings of type (2). Let \( H_0' \) denote the subgraph of \( H_0 \) that consists of \( R_{q-1}, R_q \), and the edges in \( \mathcal{B}_a \). Since \( R_1 \) is disjoint from \( H_0' \) and is not separated by \( R_{q-1} \) or \( R_q \) from the other, there is a region \( F \) of \( H_0' \) that contains \( R_1 \). The blue edges in \( H_0' \) are pairwise disjoint, and so at least \( |\mathcal{B}_a| - 2 \) vertices on \( R_0 \) are not incident with \( F \); therefore each of the at least \( |\mathcal{B}_a| - 2 \) edges that join these vertices to \( R_1 \) crosses the boundary of \( F \). We note that each of these \( |\mathcal{B}_a| - 2 \) crossings is of type (2), as required.

Proof of Theorem 1. A drawing \( \mathcal{D} \) of \( C_m \times C_n \) is red-disjoint (respectively blue-disjoint) if the red (respectively blue) cycles are pairwise disjoint in \( \mathcal{D} \). The minimum number of proper crossings in a red-disjoint (respectively blue-disjoint) drawing of \( C_m \times C_n \) is denoted by \( cr_{p,rd}(C_m \times C_n) \) (respectively \( cr_{p,bd}(C_m \times C_n) \)). Using this notation, Theorem 1 states that if \( n \geq m \geq 3 \), then \( cr_{p,rd}(C_m \times C_n) \geq (m - 2)n \) and \( cr_{p,bd}(C_m \times C_n) \geq (m - 2)n \).
We show that \( cr_{p,n}(C_m \times C_n) \geq (m - 2) n \) by induction on \( m + n \). The base case \( m + n = 6 \) (that is, \( m = n = 3 \)) follows since every drawing of \( C_3 \times C_3 \) has at least 3 proper crossings (see [8]). Let \( m, n \) be such that \( n \geq m \geq 3 \), where \( m \) and \( n \) are not both equal to three, and suppose that

\[
    cr_{p,n}(C_m \times C_n) \geq (m' - 2) n'
\]

for all \( m', \ n' \) satisfying \( n' \geq m' \geq 3 \) and \( m' + n' < m + n \). Let \( \mathcal{D} \) be a bad-drawing of \( C_m \times C_n \). We now show that the number \( cr_p(\mathcal{D}) \) of proper crossings in \( \mathcal{D} \) is at least \( (m - 2) n \).

Suppose that no blue cycle separates two blue cycles in \( \mathcal{D} \). Then we can apply Lemma 3 with \( p = n \) and \( q = m \) to obtain that \( cr_p(\mathcal{D}) \geq (n - 2) m \geq (m - 2) n \), as required. Thus we assume that a blue cycle \( B \) separates two blue cycles in \( \mathcal{D} \). In this case \( B \) crosses every red cycle, and so \( B \) has at least \( n \) proper crossings. If \( m = 3 \), then \( cr_p(\mathcal{D}) \geq n = (m - 2) n \), as required, so suppose that \( m \geq 4 \). Let \( \mathcal{D}' \) be the drawing of \( C_{m-1} \times C_n \) that results by deleting \( B \) from \( \mathcal{D} \). By the induction hypothesis, \( cr_p(\mathcal{D}') \geq ((m - 1) - 2) n \), and since \( cr_p(\mathcal{D}) \geq cr_p(\mathcal{D}') + n \), it follows that \( cr_p(\mathcal{D}) \geq (m - 2) n \), as required.

The inequality \( cr_{p,n}(C_m \times C_n) \geq (m - 2) n \) is proved analogously.

3. PROOF OF THEOREM 2

A drawing is good if every intersection of edges is a crossing rather than tangential, no edge crosses itself, no two adjacent edges cross, and no two edges cross more than once. It is a routine exercise to show that if \( \mathcal{D} \) is a drawing of \( G \) with the minimum number of crossings, then \( \mathcal{D} \) is good.

Proof of Theorem 2. Let \( \mathcal{D} \) be a good drawing of \( C_m \times C_n \). Denote by \( P_{i,j} \) the 2-path joining \( v_{i,j-1} \) to \( v_{i,j+1} \). A 2-path \( P_{i,j} \) is bad if the rotation scheme around its middle vertex is red-blue-red-blue. Let \( \mathcal{P} = \{ P_{i,j} \}_{i=0}^{m-1}, \ j=0, \ldots, n-1 \), and let \( \mathcal{R} \) be a maximal subset of pairwise disjoint 2-paths in \( \mathcal{P} \). Let \( \mathcal{B} \) be the set of those 2-paths in \( \mathcal{P} \) which are bad, and let \( \mathcal{A} \) be the set of those 2-paths in \( \mathcal{P} \setminus \mathcal{B} \) which are crossed by an edge in \( R_{i-1} \cup R_i \cup R_{i+1} \).

Let \( \mathcal{G}_j = \mathcal{A} \setminus (\mathcal{B} \cup \mathcal{R}) \). We claim that there are at least \( |\mathcal{G}_j| - 2 \) proper red-red crossings that involve only edges in \( R_{i-1} \cup R_i \cup R_{i+1} \). Since the 2-paths in \( \mathcal{G}_j \) are pairwise disjoint, and no 2-path in \( \mathcal{G}_j \) is bad or crosses an edge in \( R_{i-1} \cup R_i \cup R_{i+1} \), the ends of each 2-path \( P_{i,j} \) in \( \mathcal{G}_j \) can be joined by a green edge, drawn very close to \( P_{i,j} \), so that the resulting green-blue 3-cycles are pairwise disjoint, and such that no edge in \( R_{i-1} \cup R_i \cup R_{i+1} \) crosses an edge in a 3-cycle. Thus, the 3-cycles form, together with \( R_{i-1} \), \( R_i \), and \( R_{i+1} \), a drawing of \( C_3 \times C_{|\mathcal{G}_j|} \) where no edge in the 3-cycles is crossed. Since by Theorem 1 such a drawing must have at least \( |\mathcal{G}_j| - 2 \) proper crossings, the claim follows.
A proper crossing is associated to $W_j$ if either (i) each edge involved in the crossing belongs to $P_i,j$ for some $i$ or to $R_{j-1}, R_j, R_{j+1}$; or (ii) it occurs between a blue edge not in $W_j$ and an edge in $R_j$. Each 2-path in $\mathcal{H}_j$ contains an edge involved in a red-blue crossing of type (i) associated to $W_j$, and, by Jordan’s Curve Theorem, for each 2-path in $\mathcal{H}_j$ there is a crossing of types (i) or (ii) associated to $W_j$. These observations show that there are at least $|\mathcal{H}_j| + |\mathcal{B}_j|$ red-blue crossings associated to $W_j$. Since there are at least $m \cdot |\mathcal{P}_j|$ blue-blue and at least $|\mathcal{C}_j| \cdot 2 = (|\mathcal{P}_j| + (|\mathcal{H}_j| + |\mathcal{B}_j|)) - 2$ red-red proper crossings associated to $W_j$, it follows that there are at least $m - 2$ crossings associated to $W_j$. A moment’s thought shows that, since $n \geq 4$, no crossing is associated to $W_j$ for more than two values of $j$. Thus, the total number of crossings in $\mathcal{D}$ is at least $\left(\frac{1}{2}\right) \sum_{j=0}^{n-1} (m - 2) = (m - 2) n/2$.

ACKNOWLEDGMENTS

We thank an anonymous referee for pointing out two inaccuracies in the proofs of Theorem 1 and Lemma 3 in an earlier version of this paper. In particular, the remark after the definition of force in the proof of Lemma 3 resulted from one of the referee’s observations.

REFERENCES